Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Geometria Riemanniana - 1º semestre de 2014/15

Teste 1 - 12/11/2014 - Duration: 90 min.

Write down all calculations and relevant justifications

1) Let $Q = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$. Consider the vector fields $X, Y \in \mathcal{X}(Q)$ defined by

$$X(x,y) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad Y(x,y) = \frac{1}{2y}\frac{\partial}{\partial x} + \frac{1}{2x}\frac{\partial}{\partial y}.$$

Let ψ_t and ϕ_t be the flows of X and Y respectively.

(3 pts) a) Determine ψ_t explicitly. Is X a complete vector field ?

(2 pts) b) Show that $\alpha(x,y) = \frac{y}{x}$ is constant along the integral curves of Y.

(3 pts) c) Show that $[X,Y] = 0$.

(2 pts) d) Determine $\phi_2 \circ \psi_3 \circ \phi_{-2} \circ \psi_{-3}(1,1)$.

(2 pts) e) Determine $(D\psi_1 Y)(x,y)$.

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2) Let Ω be the standard volume-form in $\mathbb{R}^n$. Consider on $\mathbb{R}^n$ the standard Euclidean inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Let $S \subset \mathbb{R}^n$ be a compact orientable submanifold of dimension $(n-1)$.

(2 pts) a) Let $\omega \in \Omega^{n-1}(\mathbb{R}^n)$. Determine the vector field $X_\omega \in \mathcal{X}(\mathbb{R}^n)$ defined by

$$\iota_{X_\omega}\Omega = \omega,$$

where $\iota_{X_\omega}\Omega \in \Omega^{n-1}(\mathbb{R}^n)$ is given by

$$\iota_{X_\omega}\Omega(Y_1, \ldots, Y_{n-1}) = \Omega(X_\omega, Y_1, \ldots, Y_{n-1}), \text{ for } Y_1, \ldots, Y_{n-1} \in \mathcal{X}(\mathbb{R}^n).$$

(2 pts) b) Let $n : S \to \mathbb{R}^n$ be a smooth vector field of unitary normals to S, that is

$$\|n(p)\| = 1,\ n(p) \in T_pS^\perp,\ \forall p \in S.$$

Show that n defines an orientation of S given by the volume-form $\theta_n = \iota_n\Omega$.

(2 pts) c) Show that, for any $\omega \in \Omega^{n-1}(\mathbb{R}^n)$,

$$\int_S \omega = \int_S \langle X_\omega, n \rangle \theta_n.$$

(2 pts) d) Prove the divergence theorem: If $U \subset \mathbb{R}^n$ is a bounded n-dimensional submanifold with boundary $\partial U = S$ and if $Y \in \mathcal{X}(\mathbb{R}^n)$, then

$$\int_U (\operatorname{div} Y)\Omega = \int_S \langle Y, n \rangle \theta_n,$$

where, in Cartesian coordinates,

$$\operatorname{div} Y = \sum_{j=1}^{n} \frac{\partial Y^j}{\partial x^j} \in C^\infty(\mathbb{R}^n).$$