APPENDIX B

Bessel functions

z	$J_0(z)$	$J_1(z)$	$J_2(z)$	$J_3(z)$	$J_4(z)$
0	1	0	0	0	0
0.0001	0.99999 99975 00000	0.00005 00000	1.250×10^{-09}	2.083×10^{-14}	2.604×10^{-19}
0.0002	0.99999 99900 00000	0.00010 00000	5.000×10^{-09}	1.667×10^{-13}	4.167×10^{-18}
0.0005	0.99999 99375 00001	0.00025 00000	3.125×10^{-08}	2.604×10^{-12}	1.628×10^{-16}
0.001	0.99999 97500 00016	0.00049 99999	0.00000 01250	2.083×10^{-11}	2.604×10^{-15}
0.002	0.99999 90000 00250	0.00099 99995	0.00000 05000	1.667×10^{-10}	4.167×10^{-14}
0.005	0.99999 37500 09766	0.00249 99922	0.00000 31250	2.604×10^{-09}	1.628×10^{-12}
0.01	0.99997 50001 56250	0.00499 99375	0.00001 24999	2.083×10^{-08}	2.604×10^{-11}
0.02	0.99990 00024 99972	0.00999 95000	0.00004 99983	0.00000 01667	4.167×10^{-10}
0.03	0.99977 50126 55934	0.01499 83126	0.00011 24916	0.00000 05625	2.109×10^{-09}
0.05	0.99937 50976 49468	0.02499 21883	0.00031 24349	0.00000 26038	1.628×10^{-08}
0.07	0.99877 53751 05191	0.03497 85669	0.00061 22499	0.00000 71436	6.251×10^{-08}
0.10	0.99750 15620 66040	0.04993 75260	0.00124 89587	0.00002 08203	0.00000 02603
0.15	0.99438 29052 14140	0.07478 92602	0.00280 72303	0.00007 02137	0.00000 13169
0.2	0.99002 49722 39576	0.09950 08326	0.00498 33542	0.00016 62504	0.00000 41583
0.3	0.97762 62465 38296	0.14831 88163	0.01116 58619	0.00055 93430	0.00002 09990
0.4	0.96039 82266 59563	0.19602 65780	0.01973 46631	0.00132 00532	0.00006 61351
0.5	0.93846 98072 40813	0.24226 84577	0.03060 40235	0.00256 37300	0.00016 07365
0.6	0.91200 48634 97211	0.28670 09881	0.04366 50967	0.00439 96567	0.00033 14704
0.7	0.88120 08886 07405	0.32899 57415	0.05878 69444	0.00692 96548	0.00061 00970
0.8	0.84628 73527 50480	0.36884 20461	0.07581 77625	0.01024 67663	0.00103 29850
0.9	0.80752 37981 22545	0.40594 95461	0.09458 63043	0.01443 40285	0.00164 05522
1.0	0.76519 76865 57967	0.44005 05857	0.11490 34849	0.01956 33540	0.00247 66390
1.1	0.71962 20185 27511	0.47090 23949	0.13656 41540	0.02569 45286	0.00358 78203
1.2	0.67113 27442 64363	0.49828 90576	0.15934 90183	0.03287 43369	0.00502 26663
1.3	0.62008 59895 61509	0.52202 32474	0.18302 66988	0.04113 58257	0.00683 09584
1.4	0.56695 51203 74289	0.54194 77139	0.20735 58995	0.05049 77133	0.00906 28717
1.5	0.51182 76717 35918	0.55793 65079	0.23208 76721	0.06096 39511	0.01176 81324
1.6	0.45540 21676 39381	0.56989 59353	0.25696 77514	0.07252 34433	0.01499 51611
1.7	0.39798 48594 46109	0.57776 52315	0.28173 89424	0.08514 99269	0.01879 02116
1.8	0.33998 64110 42558	0.58151 69517	0.30614 35353	0.09880 20157	0.02319 65169
1.9	0.28181 85593 74385	0.58115 70727	0.32992 57277	0.11342 34066	0.02825 34512
2.0	0.22389 07791 41236	0.57672 48078	0.35283 40286	0.12894 32495	0.03399 57198
2.1	0.16660 69803 31990	0.56829 21358	0.37462 36252	0.14527 66741	0.04045 25864
2.2	0.11036 22669 22174	0.55596 30498	0.39505 86875	0.16232 54728	0.04764 71475
2.3	0.05553 97844 45602	0.53987 25326	0.41391 45917	0.17997 89313	0.05559 56638
2.4	0.00250 76832 97244	0.52018 52682	0.43098 00402	0.19811 47988	0.06430 69568
2.5	-0.04838 37764 68198	0.49709 41025	0.44605 90584	0.21660 03910	0.07378 18801
2.6	-0.09680 49543 97038	0.47081 82665	0.45897 28517	0.23529 38130	0.08401 28707
2.7	-0.14244 93700 46012	0.44160 13791	0.46956 15027	0.25404 52916	0.09498 35897
2.8	-0.18503 60333 64387	0.40970 92469	0.47768 54954	0.27269 86037	0.10666 86554
2.9	-0.24431 15457 91968	0.37542 74818	0.48322 70505	0.29109 25878	0.11903 34761
3.0	-0.26005 19549 01933	0.33905 89585	0.48609 12606	0.30906 27223	0.13203 41839
3.1	-0.29206 43476 50698	0.30092 11331	0.48620 70142	0.32644 27561	0.14561 76751
3.2	-0.32018 81696 57123	0.26134 32488	0.48352 77001	0.34306 63764	0.15972 17556
3.3	-0.34429 62603 98885	0.22066 34530	0.47803 16865	0.35876 88942	0.17427 53940
3.4	-0.36429 55967 62000	0.17922 58517	0.46972 25683	0.37338 89346	0.18919 90810
3.5	-0.38012 77399 87263	0.13737 75274	0.45862 91842	0.38677 01117	0.20440 52930

z	$J_0(z)$	$J_1(z)$	$J_2(z)$	$J_3(z)$	$J_4(z)$
3.6	-0.39176 89837 00798	0.09546 55472	0.44480 53988	0.39876 26737	0.21979 90574
3.7	-0.39923 02033 71191	0.05383 39877	0.42832 96562	0.40922 51000	0.23527 86141
3.8	-0.40255 64101 78564	0.01282 10029	0.40930 43065	0.41802 56354	0.25073 61706
3.9	-0.40182 60148 87640	-0.02724 40396	0.38785 47125	0.42504 37448	0.26605 87410
4.0	-0.39714 98098 63847	-0.06604 33280	0.36412 81459	0.43017 14739	0.28112 90650
4.1	-0.38866 96798 35854	-0.10327 32577	0.33829 24809	0.43331 47026	0.29582 65960
4.2	-0.37655 70543 67568	-0.13864 69421	0.31053 47010	0.43439 42764	0.31002 85510
4.3	-0.36101 11172 36535	-0.17189 65602	0.28105 92288	0.43334 70056	0.32361 10116
4.4	-0.34225 67900 03886	-0.20277 55219	0.25008 50982	0.43012 65203	0.33645 00658
4.5	-0.32054 25089 85121	-0.23106 04319	0.21784 89837	0.42470 39730	0.34842 29803
4.6	-0.29613 78165 74141	-0.25655 28361	0.18459 31051	0.41706 85798	0.35940 93901
4.7	-0.26933 07894 19753	-0.27908 07358	0.15057 30295	0.40722 79950	0.36929 24960
4.8	-0.24042 53272 91183	-0.29849 98581	0.11605 03864	0.39520 85134	0.37796 02554
4.9	-0.20973 83275 85326	-0.31469 46710	0.08129 15231	0.38105 50980	0.38530 65561
5.0	-0.17759 67713 14338	-0.32757 91376	0.04656 51163	0.36483 12306	0.39123 23605
5.1	-0.14433 47470 60501	-0.33709 72020	0.01213 97659	0.34661 85870	0.39564 68071
5.2	-0.11029 04397 90987	-0.34322 30059	-0.02171 84086	0.32651 65377	0.39846 82598
5.3	-0.07580 31115 85584	-0.34596 08338	-0.05474 81465	0.30464 14780	0.39962 52913
5.4	-0.04121 01012 44991	-0.34534 47908	-0.08669 53768	0.28112 59931	0.39905 75914
5.5	-0.00684 38694 17819	-0.34143 82154	-0.11731 54816	0.25611 78651	0.39671 67891
5.6	0.02697 08846 85114	-0.33433 28363	-0.14637 54691	0.22977 89298	0.39256 71796
5.7	0.05992 00097 24037	-0.32414 76802	-0.17365 60379	0.20228 37940	0.38658 63473
5.8	0.09170 25675 74816	-0.31102 77443	-0.19895 35139	0.17381 84244	0.37876 56770
5.9	0.12203 33545 92823	-0.29514 24447	-0.22208 16409	0.14457 86204	0.36911 07464
6.0	0.15064 52572 50997	-0.27668 38581	-0.24287 32100	0.11476 83848	0.35764 15948
6.1	0.17729 14222 42744	-0.25586 47726	-0.26118 15116	0.08459 82076	0.34439 28633
6.2	0.20174 72229 48904	-0.23291 65671	-0.27688 15994	0.05428 32771	0.32941 38031
6.3	0.22381 20061 32191	-0.20808 69402	-0.28987 13522	0.02404 16372	0.31276 81496
6.4	0.24331 06048 23407	-0.18163 75090	-0.30007 23264	-0.00590 76950	0.29453 38623
6.5	0.26009 46055 81606	-0.15384 13014	-0.30743 03906	-0.03534 66313	0.27480 27310
6.6	0.27404 33606 24146	-0.12498 01652	-0.31191 61379	-0.06405 99184	0.25367 98485
6.7	0.28506 47377 10576	-0.09534 21180	-0.31352 50715	-0.09183 70291	0.23128 29558
6.8	0.29309 56031 04273	-0.06521 86634	-0.31227 75629	-0.11847 40207	0.20774 16623
6.9	0.29810 20354 04820	-0.03490 20961	-0.30821 85850	-0.14377 53445	0.18319 65463
7.0	0.30007 92705 19556	-0.00468 28235	-0.30141 72201	-0.16755 55880	0.15779 81447
7.1	0.29905 13805 01550	0.02515 32743	-0.29196 59511	-0.18964 11340	0.13170 58379
7.2	0.29507 06914 00958	0.05432 74202	-0.27997 97413	-0.20987 17210	0.10508 66405
7.3	0.28821 69476 35014	0.08257 04305	-0.26559 49119	-0.22810 18891	0.07811 39072
7.4	0.27859 62326 57478	0.10962 50949	-0.24896 78286	-0.24420 22995	0.05096 59642
7.5	0.26633 96578 80378	0.13524 84276	-0.23027 34105	-0.25806 09132	0.02382 46800
7.6	0.25160 18338 49976	0.15921 37684	-0.20970 34737	-0.26958 40177	-0.00312 60139
7.7	0.23455 91395 86464	0.18131 27153	-0.18746 49278	-0.27869 70934	-0.02970 16385
7.8	0.21540 78077 46263	0.20135 68728	-0.16377 78404	-0.28534 55088	-0.05571 87049
7.9	0.19436 18448 41278	0.21917 93999	-0.13887 33892	-0.28949 50400	-0.08099 62615
8.0	0.17165 08071 37554	0.23463 63469	-0.11299 17204	-0.29113 22071	-0.10535 74349
8.1	0.14751 74540 44378	0.24760 77670	-0.08637 97338	-0.29026 44256	-0.12863 09519
8.2	0.12221 53017 84138	0.25799 85976	-0.05928 88146	-0.28691 99706	-0.15065 26274
8.3	0.09600 61008 95010	0.26573 93020	-0.03197 25341	-0.28114 77522	-0.17126 68048
8.4	0.06915 72616 56985	0.27078 62683	-0.00468 43406	-0.27301 69067	-0.19032 77356
8.5	0.04193 92518 42935	0.27312 19637	0.02232 47396	-0.26261 62039	-0.20770 08835
8.6	0.01462 29912 78741	0.27275 48445	0.48808 36792	-0.25005 32781	-0.22326 41433
8.7	-0.01252 27324 49665	0.26971 90241	0.07452 71058	-0.23545 36881	-0.23690 89597
8.8	-0.03923 38031 76542	0.26407 37032	0.09925 05539	-0.21895 98151	-0.24854 13369
8.9	-0.06525 32468 51244	0.25590 23714	0.12275 93977	-0.20072 96084	-0.25808 27293
9.0	-0.09033 36111 82876	0.24531 17866	0.14484 73415	-0.18093 51903	-0.26547 08018
9.5	-0.19392 87476 87422	0.16126 44308	0.22787 91542	-0.06531 53132	-0.26913 09309
10.0	-0.24593 57644 51348	0.04347 27462	0.25463 03137	0.05837 93793	-0.21960 26861
10.5	-0.23664 81944 62347	-0.07885 00142	0.22162 91441	0.16328 01644	-0.12832 61931
11.0	-0.17119 03004 07196	-0.17678 52990	0.13904 75188	0.22734 80331	-0.01503 95007

z	$J_0(z)$	$J_1(z)$	$J_2(z)$	$J_3(z)$	$J_4(z)$
11.5	-0.06765 39481 11665	-0.22837 86207	0.02793 59271	0.23809 54649	0.09628 77937
12.0	0.04768 93107 96834	-0.22344 71045	-0.08493 04949	0.19513 69395	0.18249 89646
12.5	0.14688 40547 00421	-0.16548 38046	-0.17336 14634	0.11000 81363	0.22616 53689
13.0	0.20692 61023 77068	-0.07031 80521	-0.21774 42642	0.00331 98170	0.21927 64875
13.5	0.21498 91658 80401	0.03804 92921	-0.20935 22337	-0.10007 95836	0.16487 24188
14.0	0.17107 34761 10459	0.13337 51547	-0.15201 98826	-0.17680 94069	0.07624 44225
14.5	0.08754 48680 10376	0.19342 94636	-0.06086 49420	-0.21021 97924	-0.02612 25583
15.0	-0.01422 44728 26781	0.20510 40386	0.04157 16780	-0.19401 82578	-0.11917 89811
15.5	-0.10923 06509 00050	0.16721 31804	0.13080 65451	-0.13345 66526	-0.18246 71848
16.0	-0.17489 90739 83629	0.09039 71757	0.18619 87209	-0.04384 74954	-0.20264 15317
16.5	-0.19638 06929 36861	-0.00576 42137	0.19568 20004	0.05320 22744	-0.17633 57188
17.0	-0.16985 42521 51184	-0.09766 84928	0.15836 38412	0.13493 05730	-0.11074 12860
17.5	-0.10311 03982 28686	-0.16341 99694	0.08443 38303	0.18271 91306	-0.02178 72712
18.0	-0.01335 58057 21984	-0.18799 48855	-0.00753 25149	0.18632 09933	0.06963 95127
18.5	0.07716 48214 22555	-0.16663 36400	-0.09517 92690	0.14605 43386	0.14254 82437
19.0	0.14662 94396 59651	-0.10570 14311	-0.15775 59061	0.07248 96614	0.18064 73781
19.5	0.17885 38270 40173	-0.02087 70701	-0.18099 50650	-0.01625 01227	0.17599 50273
20.0	0.16702 46643 40583	0.06683 31242	-0.16034 13519	-0.09890 13946	0.13067 09336

z	$J_5(z)$	$J_6(z)$	$J_7(z)$	$J_8(z)$	$J_9(z)$
0	0	0	0	0	0
0.1	2.603×10^{-09}	2.169×10^{-11}	1.550×10^{-13}	9.685×10^{-16}	5.380×10^{-18}
0.2	8.319×10^{-08}	1.387×10^{-09}	1.982×10^{-11}	2.477×10^{-13}	2.753×10^{-15}
0.3	0.00000 06304	1.577×10^{-08}	3.381×10^{-10}	6.341×10^{-12}	1.057×10^{-13}
0.4	0.00000 26489	8.838×10^{-08}	2.527×10^{-09}	6.321×10^{-11}	1.405×10^{-12}
0.5	0.00000 80536	0.00000 03361	1.202×10^{-08}	3.758×10^{-10}	1.045×10^{-11}
0.6	0.00001 99482	0.00000 09996	4.291×10^{-08}	1.611×10^{-09}	5.375×10^{-11}
0.7	0.00004 28824	0.00000 25088	0.00000 01257	5.509×10^{-09}	2.145×10^{-10}
0.8	0.00008 30836	0.00000 55601	0.00000 03186	1.597×10^{-08}	7.109×10^{-10}
0.9	0.00014 86580	0.00001 12036	0.00000 07229	4.077×10^{-08}	2.043×10^{-09}
1.0	0.00024 97577	0.00002 09383	0.00000 15023	9.422×10^{-08}	5.249×10^{-09}
1.1	0.00039 87099	0.00003 68150	0.00000 29084	0.00000 02008	1.231×10^{-08}
1.2	0.00061 01049	0.00006 15414	0.00000 53093	0.00000 04002	2.679×10^{-08}
1.3	0.00090 08414	0.00009 85905	0.00000 92248	0.00000 07540	5.471×10^{-08}
1.4	0.00129 01251	0.00015 23073	0.00001 53661	0.00000 13538	0.00000 01059
1.5	0.00179 94218	0.00022 80127	0.00002 46798	0.00000 23321	0.00000 01956
1.6	0.00245 23620	0.00033 21012	0.00003 83972	0.00000 38744	0.00000 03469
1.7	0.00327 45981	0.00047 21304	0.00005 80872	0.00000 62348	0.00000 05936
1.8	0.00429 36149	0.00065 68991	0.00008 57125	0.00000 97534	0.00000 09843
1.9	0.00553 84930	0.00089 65121	0.00012 36884	0.00001 48764	0.00000 15863
2.0	0.00703 96298	0.00120 24290	0.00017 49441	0.00002 21796	0.00000 24923
2.1	0.00882 84171	0.00158 74951	0.00024 29833	0.00003 23938	0.00000 38266
2.2	0.01093 68819	0.00206 59518	0.00033 19463	0.00004 64337	0.00000 57535
2.3	0.01339 72905	0.00265 34256	0.00044 66689	0.00006 54286	0.00000 84866
2.4	0.01624 17239	0.00336 68927	0.00059 27398	0.00009 07560	0.00001 23002
2.5	0.01950 16251	0.00422 46205	0.00077 65532	0.00012 40774	0.00001 75420
2.6	0.02320 73276	0.00524 60815	0.00100 53563	0.00016 73755	0.00002 46466
2.7	0.02738 75668	0.00645 18427	0.00128 72898	0.00022 29934	0.00003 41524
2.8	0.03206 89832	0.00786 34275	0.00163 14204	0.00029 36744	0.00004 67189
2.9	0.03727 56220	0.00950 31514	0.00204 77633	0.00038 26023	0.00006 31459
3.0	0.04302 84349	0.01139 39323	0.00254 72945	0.00049 34418	0.00008 43950
3.1	0.04934 47926	0.01355 90753	0.00314 19503	0.00063 03778	0.00011 16123
3.2	0.05623 80126	0.01602 20338	0.00384 46142	0.00079 81533	0.00014 61522
3.3	0.06371 69093	0.01880 61494	0.00466 90886	0.00100 21053	0.00018 96036
3.4	0.07178 53735	0.02193 43706	0.00563 00521	0.00124 81970	0.00024 38159
3.5	0.08044 19866	0.02542 89545	0.00674 30003	0.00154 30467	0.00031 09276
4.0	0.13208 66560	0.04908 75752	0.01517 60694	0.00402 86678	0.00093 86019
4.5	0.19471 46586	0.08427 62611	0.03002 20377	0.00912 56340	0.00242 46609
5.0	0.26114 05461	0.13104 87318	0.05337 64102	0.01840 52167	0.00552 02831

z	$J_5(z)$	$J_6(z)$	$J_7(z)$	$J_8(z)$	$J_9(z)$
5.5	0.32092 47371	0.18678 27330	0.08660 12258	0.03365 67508	0.01130 93220
6.0	0.36208 70749	0.24583 68634	0.12958 66518	0.05653 19909	0.02116 53240
6.5	0.37356 53771	0.29991 32338	0.18012 05930	0.08803 88126	0.03659 03304
7.0	0.34789 63248	0.33919 66050	0.23358 35695	0.12797 05340	0.05892 05083
7.5	0.28347 39052	0.35414 05269	0.28315 09379	0.17440 78905	0.08891 92285
8.0	0.18577 47722	0.33757 59001	0.32058 90780	0.22345 49864	0.12632 08947
8.5	0.06713 30194	0.28668 09063	0.33759 29660	0.26935 45671	0.16942 73956
9.0	-0.05503 88557	0.20431 65177	0.32746 08792	0.30506 70723	0.21488 05825
9.5	-0.16132 12602	0.09931 90781	0.28677 69378	0.32329 95671	0.25772 75962
10.0	-0.23406 15282	-0.01445 88421	0.21671 09177	0.31785 41268	0.29185 56853
10.5	-0.26105 25019	-0.12029 52374	0.12357 22307	0.28505 82116	0.31080 21870
11.0	-0.23828 58518	-0.20158 40009	0.01837 60326	0.22497 16788	0.30885 55001
11.5	-0.17111 26519	-0.24508 14040	-0.08462 44654	0.14206 03158	0.28227 36003
12.0	-0.07347 09631	-0.24372 47672	-0.17025 38041	0.04509 53291	0.23038 09096
12.5	0.03473 76998	-0.19837 52091	-0.22517 79005	-0.05382 40395	0.15628 31300
13.0	0.13161 95599	-0.11803 06721	-0.24057 09496	-0.14104 57351	0.06697 61987
13.5	0.19778 17577	-0.01836 74131	-0.21410 83471	-0.21410 83471	-0.20367 08728
14.0	0.22037 76483	0.08116 81834	-0.15080 49196	-0.23197 31031	-0.11430 71981
14.5	0.19580 73465	0.16116 21076	-0.06243 18091	-0.22144 10957	-0.18191 69861
15.0	0.13045 61346	0.20614 97375	0.03446 36554	-0.17398 36591	-0.22004 62251
15.5	0.03928 00410	0.20780 91468	0.12160 44597	-0.09797 28606	-0.22273 77352
16.0	-0.05747 32704	0.16672 07377	0.18251 38237	-0.00702 11420	-0.18953 49657
16.5	-0.13869 83805	0.09227 60942	0.20580 82672	0.08234 91022	-0.12595 45923
17.0	-0.18704 41194	0.00071 53334	0.18754 90607	0.15373 68342	-0.04285 55697
17.5	-0.19267 90261	-0.08831 50294	0.13212 01488	0.19401 11484	0.04526 14726
18.0	-0.15537 00988	-0.15595 62342	0.05139 92760	0.19593 34488	0.12276 37897
18.5	-0.08441 18549	-0.18817 62733	-0.03764 84305	0.15968 55691	0.17575 48687
19.0	0.00357 23925	-0.17876 71715	-0.11647 79745	0.09294 12956	0.19474 43287
19.5	0.08845 32108	-0.13063 44063	-0.16884 36147	0.00941 33496	0.17656 73888
20.0	0.15116 97680	-0.05508 60496	-0.18422 13977	-0.07386 89288	0.12512 62546

z	$J_{10}(z)$	$J_{11}(z)$	$J_{12}(z)$	$J_{13}(z)$	$J_{14}(z)$
0	0	0	0	0	0
0.1	2.691×10^{-20}	1.223×10^{-22}	5.096×10^{-25}	1.960×10^{-27}	7.000×10^{-30}
0.2	2.753×10^{-17}	2.503×10^{-19}	2.086×10^{-21}	1.605×10^{-23}	1.146×10^{-25}
0.3	1.586×10^{-15}	2.163×10^{-17}	2.704×10^{-19}	3.120×10^{-21}	3.344×10^{-23}
0.4	2.812×10^{-14}	5.114×10^{-16}	8.525×10^{-18}	1.312×10^{-19}	1.874×10^{-21}
0.5	2.613×10^{-13}	5.942×10^{-15}	1.238×10^{-16}	2.382×10^{-18}	4.255×10^{-20}
0.6	1.614×10^{-12}	4.405×10^{-14}	1.102×10^{-15}	2.544×10^{-17}	5.454×10^{-19}
0.7	7.518×10^{-12}	2.394×10^{-13}	6.989×10^{-15}	1.883×10^{-16}	4.710×10^{-18}
0.8	2.848×10^{-11}	1.037×10^{-12}	3.460×10^{-14}	1.065×10^{-15}	3.046×10^{-17}
0.9	9.212×10^{-11}	3.774×10^{-12}	1.417×10^{-13}	4.911×10^{-15}	1.580×10^{-16}
1.0	2.631×10^{-10}	1.198×10^{-11}	5.000×10^{-13}	1.925×10^{-14}	6.885×10^{-16}
1.1	6.791×10^{-10}	3.403×10^{-11}	1.563×10^{-12}	6.623×10^{-14}	2.606×10^{-15}
1.2	1.613×10^{-09}	8.820×10^{-11}	4.420×10^{-12}	2.044×10^{-13}	8.776×10^{-15}
1.3	3.570×10^{-09}	2.116×10^{-10}	1.149×10^{-11}	5.761×10^{-13}	2.680×10^{-14}
1.4	7.444×10^{-09}	4.755×10^{-10}	2.783×10^{-11}	1.502×10^{-12}	7.529×10^{-14}
1.5	1.474×10^{-08}	1.010×10^{-09}	6.333×10^{-11}	3.665×10^{-12}	1.969×10^{-13}
1.6	2.791×10^{-08}	2.040×10^{-09}	1.366×10^{-10}	8.433×10^{-12}	4.834×10^{-13}
1.7	5.080×10^{-08}	3.947×10^{-09}	2.809×10^{-10}	1.844×10^{-11}	1.123×10^{-12}
1.8	8.924×10^{-08}	7.347×10^{-09}	5.539×10^{-10}	3.852×10^{-11}	2.486×10^{-12}
1.9	0.00000 01520	1.321×10^{-08}	1.052×10^{-09}	7.728×10^{-11}	5.267×10^{-12}
2.0	0.00000 02515	2.304×10^{-08}	1.933×10^{-09}	1.495×10^{-10}	1.073×10^{-11}
2.1	0.00000 04059	3.907×10^{-08}	3.443×10^{-09}	2.798×10^{-10}	2.110×10^{-11}
2.2	0.00000 06400	6.460×10^{-08}	5.968×10^{-09}	5.084×10^{-10}	4.018×10^{-11}
2.3	0.00000 09880	0.00000 01043	1.009×10^{-08}	8.987×10^{-10}	7.430×10^{-11}
2.4	0.00000 14958	0.00000 01650	1.665×10^{-08}	1.550×10^{-10}	1.338×10^{-10}
2.5	0.00000 22247	0.00000 02559	2.693×10^{-08}	2.612×10^{-09}	2.349×10^{-10}
2.6	0.00000 32547	0.00000 03897	4.268×10^{-08}	4.309×10^{-09}	4.034×10^{-10}

z	$J_{10}(z)$	$J_{11}(z)$	$J_{12}(z)$	$J_{13}(z)$	$J_{14}(z)$
2.7	0.00000 46894	0.00000 05837	6.645×10^{-08}	6.971×10^{-09}	6.781×10^{-10}
2.8	0.00000 66611	0.00000 08607	0.00000 01017	1.107×10^{-08}	1.118×10^{-09}
2.9	0.00000 93376	0.00000 12511	0.00000 01533	1.729×10^{-08}	1.810×10^{-09}
3.0	0.00001 29284	0.00000 17940	0.00000 02276	2.659×10^{-08}	2.880×10^{-09}
3.1	0.00001 76936	0.00000 25402	0.00000 03333	4.028×10^{-08}	4.512×10^{-09}
3.2	0.00002 39530	0.00000 35542	0.00000 04819	6.017×10^{-08}	6.962×10^{-09}
3.3	0.00003 20960	0.00000 49177	0.00000 06884	8.872×10^{-08}	1.059×10^{-08}
3.4	0.00004 25933	0.00000 67328	0.00000 09721	0.00000 01292	1.591×10^{-08}
3.5	0.00005 60095	0.00000 91267	0.00000 13581	0.00000 01860	2.360×10^{-08}
4.0	0.00019 50406	0.00003 66009	0.00000 62645	0.00000 09859	0.00000 01436
4.5	0.00057 30098	0.00012 20492	0.00002 36751	0.00000 42179	0.00000 06950
5.0	0.00146 78026	0.00035 09274	0.00007 62781	0.00001 52076	0.00000 28013
5.5	0.00335 55759	0.00089 27721	0.00021 55123	0.00004 76455	0.00000 97207
6.0	0.00696 39810	0.00204 79460	0.00054 51544	0.00013 26717	0.00002 97564
6.5	0.01328 82562	0.00429 66118	0.00125 41220	0.00033 39927	0.00008 18487
7.0	0.02353 93444	0.00833 47614	0.00265 56200	0.00077 02216	0.00020 52029
7.5	0.03899 82579	0.01507 61259	0.00522 50447	0.00164 40171	0.00047 42147
8.0	0.06076 70268	0.02559 66722	0.00962 38218	0.00327 47932	0.00101 92562
8.5	0.08943 28589	0.04100 28606	0.01669 21921	0.00612 80346	0.00205 23844
9.0	0.12469 40928	0.06221 74015	0.02739 28887	0.01083 03016	0.00398 46493
9.5	0.16502 64047	0.08969 64137	0.04269 16060	0.01815 60646	0.00699 86761
10.0	0.20748 61066	0.12311 65280	0.06337 02550	0.02897 20839	0.01195 71632
10.5	0.24774 55375	0.16109 40750	0.08978 49053	0.04412 85657	0.01948 58287
11.0	0.28042 82305	0.20101 40099	0.12159 97893	0.06429 46213	0.03036 93155
11.5	0.29975 92326	0.23904 68041	0.15754 76971	0.08974 83898	0.04536 17059
12.0	0.30047 60353	0.27041 24826	0.19528 01827	0.12014 78829	0.06504 02303
12.5	0.27887 17466	0.28991 16646	0.23137 27831	0.15432 40789	0.08962 13011
13.0	0.23378 20102	0.29268 84324	0.26153 68754	0.19014 88760	0.11876 08767
13.5	0.16729 84008	0.27512 88367	0.28105 97034	0.22453 28582	0.15137 39495
14.0	0.08500 67054	0.23574 53488	0.28545 02712	0.25359 79733	0.18551 73935
14.5	-0.00438 68871	0.17586 61074	0.27121 82225	0.27304 68125	0.21838 29586
15.0	-0.09007 18110	0.09995 04771	0.23666 58441	0.27871 48734	0.24643 99366
15.5	-0.16069 03157	0.01539 53923	0.18254 18403	0.26725 00378	0.26574 85457
16.0	-0.20620 56944	-0.06822 21524	0.11240 02349	0.23682 25048	0.27243 63353
16.5	-0.21975 41120	-0.14041 40283	0.03253 54076	0.18773 82576	0.26329 45740
17.0	-0.19911 33197	-0.19139 53947	-0.04857 48381	0.12281 91527	0.23641 58951
17.5	-0.14745 64908	-0.21378 31764	-0.12129 95024	0.04742 95731	0.19176 62968
18.0	-0.07316 96592	-0.20406 34110	-0.17624 11765	-0.03092 48243	0.13157 19858
18.5	0.11319 16799	-0.16351 79303	-0.20577 29230	-0.10343 07265	0.06041 08209
19.0	0.09155 33316	-0.09837 24007	-0.20545 82166	-0.16115 37677	-0.01506 79918
19.5	0.15357 19323	-0.01905 77146	-0.17507 29436	-0.19641 66776	-0.08681 59598
20.0	0.18648 25580	0.06135 63034	-0.11899 06243	-0.20414 50525	-0.14639 79440

Table of zeros of Bessel functions

Note: The kth zero of J_n is denoted $j_{n,k}$.

k	J_0	J_1	J_2	J_3	J_4	J_5	J_6	J_7
1	2.40482 55577	3.831706	5.135622	6.380162	7.588342	8.771484	9.936110	11.08637
2	5.52007 81103	7.015587	8.417244	9.761023	11.06471	12.33860	13.58929	14.82127
3	8.65372 79129	10.17347	11.61984	13.01520	14.37254	15.70017	17.00382	18.28758
4	11.79153 44391	13.32369	14.79595	16.22347	17.61597	18.98013	20.32079	21.64154
5	14.93091 77086	16.47063	17.95982	19.40942	20.82693	22.21780	23.58608	24.93493
6	18.07106 39679	19.61586	21.11700	22.58273	24.01902	25.43034	26.82015	28.19119
7	21.21163 66299	22.76008	24.27011	25.74817	27.19909	28.62662	30.03372	31.42279
8	24.35247 15308	25.90367	27.42057	28.90835	30.37101	31.81172	33.23304	34.63709
9	27.49347 91320	29.04683	30.56920	32.06485	33.53714	34.98878	36.42202	37.83872
10	30.63460 64684	32.18968	33.71652	35.21867	36.69900	38.15987	39.60324	41.03077
11	33.77582 02136	35.33231	36.86286	38.37047	39.85763	41.32638	42.77848	44.21541
12	36.91709 83537	38.47477	40.00845	41.52072	43.01374	44.48932	45.94902	47.39417
13	40.05842 57646	41.61709	43.15345	44.66974	46.16785	47.64940	49.11577	50.56818
14	43.19979 17132	44.75932	46.29800	47.81779	49.32036	50.80717	52.27945	53.73833
15	46.34118 83717	47.90146	49.44216	50.96503	52.47155	53.96303	55.44059	56.90525

k	J_8	J_9	J_{10}	J_{11}	J_{12}	J_{13}	J_{14}	J_{15}
1	12.22509	13.35430	14.47550	15.58985	16.69825	17.80144	18.90000	19.99443
2	16.03777	17.24122	18.43346	19.61597	20.78991	21.95624	23.11578	24.26918
3	19.55454	20.80705	22.04699	23.27585	24.49489	25.70510	26.90737	28.10242
4	22.94517	24.23389	25.50945	26.77332	28.02671	29.27063	30.50595	31.73341
5	26.26681	27.58375	28.88738	30.17906	31.45996	32.73105	33.99318	35.24709
6	29.54566	30.88538	32.21186	33.52636	34.82999	36.12366	37.40819	38.68428
7	32.79580	34.15438	35.49991	36.83357	38.15638	39.46921	40.77283	42.06792
8	36.02562	37.40010	38.76181	40.11182	41.45109	42.78044	44.10059	45.41219

k	J_{16}	J_{17}	J_{18}	J_{19}	J_{20}	J_{21}	J_{22}	J_{23}
1	21.08515	22.17249	23.25678	24.33825	25.41714	26.49365	27.56794	28.64019
2	25.41701	26.55979	27.69790	28.83173	29.96160	31.08780	32.21059	33.33018
3	29.29087	30.47328	31.65012	32.82180	33.98870	35.15115	36.30943	37.46381
4	32.95366	34.16727	35.37472	36.57645	37.77286	38.96429	40.15105	41.33343
5	36.49340	37.73268	38.96543	40.19210	41.41307	42.62870	43.83932	45.04521

k	J_{24}	J_{25}	J_{26}	J_{27}	J_{28}	J_{29}	J_{30}	J_{31}
1	29.71051	30.77904	31.84589	32.91115	33.97493	35.03730	36.09834	37.15811
2	34.44678	35.56057	36.67173	37.78040	38.88671	39.99080	41.09278	42.19275

Fourier series

$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta$$

$$\cos(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta$$

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)\,d\theta.$$

Differential equation

$$J_n''(z) + \frac{1}{z}J_n'(z) + \left(1 - \frac{n^2}{z^2}\right) J_n(z) = 0$$

Power series

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2})^{n+2k}}{k!(n+k)!}$$

Generating function

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(z)t^n$$

Limiting values

If n is constant, z is real and $|z| \to \infty$,

$$J_n(z) = \sqrt{\tfrac{2}{\pi z}} \cos(z - \tfrac{1}{2}(n + \tfrac{1}{2})\pi) + O(|z|^{-3/2}).$$

[Here, $O(|z|^{-3/2})$ represents an error term which is bounded by some constant multiple of $|z|^{-3/2}$]

If z is constant and $n \to \infty$, $J_n(z) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{2n}\right)^n$.

For n fixed, as $k \to \infty$, $j_{n,k} \sim (k + \frac{1}{2}n - \frac{1}{4})\pi$.

Other formulas

$$J_{-n}(z) = (-1)^n J_n(z)$$

$$J_n'(z) = \tfrac{1}{2}(J_{n-1}(z) - J_{n+1}(z))$$

$$J_n(z) = \tfrac{z}{2n}(J_{n-1}(z) + J_{n+1}(z))$$

$$\frac{d}{dz}(z^n J_n(z)) = z^n J_{n-1}(z)$$

$$1 = \sum_{n=-\infty}^{\infty} J_n(z) = J_0(z) + 2J_2(z) + 2J_4(z) + 2J_6(z) + \dots$$

$$1 = \sum_{n=-\infty}^{\infty} J_n(z)^2 = J_0(z)^2 + 2J_1(z)^2 + 2J_2(z)^2 + 2J_3(z)^2 + \dots$$

In particular, $|J_n(z)| \leq 1$ for all n and z, and if $n \neq 0$ then $|J_n(z)| \leq \frac{1}{\sqrt{2}}$.

Computation

Although the power series converges very quickly for small values of z, and converges for all values of z, rounding errors tend to accumulate for larger z because a small number is resulting from addition and subtraction of very large numbers.

Instead, a computer program for calculating the Bessel functions can be based on the recurrence relation $J_n(z) = (2(n+1)/z)J_{n+1}(z) - J_{n+2}(z)$ and normalizing via the relation $J_0(z) + 2J_2(z) + 2J_4(z) + \cdots = 1$. This is called *Miller's backwards recurrence algorithm* (J. C. P. Miller, *The Airy integral*, 1946). Build an array indexed by n and make the last two entries 1 and 0, use the recurrence relation to calculate the remaining entries, and then normalize. An array containing 100 entries gives reasonable accuracy, and does not consume much memory. Here is a simple C++ program which implements this method. I haven't put in any exception checking.

```
/* file bessel.cpp */
#include <iostream.h>
#include <stdio.h>
#define length 100
void main() {
  long double X[length], z, sum;
  int n=0, j=0;
  X[length - 2]=1; X[length - 1]=0;
  while (1)
  {
    printf("\n\nOrder (integer); -1 to exit: ");
    cin>>n;
    if (n<0)
      break;
    printf("Argument (real): ");
    cin>>z;
    if (z==0)        // prevent divide by zero
      {printf("J_0(0)=1; J_n(0)=0 (n>0)");}
    else
      {for(j=length - 3; j>=0; --j)
        {X[j]=(2*(j+1)/z)*X[j+1] - X[j+2];}
      sum=X[0];
      for(j=2; j < length; j=j+2)
        {sum+=2*X[j];}
      printf("J_%d(%Lf)= %11.10Lf",n,z,X[n]/sum);
    }
  }
}
```

I compiled this program using Borland C++. It prints out the answer to 10 decimal places, and at least for reasonably small values of n and z, up to about 50, the answers it gives agree with published tables to this accuracy. If you need more accuracy, I recommend the standard Unix multiple precision arithmetic utility bc. If invoked with the option -l (which loads the library mathlib of mathematical functions), it recognises the syntax j(n,z) and calculates $J_n(z)$ using the above algorithm. The number of digits after the decimal point is set to 50, for example, by using the command scale=50. Windows users can use bc in the free Unix environment Cygwin (http://www.cygwin.com); there is also a (free) version compiled for MS-DOS in UnxUtils.zip (http://unxutils.sourceforge.net). Here is a sample session:

```
$ bc -l
j(1,1)
.44005058574493351595
scale=50
for (n=0;n<5;n++) {j(n,1)}
.76519768655796655144971752610266322090927428975532
.44005058574493351595968220371891491312737230199276
.11490348493190048046964688133516660534547031423020
.01956335398266840591890532162175150825450895492805
.00247663896410995504378504839534244418158341533812
quit
$
```

FM Synthesis

$$\sin(\phi + z\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(z)\sin(\phi + n\theta)$$

The following table shows how index of modulation (z) varies as a function of operator output level (an integer in the range 0–99) on the Yamaha six operator synthesizers DX7, DX7IID, DX7IIFD, DX7S, DX5, DX1, TX7, TX816, TX216, TX802 and TF1:

	0	1	2	3	4	5	6	7	8	9
0	0.0002	0.0003	0.0005	0.0007	0.0010	0.0012	0.0016	0.0019	0.0023	0.0027
10	0.0032	0.0038	0.0045	0.0054	0.0064	0.0076	0.0083	0.0091	0.0108	0.0118
20	0.0140	0.0152	0.0166	0.0181	0.0198	0.0216	0.0235	0.0256	0.0280	0.0305
30	0.0332	0.0362	0.0395	0.0431	0.0470	0.0513	0.0559	0.0610	0.0665	0.0725
40	0.0791	0.0862	0.0940	0.1025	0.1118	0.1219	0.1330	0.1450	0.1581	0.1724
50	0.1880	0.2050	0.2236	0.2438	0.2659	0.2900	0.3162	0.3448	0.3760	0.4101
60	0.4472	0.4877	0.5318	0.5799	0.6324	0.6897	0.7521	0.8202	0.8944	0.9754
70	1.0636	1.1599	1.2649	1.3794	1.5042	1.6403	1.7888	1.9507	2.1273	2.3198
80	2.5298	2.7587	3.0084	3.2807	3.5776	3.9014	4.2545	4.6396	5.0595	5.5174
90	6.0168	6.5614	7.1552	7.8028	8.5090	9.2792	10.119	11.035	12.034	13.123

The following table shows how index of modulation (z) varies as a function of operator output level (an integer in the range 0–99) on the Yamaha four operator synthesizers DX11, DX21, DX27, DX27S, DX100 and TX81Z:

	0	1	2	3	4	5	6	7	8	9
0	0.0004	0.0006	0.0009	0.0013	0.0018	0.0024	0.0031	0.0036	0.0043	0.0052
10	0.0061	0.0073	0.0087	0.0103	0.0123	0.0146	0.0159	0.0174	0.0206	0.0225
20	0.0268	0.0292	0.0318	0.0347	0.0379	0.0413	0.0450	0.0491	0.0535	0.0584
30	0.0637	0.0694	0.0757	0.0826	0.0900	0.0982	0.1071	0.1168	0.1273	0.1388
40	0.1514	0.1651	0.1801	0.1963	0.2141	0.2335	0.2546	0.2777	0.3028	0.3302
50	0.3601	0.3927	0.4282	0.4670	0.5093	0.5554	0.6056	0.6604	0.7202	0.7854
60	0.8565	0.9340	1.0185	1.1107	1.2112	1.3209	1.4404	1.5708	1.7130	1.8680
70	2.0371	2.2214	2.4225	2.6418	2.8809	3.1416	3.4259	3.7360	4.0741	4.4429
80	4.8450	5.2835	5.7617	6.2832	6.8519	7.4720	8.1483	8.8858	9.6900	10.567
90	11.523	12.566	13.704	14.944	16.297	17.772	19.380	21.134	23.047	25.133

APPENDIX C

Complex numbers

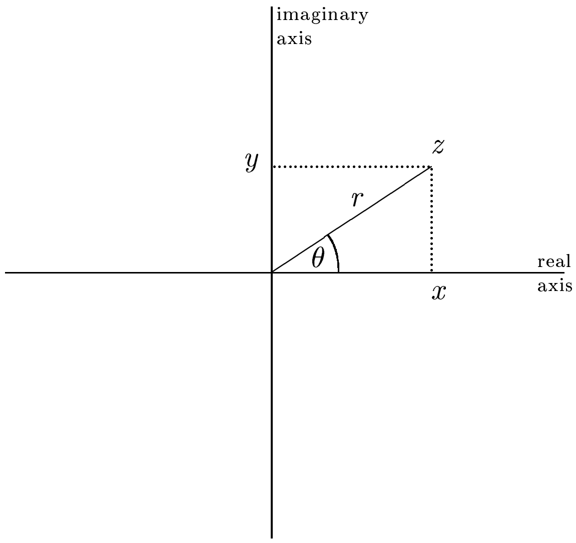
We use i to denote $\sqrt{-1}$, and the general complex number is of the form $a + ib$ where a and b are real numbers. Addition and multiplication are given by

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$
$$(a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2).$$

These formulas follow from the equation $i^2 = -1$ and the usual rules of multiplication and addition, such as the distributivity of multiplication over addition.

The real numbers a and b can be thought of as the Cartesian coordinates of the complex number $a + ib$, so that complex numbers correspond to points on the plane. In this language, the real numbers are contained in the complex numbers as the x axis, and the points on the y axis are called pure imaginary numbers.

For the purpose of multiplication, it is easier to work in polar coordinates. If $z = x + iy$ is a complex number, we define the *absolute value* of z to be $|z| = \sqrt{x^2 + y^2}$. The *argument* of z is the angle θ formed by the line from zero to z. Angle is measured counterclockwise from the x axis.



The *complex conjugate* of $z = x + iy$ is defined to be $\bar{z} = x - iy$, so that

$$z\bar{z} = |z|^2 = x^2 + y^2.$$

So division by a nonzero complex number z is achieved by multiplying by

$$\frac{\bar{z}}{|z|^2} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2},$$

which is the *multiplicative inverse* of z.

The *exponential function* is defined for a complex argument $z = x + iy$ by

$$e^z = e^x(\cos y + i \sin y).$$

This means that convertion from Cartesian coordinates to polar coordinates is given by

$$z = x + iy = re^{i\theta},$$

where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$. Translation in the other direction is given by $x = r\cos\theta$ and $y = r\sin\theta$. The trigonometric identities

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

are equivalent to the statement that if z_1 and z_2 are complex numbers then

$$e^{z_1}e^{z_2} = e^{z_1+z_2}.$$

So we have Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{C.1}$$

and

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \tag{C.2}$$

$$\sin\theta = \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}). \tag{C.3}$$

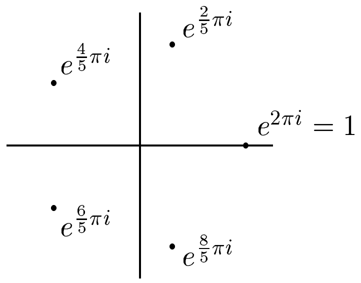
Using (C.1), the relation $(e^{i\theta})^n = e^{in\theta}$ translates as de Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

The complex nth roots of unity (i.e., of the number one) are the numbers

$$e^{2\pi im/n} = \cos 2\pi m/n + i \sin 2\pi m/n$$

for $0 \le m \le n-1$. These are equally spaced around the unit circle in the complex plane. For example, here is a picture of the complex fifth roots of unity.



Remark. Engineers use the letter j instead of i.

Hyperbolic functions: In Section 3.7 the analysis of the xylophone involves the *hyperbolic functions* $\cosh x$ and $\sinh x$. These are defined by analogy with equations (C.2) and (C.3) via

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}) \tag{C.4}$$

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}). \tag{C.5}$$

The standard identities for these functions are

$$\cosh^2 x - \sinh^2 x = 1,$$

and

$$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$$
$$\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B.$$

The values at zero are given by

$$\sinh(0) = 0, \qquad \cosh(0) = 1.$$

The derivatives are given by

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \frac{d}{dx}\cosh x = \sinh x.$$

Note the changes in sign from the corresponding trigonometric formulas.

APPENDIX D

Dictionary

As an aide to reading the literature on the subject in French, German, Italian, Latin and Spanish, as well as the literature on ancient Greek music, here is a dictionary of common terms. I have tried to avoid including words whose meaning is obvious.

abaissé (Fr.), *lowered*
abdämpfen (G.), *to damp, mute*
Abklingen (G.), *decay*
Abgeleiteter Akkord (G.), *inversion of a chord*
Absatz (G.), *cadence*
Abstimmung (G.), *tuning*
accord (Fr.), *chord*
accordage (Fr.), accordatura (It.), *tuning, intonation*
accordo (It.), *chord*
Achtelnote (G.), *eighth note* (USA), *quaver* (GB)
acorde (Sp.), *chord*
afinación (Sp.), *tuning*
affaiblissement (Fr.), *decay*
aigu (Fr.), *acute, high*
Akkord (G.), *chord*
allgemein (G.), *general*
alma (Sp.), âme (Fr.), *sound post*
anima (It.), *sound post*
Anklang (G.), *tune, harmony, accord*
archet (Fr.), arco (It., Sp.), *bow*
armoneggiare (It.), *to harmonize*
armonica (It.), armónico (Sp.), *harmonic*
armure (Fr.), *key signature*
atenuamiento (Sp.), attenuazione (It.), *decay*
audición (Sp.), audition (Fr.), *hearing*
auferions (archaic Eng.), *wire strings*
Aufhaltung (G.), *suspension (harmony)*
aufzählen (G.), *to enumerate*
aulos (Gk.), *ancient Greek reed instrument*
Ausdruck (G.), *expression*

B (G.), *B♭ (in German H denotes B)*
barre (Fr.), *bar line*
battements (Fr.), battimenti (It.), *beats*
battuta (It.), *beat*
bec (Fr.), becco (It.), *mouthpiece*
bécarre (Fr.), becuardo (Sp.), *natural* (♮)
Bedingung (G.), *condition*
Beispiel (G.), *example*
beliebig (G.), *arbitrary*
bémol (Fr.), bemol (Sp.), bemolle (It.), *flat* (♭)
bequadro (It.), *natural* (♮)
beweisen (G.), *to prove*
Beziehung (G.), *relation*
blanche (Fr.), *half note* (USA), *minim* (GB)
Blasinstrument (G.), *wind instrument*
Bogen (G.), *bow*
bois (Fr.), *wood, (pl.) woodwind*
bruit (Fr.), *noise*
Bund (G.), *fret*

cadenza d'inganno (It.), *deceptive cadence*
caisse (Fr.), *drum*
canon (Gk.), *monochord*
Canonici, *followers of the Pythagorean system of music, where consonance is based on ratios, see also* Musici
chevalet (Fr.), *bridge of stringed instrument*
cheville (Fr.), *peg, pin*
chiave (It.), clave (Sp.), clavis (L.), *clef, key*
chiffrage (Fr.), *time signature*
chiuso (It.), *closed*
clavecin (Fr.), *harpsichord*

cloche (Fr.), *bell*
comma enharmonique (Fr.), *great diesis*
concento (It.), concentus (L.), *harmony*
controreazione (It.), *feedback*
conversio (L.), *inversion*
cor (Fr.), *horn*
corde (Fr.), *string*
crotchet (GB), *quarter note* (USA)
cuarta (Sp.), *fourth*
cuerda (Sp.), *string*

Dach (G.), *sounding board*
daher (G.), *hence*
Darstellung (G.), *representation*
demi-ton (Fr.), *semitone*
denarius (L.), *numbers* 1–10
diapason (Fr., It.), diapasón (Sp.), *pitch*
diapason (Gk.), *octave*
diapente (Gk.), *fifth*
diastema (Gk.), *interval*
diatessaron (Gk.), *fourth*
diazeuxis (Gk.), *separation of two tetrachords by a tone*
dièse (Fr.), diesis (It.), *sharp* (♯)
disdiapason (Gk.), *two octaves*
dodécaphonique (Fr.), *twelve tone*
Doppelbee (G.), *double flat* (♭♭)
Doppelkreuz (G.), *double sharp* (𝄪)
Dreiklang (G.), *triad*
Dur (G.), *major*
durchgehend (G.), *transient*

échantilloneur (Fr.), *sampler*
échelle (Fr.), *scale*
écouter (Fr.), *to hear*
égale (Fr.), *equal*
eighth note (USA), *quaver* (GB)
einfach (G.), *simple*
Einführung (G.), *introduction*
Einheit (G.), *unity*
Einklang (G.), *consonance*
Einselement (G.), *identity element*
emmeleia (Gk.), *consonance*
enmascaramiento (Sp.), *masking*
ensemble (Fr.), *set*
entier (Fr.), *integer*
entonación (Sp.), *intonation*
entsprechen (G.), *to correspond to*
epimoric, *ratio* $n+1:n$
erhöhen (G.), *to raise, increase*
erweitern (G.), *to extend, augment*
escala (Sp.), *scale*
espectro (Sp.), *spectrum*
estribo (Sp.), étrier (Fr.), *stapes*
étroit (Fr.), *narrow*

faux (Fr.), *out of tune*
feinte brisée (Fr.), *split key*
fistula (L.), *pipe, flute*
Folge (G.), *sequence, series*

gama (Sp.), gamma (It.), gamme (Fr.), *scale*
ganancia (Sp.), *gain*
ganze Note (G.), *whole note* (USA), *semibreve* (GB)
ganze Zahl (G.), *integer*
ganzer Ton (G.), *whole tone*
Gegenpunkt (G.), *counterpoint*
Geige (G.), *violin*
gerade (G.), *even, just, exactly*
Geräusch (G.), *noise*
Gesetz (G.), *law, rule*
giusto (It.), *just, precise*
gleichschwebende (G.), *equal beating*
gleichstufige (G.), *equal (temperament)*
Gleichung (G.), *equation*
gleichzeitig (G.), *simultaneous*
Glied (G.), *term*
Grundlage (G.), *foundation*
Grundton (G.), *fundamental*
guadagno (It.), *gain*

H (G.), *B (in German B denotes B♭)*
Halbton (G.), *semitone*
half note (USA), *minim* (GB)
hautbois (Fr.), *oboe*
hauteur (Fr.), *pitch*
helicon (Gk.), *instrument used for calculating ratios*
hemiolios (Gk.), *ratio* 3:2
Höhe (G.), *pitch*
Hörbar (G.), *audible*
Hören (G.), *hearing*

impair (Fr.), *odd*
inégale (Fr.), *unequal*

Kettenbruch (G.), *continued fractions*
Klang(farbe) (G.), *timbre*
Klangstufe (G.), *degree of scale*
Klappe (G.), *key (wind instruments)*

klein (G.), *small, minor*
Kombinationston (G.), *combination tone*
Komma (G.), *comma*
Kraft (G.), *energy*
Kreuz (G.), *sharp* (♯)

laud (Sp.), Laute (G.), *lute*
Leistung (G.), *power*
leiten (G.), *to derive, deduce*
Leiter (G.), *scale*
ley (Sp.), *law*
limaçon (Fr.), *cochlea*
llave (Sp.), *key (wind instruments)*
Lösung (G.), *solution*
loup (Fr.), *wolf*

maggiore (It.), majeur (Fr.), mayor (Sp.), *major*
marche d'harmonie (Fr.), *harmonic sequence*
Menge (G.), *set*
menor (Sp.), *minor*
mehrstimmig (G.), *polyphonic*
mesolabium, *mechanical means for producing ratio* 18:17, *approximation to equal tempered semitone for lutes*
mésotonique (Fr.), *meantone*
minim (GB), *half note* (USA)
minore (It.), *minor*
mitteltönig (G.), *meantone*
Moll (G.), *flat* (♭), *minor*
Mundstück (G.), *mouthpiece*
Musici, *followers of the Aristoxenian system of music, in which the ear is the judge of consonance, see also* Canonici
Muster (G.), *pattern*

Nachhall (G.), *reverberation*
Naturseptime (G.), *natural seventh*
Nebendreiklang (G.), *secondary triad (not* I, IV *or* V*)*
Nenner (G.), *denominator*
neuvième (Fr.), *ninth*
nœud (Fr.), *node (vibration)*
None (G.), *nineth (interval)*
Notenschlussel (G.), *clef*
numérique (Fr.), *digital*

Oberwelle (G.), *harmonic*
offen (G.), *open*
Ohr (G.), *ear*
Ohrmuschel (G.), *auricle*
oído (Sp.), *ear*
onda (It., Sp.), *wave*
onda portante (It.), onda portadora (Sp.), *carrier*
onde (Fr.), *wave*
ordinateur (Fr.), *computer*
orecchio (It.), oreille (Fr.), *ear*
organo (It)., órgano (Sp.), Orgel (G.), orgue (Fr.), *organ*
ouïe (Fr.), *hearing; sound-hole*

padigione (It.), *auricle*
pair (Fr.), par (Sp.), *even*
paraphonia (Gk., L.), *Intervals of fourth and fifth*
parfait (Fr.), *perfect*
pavillon (Fr.), *auricle*
plagal cadence, *the cadence* IV–I
point d'orgue (Fr.), *fermata*
portée (Fr.), *staff, stave*
porteuse (Fr.), *carrier*
potencia (Sp.), potenza (It.), *power*
profondeur (Fr.), *depth*
puissance (Fr.), *power*
pulsaciones (Sp.), *beats*

Quadrat (G.), *natural* (♮)
quadrivium (L.), *The four disciplines: arithmetic, geometry, astronomy and music*
quarta (It., L.), quarte (Fr.), Quarte (G.), *fourth*
quarter note (USA), *crotchet* (GB)
quaternarius (L.), *numbers* 1–4
quaver (GB), *eighth note* (USA)
quinta (It., L., Sp.), quinte (Fr.), Quinte (G.), *fifth*

réaction (Fr.), *feedback*
reine (G.), *pure*
renversement (Fr.), *inversion*
résoudre (Fr.), *to resolve*
retard (Fr.), *delay*
retroalimentación (Sp.), *feedback*
ronde (Fr.), *whole note* (USA), *semibreve* (GB)
Rückkopplung (G.), *feedback*

Saite (G.), *string*

Satz (G.), *theorem; movement*
Schall (G.), *sound*
Scheibe (G.), *disc*
Schlag (G.), *beat*
Schlüssel (G.), *clef*
Schnecke (G.), *cochlea*
Schwebungen (G.), *beats*
Schwelle (G.), *threshold, limen*
Schwingungen (G.), *vibrations*
semibreve (GB), *whole note* (USA)
semiquaver (GB), *sixteenth note* (USA)
senarius (L.), *numbers* 1–6
sensible (Fr.), *leading note*
septenarius (L.), *numbers* 1–7
septima (L.), Septime (G.), *seventh*
Septimenakkord (G.), *seventh chord*
série de hauteurs (Fr.), *tone row*
sesquialtera (L.), *ratio* 3:2
sesquitertia (L.), *ratio* 4:3
settima (It.), *seventh*
seuil (Fr.), *threshold, limen*
Sext (G.), sexta (L.), *sixth*
sibilo (It.), sifflement (Fr.), silbo (Sp.), *hiss*
siècle (Fr.), *century*
sillet (Fr.), *bridge*
sixteenth note (USA), *semiquaver* (GB)
Skala (G.), *scale*
soglia (It.), *threshold, limen*
son (Fr.), *sound*
son combiné (Fr.), *combination tone*
son différentiel (Fr.), *difference tone*
sonido (Sp.), *sound*
sonido de combinación (Sp.), *combination tone*
sonorità (It.), *harmony, resonance*
sonus (L.), *sound*
sostenido (Sp.), *sharp* (♯)
spectre (Fr.), *spectrum*
staffa (It.), *stapes*
stanghetta (It.), *bar line*
stark (G.), *loud*
Stege (G.), *bridge*
Steigbügel (G.), *stapes*
Stimmstock (G.), *sound post*
Stimmung (G.), *tuning, key, pitch*
Stufe (G.), *scale degree*
subsemitonia (L.), *split keys*
suono (It.), *sound*
suono di combinazione (It.), *combination tone*
superparticular, *ratio* $n{+}1:n$
synaphe (Gk.), *conjunction, or overlapping of two tetrachords*

Takt (G.), *time, measure, bar*
Taktstrich (G.), *bar line*
tambour (Fr.), tamburo (It.), tambor (Sp.), *drum*
Tastame (It.), Tastatur, Tastenbrett, Tastenleiter (G.), Tastatura, Tastiera (It.), *keyboard of piano or organ*
tasto (It.), tecla (Sp.), *fret*
teilbar (G.), *divisible*
Teilmenge (G.), *subset*
Teilung (G.), *division*
Temperatur (G.), *temperament*
temperiert (G.), *tempered*
temps (Fr.), *time, beat, measure*
tercera (Sp.), tertia (L.), Terz (G.), terza (It.), *third*
tiempo (Sp.), *beat*
tierce (Fr.), *third*
ton (Fr.), *pitch, tone, key*
tonalité (Fr.), Tonart (G.), *key*
Tonausweichung (G.), *modulation*
Tonhöhe (G.), *pitch*
tono medio (It., Sp.), *meantone*
Tonschluss (G.), *cadence*
Tonstufe (G.), *scale degree*
touche (Fr.), *fret, key*
Träger (G.), *carrier*
traité (Fr.), *treatise*
tripla (L.), *ratio* 3:1
Trommel (G.), *drum*
tuyau (Fr.), *pipe*
tuyau à bouche (Fr.), *open pipe*
tuyau d'orgue (Fr.), *organ pipe*
tympan (Fr.), *eardrum*

überblasen (G.), *to overblow*
Übereinstimmung (G.), *consonance, harmony*
übermässig (G.), *augmented*
udibile (It.), *audible*
udito (It.), *hearing*
uguale (It.), *equal*

umbral (S.), *threshold, limen*
Umkehrung (G.), *inversion*
Unterdominant (G.), *subdominant*
Unterhalbton (G.), *leading note*
Unterleitton (G.), *dominant seventh*
Untergruppe (G.), *subgroup*
Untertaste (G.), *white key*

valeur propre (Fr.), *eigenvalue*
vent (Fr.), *wind*
Ventil (G.), ventile (It.), *valve (wind instruments)*
ventre (Fr.), *antinode (vibration)*
vents (Fr.), *wind instruments*
Verbindung (G.), *combination, union*
Verdeckung (G.), *masking*
vergleichen (G.), *to compare*
Verhältnis (G.), *ratio, proportion*
Verknüpfung (G.), *operation*
verlängertes Intervall (G.), *augmented interval*
vermindert (G.), *diminished*
versetzen (G.), *to transpose*
Versetzungszeichen (G.), *accidentals*
Verspätung (G.), *delay*
Verstärker (G.), *amplifier*
Verstärkung (G.), *gain*
verstimmt (G.), *out of tune*
verwandt (G.), *related*
Verzerrung (G.), *distortion*
Viertel (G.), *quarter*
voix (Fr.), *voice*
Vollkommenheit (G.), *perfection*

Welle (G.), *wave*
wenig (G.), *little, slightly*
whole note (USA), *semibreve* (GB)
wohltemperirte (G.), *well tempered*

Zahl (G.), *number*
Zählzeit (G.), *beat*
Zeichen (G.), *sign, note*
Zeit (G.), *time*
Zischen (G.), *hiss*
Zuklang (G.), *unison, consonance*

APPENDIX E

Equal tempered scales

q	p_3	e_3	p_5	e_5	p_7	e_7	e_{35}	e_{357}	$e_5 . q^2$	$e_{35} . q^{\frac{3}{2}}$	$e_{357} . q^{\frac{4}{3}}$
2	1	+213.686	1	−101.955	2	+231.174	166.245	190.365	392	470	480
3	1	+13.686	2	+98.045	2	−168.826	70.000	112.993	882	**364**	489
4	1	−86.314	2	−101.955	3	−68.826	94.459	86.760	1631	756	551
5	2	+93.686	3	+18.045	4	−8.826	67.464	55.319	451	754	473
6	2	+13.686	4	+98.045	5	+31.174	70.000	59.922	3530	1029	653
7	2	−43.457	4	−16.241	6	+59.746	32.804	43.672	796	608	585
8	3	+63.686	5	+48.045	6	−68.826	56.410	60.831	3075	1276	973
9	3	+13.686	5	−35.288	7	−35.493	26.764	23.104	2858	723	433
10	3	−26.314	6	+18.045	8	−8.826	22.561	19.113	1804	713	412
11	4	+50.050	6	−47.410	9	+12.992	48.748	40.503	5737	1778	991
12	4	+13.686	7	−1.955	10	+31.174	9.776	19.689	**282**	**406**	541
13	4	−17.083	8	+36.507	10	−45.749	28.500	35.202	6170	1336	1076
14	5	+42.258	8	−16.241	11	−25.969	32.012	30.132	3183	1677	1017
15	5	+13.686	9	+18.045	12	−8.826	16.015	14.034	4060	930	519
16	5	−11.314	9	−26.955	13	+6.174	20.671	17.250	6900	1323	695
17	5	−33.373	10	+3.927	14	+19.409	23.761	22.404	1135	1665	979
18	6	+13.686	11	+31.378	15	+31.174	24.207	26.732	10167	1849	1261
19	6	−7.366	11	−7.218	15	−23.457	7.293	13.745	2606	604	697
20	6	−26.314	12	+18.045	16	−8.826	22.561	19.113	7218	2018	1038
21	7	+13.686	12	−16.241	17	+2.603	15.018	12.354	7162	1445	716
22	7	−4.496	13	+7.136	18	+12.992	5.964	8.943	3454	615	551
31	10	+0.783	18	−5.181	25	−1.084	3.705	3.089	4979	639	**301**
41	13	−5.826	24	+0.484	33	−2.972	4.134	3.786	814	1085	535
53	17	−1.408	31	−0.068	43	+4.759	0.997	2.866	**192**	**385**	570
65	21	+1.379	38	−0.417	52	−8.826	1.018	5.163	1760	534	1349
68	22	+1.922	40	+3.927	55	+1.762	3.092	2.722	18160	1734	755
72	23	−2.980	42	−1.955	58	−2.159	2.520	2.406	10135	1540	721
84	27	−0.599	49	−1.955	68	+2.603	1.446	1.911	13794	1113	703
99	32	+1.565	58	+1.075	80	−0.871	1.343	1.206	10539	1323	552
118	38	+0.127	69	−0.260	95	−2.724	0.205	1.582	3621	**262**	915
130	42	+1.379	76	−0.417	105	+0.405	1.018	0.864	7040	1509	569
140	45	−0.599	82	+0.902	113	−0.254	0.766	0.642	17682	1269	467
171	55	−0.349	100	−0.201	138	−0.405	0.285	0.330	5866	636	**313**
441	142	+0.081	258	+0.086	356	−0.118	0.083	0.096	16689	772	**324**
494	159	−0.079	289	+0.069	399	+0.405	0.074	0.241	16909	815	943
612	197	−0.039	358	+0.006	494	−0.198	0.028	0.117	2166	**424**	607
665	214	−0.148	389	−0.0001	537	+0.197	0.105	0.142	**50**	1798	825

This table shows how well the scales based around equal divisions of the octave approximate the 5:4 major third, the 3:2 perfect fifth and the 7:4 seventh harmonic. The first column (q) gives the number of divisions to the octave. The second column (p_3) shows the scale degree closest to the 5:4 major third (counting from zero for the tonic), and the next column (e_3) shows the error in cents:

$$e_3 = 1200 \left(\frac{p_3}{q} - \log_2 \left(\frac{5}{4} \right) \right).$$

Similarly, the next two columns (p_5 and e_5) show the scale degree closest to the 3:2 perfect fifth and the error in cents:

$$e_5 = 1200 \left(\frac{p_5}{q} - \log_2 \left(\frac{3}{2} \right) \right).$$

The two columns after that (p_7 and e_7) show the scale degree closest to the 7:4 seventh harmonic and the error in cents:

$$e_7 = 1200\left(\frac{p_7}{q} - \log_2\left(\frac{7}{4}\right)\right).$$

We write e_{35} for the root mean square (RMS) error of the major third and perfect fifth:

$$e_{35} = \sqrt{(e_3^2 + e_5^2)/2}$$

and e_{357} for the RMS error for the major third, perfect fifth and seventh harmonic:

$$e_{357} = \sqrt{(e_3^2 + e_5^2 + e_7^2)/3}.$$

Theorem 6.2.3 shows that the quantity $e_5.q^2$ is a good measure of how well the perfect fifth is approximated by p_5/q of an octave, with respect to the number of notes in the scale. This theorem shows that there are infinitely many values of q for which $e_5.q^2 < 1200$, while on average we should expect this quantity to grow linearly with q.

Similarly, Theorem 6.2.5 with $k = 2$ shows that the quantity $e_{35}.q^{\frac{3}{2}}$ is a good measure of how well the major third and perfect fifth are simultaneously approximated, and shows that there are infinitely many values of q for which $e_{35}.q^{\frac{3}{2}} < 1200$, while on average we should expect this quantity to grow like the square root of q. Theorem 6.2.5 with $k = 3$ shows that the quantity $e_{357}.q^{\frac{4}{3}}$ is a good measure of how well all three intervals: major third, perfect fifth and seventh harmonic are simultaneously approximated, and shows that there are infinitely many values of q for which $e_{357}.q^{\frac{4}{3}} < 1200$, while on average we should expect this quantity to grow like the cube root of q.

Particularly good values of $e_5.q^2$, $e_{35}.q^{\frac{3}{2}}$ and $e_{357}.q^{\frac{4}{3}}$ are indicated in bold face in the last three columns of the table.

APPENDIX F

Frequency and MIDI chart

This table shows the frequencies and MIDI numbers of the notes in the standard equal tempered scale, based on the standard A4 = 440 Hz.

	MIDI	Hz	USA	Eur
piano ↑	108	4186.01	C8	c'''''
violin ↑	107	3951.07	B7	
	106	3729.31		
	105	3520.00	A7	
	104	3322.44		
	103	3135.96	G7	
	102	2959.96		
	101	2793.83	F7	
	100	2637.02	E7	
	99	2489.02		
flute ↑	98	2349.32	D7	
	97	2217.46		
	96	2093.00	C7	c''''
	95	1975.53	B6	
	94	1864.66		
	93	1760.00	A6	
	92	1661.22		
–	91	1567.98	G6	
	90	1479.98		
	89	1396.91	F6	
–	88	1318.51	E6	
leger	87	1244.51		
lines	86	1174.66	D6	
	85	1108.73		
–	84	1046.50	C6	c'''
	83	987.767	B5	
	82	932.328		
–	81	880.000	A5	
	80	830.609		
	79	783.991	G5	
	78	739.989		
┌	77	698.456	F5	
│	76	659.255	E5	
│	75	622.254		
⊢	74	587.330	D5	
│	73	554.365		
│ treble	72	523.251	C5	c''
⊢ clef	71	493.883	B4	
│	70	466.164		
│	69	440.000	A4	
│	68	415.305		
⊢	67	391.995	G4	
│	66	369.994		
│	65	349.228	F4	
└	64	329.628	E4	
	63	311.127		
	62	293.665	D4	
	61	277.183		
middle c	60	261.626	C4	c'

	MIDI	Hz	USA	Eur
flute ↓	59	246.942	B3	
	58	233.082		
┌	57	220.000	A3	
│	56	207.652		
violin ↓	55	195.998	G3	
│	54	184.997		
⊢	53	174.614	F3	
│	52	164.814	E3	
│ bass	51	155.563		
⊢ clef	50	146.832	D3	
│	49	138.591		
│	48	130.813	C3	c
⊢	47	123.471	B2	
│	46	116.541		
│	45	110.000	A2	
│	44	103.826		
└	43	97.9989	G2	
	42	92.4986		
	41	87.3071	F2	
–	40	82.4069	E2	
	39	77.7817		
	38	73.4162	D2	
	37	69.2957		
–	36	65.4064	C2	C
leger	35	61.7354	B1	
lines	34	58.2705		
–	33	55.0000	A1	
	32	51.9131		
	31	48.9994	G1	
	30	46.2493		
–	29	43.6535	F1	
	28	41.2034	E1	
	27	38.8909		
	26	36.7081	D1	
	25	34.6478		
	24	32.7032	C1	C_1
	23	30.8677	B0	
	22	29.1352		
piano ↓	21	27.5000	A0	
	20	25.9565		
	19	24.4997	G0	
	18	23.1247		
	17	21.8268	F0	
	16	20.6017	E0	
	15	19.4454		
	14	18.3540	D0	
	13	17.3239		
	12	16.3516	C0	C_2
	11	15.4339		

APPENDIX G

Getting stuff from the internet

This appendix is about software and other resources which may be found online. The information is, of course, very volatile. So it is likely that by the time you are reading this, a lot of the information will already be out of date.

Scales and Temperaments: The best internet resource on the subject of scales, temperaments and tunings is

http://www.xs4all.nl/~huygensf/doc/bib.html

This is part of the Huygens-Fokker Foundation website, maintained by Manuel Op de Coul, and consists of a giant bibliography together with links to other internet resources on the subject. The front page of the website is at

http://www.xs4all.nl/~huygensf/english/

Also on the same website, a discography of microtonal music can be found at

http://www.xs4all.nl/~huygensf/doc/discs.html

A large collection of scales and temperaments can be found at

http://www.xs4all.nl/~huygensf/doc/scales.zip

and the Scala scales and temperaments software can be found at

http://www.xs4all.nl/~huygensf/scala/

To subscribe to the alternate tunings email discussion group, send an empty email message to tuning-subscribe@onelist.com.

Just Intonation Network: http://www.dnai.com/~jinetwk/

Bohlen–Pierce scale: http://members.aol.com/bpsite/index.html

Music Theory: Sites offering free music theory tuition online include

Easy Music Theory (Gary Ewer): http://www.musictheory.halifax.ns.ca/

Java Music Theory: http://academics.hamilton.edu/music/spellman/JavaMusic/

Online Music Instruction Page (Ken Fansler):
http://orathost.cfa.ilstu.edu/~kwfansle/onlinemusicpage.htm

Practical Music Theory: http://www.teoria.com/java/eng/java.htm

Sound editors: There are some good shareware sound editors. Among the best are:

Cool Edit: http://www.syntrillium.com/cooledit/index.html

Goldwave: http://www.goldwave.com/

Acid Wav: http://www.polyhedric.com/software/acid/

There are two freeware audio frequency analysers for the PC called

Spectrogram: http://www.monumental.com/rshorne/gram.html

Frequency analyzer: http://www.hitsquad.com/smm/programs/Frequency/

CSound: This free software is described in §8.10. Versions for various platforms (PC, Mac, Unix, Atari, NeXT) are available from

ftp://ftp.maths.bath.ac.uk/pub/dream/

To subscribe to the email discussion group for CSound, send an empty message to csound-subscribe@lists.bath.ac.uk. Further information about CSound can be found at the following www pages:

http://www.mitpress.com/e-books/csound/frontpage.html
(the CSound front page, MIT Press)

http://www.bright.net/~dlphilp/dp_csound.html
(Dave Phillips' PC CSound page)

http://www.bright.net/~dlphilp/linux_csound.html
(Dave Phillips' Linux CSound page)

http://music.dartmouth.edu/~dupras/wCsound/csoundpage.html
(Martin Dupras' CSound page)

A utility for PC and Unix called MIDI2CS, written by Rudiger Borrmann, converts MIDI files to Csound scores. It is available from

http://www.snafu.de/~rubo/songlab/midi2cs/csound.html

A utility for emulating the Yamaha DX7 with CSound can be found at Jeff Harrington's site

http://www.parnasse.com/dx72csnd.shtml

Other synthesis software: This is a rapidly expanding field, and new products turn up almost every week. The ones I know of are as follows.

Audio Architect (PC): http://www.audiarchitect.com/

Bitheadz Retro AS-1 (Mac): http://www.bitheadz.com (free demo)

CLM (Common Lisp Music, freeware):
http://www-ccrma.stanford.edu/CCRMA/Software/clm/clm.html

CMix (Next, Linux, Sparc, SGI, PowerMac; freeware):
http://www.music.princeton.edu/winham/cmix.html

Cybersound Studio (Mac, Win 95/98/ME): http://www.cybersound.com

Cycling '74 (Mac + Opcode Max): http://www.cycling74.com (free demo)

Grain Wave (Mac shareware): http://www.nmol.com/users/mikeb/

Ik Multimedia's Groovemaker and Axé (Mac, Win 95/98/ME):
http://www.ikmultimedia.com (free demo)

Lemur (Mac): http://datura.cerl.uiuc.edu/Lemur/AboutLemur.html

Native Instruments Reaktor/Generator/Transformator (Win 95/98/ME):
http://www.native-instruments.com/ (free demo)

Nemesis GigaSampler (Win 95/98/ME): http://www.nemesismusic.com

Nyquist (freeware):
http://www.cs.cmu.edu/afs/cs.cmu.edu/project/music/web/music.software.html

Seer Systems Reality: http://www.seersystems.com

Steinberg Rebirth RB-338 (Mac, Win 95/98/ME/NT):
http://www.us.steinberg.net (free demo)

Synthesis Toolkit (C++ code):
http://www-ccrma.stanford.edu/CCRMA/Software/STK/

Virtual Sampler (Win 95/98/ME/NT):
This can be found at Sonic Spot, http://www.sonicspot.com/, or at MAZ, http://www.maz-sound.com/. It is shareware, and the unregistered version does everything but save sounds. It includes a complete Yamaha DX7 emulation.

The most impressive site for information on the processes and control of synthesis is Electronic Music Interactive, at

http://nmc.uoregon.edu/emi/emi.html

Synthesizers and patches: The best general websites for synthesizers and patches are

Synthesizer and Midi Links Page:
http://www.interlog.com/~spinner/lbquirke/synthesis/links/

Synth Site: http://www.sonicstate.com/bbsonic/synth/index.cfm

At the anonymous ftp site ftp.ucsd.edu, in the subdirectory /midi/patches, there are patches for Casio CZ-1, CZ-2, Ensoniq ESQ1, SQ1, Kawai K1, K4, K5, XD-5, Korg M1, T3, WS (Wavestation), Roland D10, D5, D50, D70, SC55, U20, and Yamaha DX7, FB01, TX81Z, SY22, SY55, SY77, SY85.

For the Yamaha DX7, there is a web page which I maintain at

http://www.math.uga.edu/~djb/dx7.html

which contains, among other things, a patch archive and instructions for joining the email discussion group.

Typesetting software:

CMN (Common Music Notation, freeware for NeXT and SGI machines):
http://ccrma-www.stanford.edu/CCRMA/Software/cmn/cmn.html

Finale is a commercial music notation package for the Mac and Windows (current version Finale 2002), and is available from Coda Music Software. Their web site

http://www.codamusic.com/

has more information. A free demonstration version of the program is available on this web site. Without academic discount, Finale is very expensive, but with academic discount it costs about $200–$250. To subscribe to the email discussion group for Finale, send an email message to listserv@shsu.edu with the phrase "subscribe Finale" or "subscribe Finale-Digest" in the body of the message. To be removed from the list, send "signoff Finale" or "signoff Finale-Digest" to the same address.

Finale forum (not sanctioned by Coda Music): http://www.cmp.net/finale/

Finale Resource Page: http://www.peabody.jhu.edu/~skot/finale/fin_home.html

Ftp site for Finale users: ftp://ftp.shsu.edu/pub/finale/

Keynote is a public domain textual, graphical and algorithmic music editor for the Unix X Window system, the Mac or the Amiga, available from

ftp://xcf.berkeley.edu

LilyPond is a GNU project (and hence free) music typesetter for Unix systems. It is available from

http://www.cs.uu.nl/~hanwen/lilypond/index.html

Lime (Mac, Windows): http://www.cerlsoundgroup.org/

Mozart: http://www.mozart.co.uk/

Muzika 3 is a public domain (freeware) music notation package for Windows, available from

ftp://garbo.uwasa.fi/windows/sound/muzika3.zip or from

ftp://ftp.cica.indiana.edu/ftp/pub/win3/sounds/muzika3.zip

Nutation (NeXT, freeware): ftp://ccrma-ftp.stanford.edu/pub/Nu.pkg.tar

Overture is a Mac based commercial music notation package.

Score: http://ace.acadiau.ca/score/links3.htm

Sibelius is a notation package for the PC: http://www.sibelius.com/

MusicTEX: MusicTEX, written by the french organist Daniel Taupin, and its successor MusixTEX are public domain music typesetting packages to run under Donald Knuth's TEX program. The necessary files may be found on

ftp://rsovax.ups.circe.fr/TeX/musictex/

See also: http://www.gmd.de/Misc/Music/

A public domain version of TEX for Windows 95 or higher, called MikTeX, and can be found at http://www.miktex.de. Versions for all platforms are available from CTAN at ftp.tex.ac.uk, ftp.dante.de or ctan.tug.org. See also TUG (the TEX user's group) at http://tug.org.

Goldberg Variation 25, J. S. Bach

Example of Output from MusicTEX

MuTEX is the precursor of MusicTEX, written by Andrea Steinbach and Angelica Schofer. It is in the public domain, and is available by anonymous ftp from ymir.claremont.edu in [anonymous.tex.music.mtex] (VMS).

MIDI2TEX is a program written by Hans Kuykens for converting MIDI files into MusicTEX files. The latest version can be found on CTAN (see page 315).

ABC2MTEX is a program for converting tunes from its own text-based format into MusicTEX files. It is designed primarily for folk and traditional music of Western European origin written on one stave in standard classical notation. It can be obtained directly from its author, Chris Walshaw, via email: C.Walshaw@gre.ac.uk, or from

ftp://celtic.stanford.edu/pub/tunes/abc2mtex/

Sequencers: Cakewalk and Cubase are competing commercial Windows based sequencers, neither of which is cheap, but both of which are packed with features. To subscribe to the Cakewalk users' group, send a message to listserv@lists.colorado.edu with the phrase "subscribe cakewalk" in the body of the message. To subscribe to the Cubase users' group, send a message to cubase-users-request@nessie.mcc.ac.uk. Messages for the group should be sent to cubase-users@mcc.ac.uk.

Power Tracks Pro Audio is a very cheap, but fully functional commercial Windows based sequencer, available from PG Music for $29. More information can be found at

http://www.pgmusic.com/

Rosegarden is an integrated MIDI sequencer and musical notation editor. It is free software for Unix and X, and it may be found at

http://www.bath.ac.uk/~masjpf/rose.html

WinJammer is a shareware Windows based sequencer, which may be found at

ftp://ftp.cnr.it/pub/msdos/win3/sounds/wjmr23.zip

WinJammer Pro (I'm not sure what the difference is) is in the same directory, as wjpro.zip.

Random music: There are a number of freeware/shareware probabilistic music programs designed to run under Windows.

Aleatoric composer (shareware):
ftp://oak.oakland.edu/msdos/music/alcomp11.zip

Art Song 2.3 (shareware): http://members.aol.com/strohbeen/fmlsw.html

FMusic 1.9 (freeware): http://members.aol.com/dsinger594/caman/fmusic19.zip

FractMus 2.3 (freeware): ftp://ftp.cdrom.com/pub/win95/music/frctmu25.zip

Fractal Tune Smithy (freeware/shareware):
http://matrix.crosswinds.net/~fractalmelody/index.htm

Improvise 1.2 (shareware):
ftp://ftp.cnr.it/pub/msdos/win3/sounds/impvz120.zip

Make-Prime-Music (freeware):
http://members.tripod.de/Latrodectus98/index.html

Mandelbrot Music (freeware): http://www.fin.ne.jp/~yokubota/mandele.shtml

MusiNum 2.08 (freeware):
http://www.forwiss.uni-erlangen.de/~kinderma/musinum/musinum.html

QuasiFractalComposer 2.01 (freeware):
http://members.tripod.com/~paulwhalley/

Tangent (free/shareware): http://www.randomtunes.com/

The Well Tempered Fractal 3.0 (freeware):
http://www-ks.rus.uni-stuttgart.de/people/schulz/fmusic/wtf/wtf30.zip

MIDI: The MIDI specification can be obtained via email by sending a message with the phrase GET MIDISPEC PACKAGE in the message body, to listserv@auvm.american.edu. There are archives of MIDI files available at

ftp://ftp.cs.ruu.nl/MIDI/DOC/archives/

ftp://ftp.waldorf-gmbh.de/pub/midi/

There are two programs called mf2t and t2mf which convert standard MIDI files into human readable ASCII text and back again. The MIDI home page on the WWW is

http://www.eeb.ele.tue.nl/midi/index.html

A good starting point for information about MIDI is the Northwestern University site

http://nuinfo.nwu.edu/musicschool/links/projects/midi/expmidiindex.html

Academic Computer Music: The following departments in American universities have programs in computer music. CalArts (David Rosenboom, Morton Subotnick), Carnegie Mellon (Roger Dannenberg), MIT (Tod Machover, Barry Vercoe), Princeton (Paul Lansky), Stanford (John Chowning, Chris Chaffe, Perry Cook, etc.), SUNY Buffalo (David Felder, Cort Lippe), UC Berkeley (David Wessel), UCSD (Miller Puckett, F. Richard Moore, George Lewis, Peter Otto).

IRCAM is an institution in Paris for computer music, which has an anonymous ftp site at ftp.ircam.fr. In particular, the music/programming environment MAX can be found there.

Music Theory Online (the Online Journal of the Society for Music Theory) can be found at

http://boethius.music.ucsb.edu/mto/mtohome.html

FAQs: There are several FAQs ("Frequently Asked Questions" and their answers) available on the internet. Two that I know of are available from the site xcf.berkeley.edu, either by anonymous ftp or by email. They are the electronic and computer music FAQ, in /pub/misc/netjam/doc/ECMFAQ and the composition FAQ, in /pub/misc/netjam/doc/FAQ/composition/compositionFAQ.entire. Or send an email message to netjam-request@xcf.berkeley.edu with the subject line "request for ECM FAQ", respectively "request for composition FAQ".

Other resources: The following are some interesting WWW pages:

Everyone seems to want to know more about the infamous "Mozart effect". Volume VII, Issue 1 (Winter 2000) of MuSICA Research Notes is devoted to this much overpublicized and misunderstood topic, and can be found at http://www.musica.uci.edu/mm/V7I1W00.html

http://www.oulu.fi/music.html is a directory of music sites.

http://www.music.indiana.edu/misc/music-resources.html is a catalog of music resources.

http://sunsite.unc.edu/pub/ianc/index.html is the Internet Underground Music Archive.

To subscribe to the electronic music email discussion group, send a message to listserv@auvm.bitnet with the line "SUB EMUSIC-L" in the body. Messages for the group should be sent to emusic-l@auvm.bitnet. For the digests only, replace EMUSIC-L with EMUSIC-D.

Online papers: See Appendix O for a selection of relevant papers which can be downloaded from academic journals.

APPENDIX I

Intervals

This is a table of intervals not exceeding one octave (or a tritave in the case of the Bohlen–Pierce, or BP scale). A much more extensive table may be found in Appendix XX to Helmholtz [**48**] (page 453), which was added by the translator, Alexander Ellis. Names of notes in the BP scale are denoted with a subscript BP, to save confusion with notes which may have the same name in the octave based scale.

The first column is equal to 1200 times the logarithm to base two of the ratio given in the second column. Logarithms to base two can be calculated by taking the natural logarithm and dividing by $\ln 2$. So the first column is equal to

$$\frac{1200}{\ln 2} \approx 1731.234$$

times the natural logarithm of the second column.

We have given all intervals to three decimal places for theoretical purposes. While intervals of less than a few cents are imperceptible to the human ear in a melodic context, in harmony very small changes can cause large changes in beats and roughness of chords. Three decimal places gives great enough accuracy that errors accumulated over several calculations should not give rise to perceptible discrepancies.

If more accuracy is needed, I recommend using the multiple precision package bc (see page 298) with the -l option. The following lines can be made into a file to define some standard intervals in cents. For example, if the file is called music.bc then the command “bc -l music.bc” will load them at startup.

```
scale=50 /* fifty decimal places - seems like plenty but you never know */
octave=1200
savart=1.2*l(10)/l(2)
syntoniccomma=octave*l(81/80)/l(2)
pythagoreancomma=octave*l(3^12/2^19)/l(2)
septimalcomma=octave*l(64/63)/l(2)
schisma=pythagoreancomma-syntoniccomma
diaschisma=syntoniccomma-schisma
perfectfifth=octave*l(3/2)/l(2)
equalfifth=700
meantonefifth=octave*l(5)/(4*l(2))
perfectfourth=octave*l(4/3)/l(2)
justmajorthird=octave*l(5/4)/l(2)
justminorthird=octave*l(6/5)/l(2)
justmajortone=octave*l(9/8)/l(2)
justminortone=octave*l(10/9)/l(2)
```

Cents	Interval ratio	Eitz	Name, etc.	Ref
0.000	1:1	C^0, C^0_{BP}	Fundamental	§4.1
1.000	$2^{\frac{1}{1200}}$:1		Cent	§5.4
1.805	$2^{\frac{1}{665}}$:1		Degree of 665 tone scale	§6.4
1.953	32805:32768	$B\sharp^{-1}$	Schisma	§5.8
3.986	$10^{\frac{1}{1000}}$:1		Savart	§5.4
14.191	245:243	C^{+1}_{BP}	BP-minor diesis	§6.7
19.553	2048:2025	$D\flat\flat^{+2}$	Diaschisma	§5.8
21.506	81:80	C^{+1}	Syntonic, or ordinary comma	§5.5
22.642	$2^{\frac{1}{53}}$:1		Degree of 53 tone scale	§6.3
23.460	3^{12}:2^{19}	$B\sharp^0$	Pythagorean comma	§5.2
27.264	64:63		Septimal comma	§5.8
35.099			Carlos' γ scale degree	§6.6
41.059	128:125	$D\flat\flat^{+3}$	Great diesis	§5.12
49.772	7^{13}:3^{23}	$D\flat\flat^0_{BP}$	BP 7/3 comma	§6.7
63.833			Carlos' β scale degree	§6.6
70.672	25:24	$C\sharp^{-2}$	Small (just) semitone	§5.5
77.965			Carlos' α scale degree	§6.6
90.225	256:243	$D\flat^0$	Diesis or Limma	§5.2
100.000	$2^{\frac{1}{12}}$:1	$\approx C\sharp^{-\frac{7}{11}}$	Equal semitone	§5.14
111.731	16:15	$D\flat^{+1}$	Just minor semitone (ti–do, mi–fa)	§5.5
113.685	2187:2048	$C\sharp^0$	Pythagorean apotomē	§5.2
133.238	27:25	$D\flat^{-2}_{BP}$		§6.7
146.304	$3^{\frac{1}{13}}$:1		BP-equal semitone	§6.7
182.404	10:9	D^{-1}	Just minor tone (re–mi, so–la)	§5.5
193.157	$\sqrt{5}$:2	$D^{-\frac{1}{2}}$	Meantone whole tone	§5.12
200.000	$2^{\frac{1}{6}}$:1	$\approx D^{-\frac{2}{11}}$	Equal whole tone	§5.14
203.910	9:8	D^0	Just major tone (do–re, fa–so, la–ti);	§5.5
			Pythagorean major tone;	§5.2
			Nineth harmonic	§4.1
294.135	32:27	$E\flat^0$	Pythagorean minor third	§5.2
300.000	$2^{\frac{1}{4}}$:1	$\approx E\flat^{+\frac{3}{11}}$	Equal minor third	§5.14
315.641	6:5	$E\flat^{+1}$	Just minor third (mi–so, la–do, ti–re)	§5.5
386.314	5:4	E^{-1}	Just major third (do–mi, fa–la, so–ti);	§5.5
			Meantone major third;	§5.12
			Fifth harmonic	§4.1
400.000	$2^{\frac{1}{3}}$:1	$\approx E^{-\frac{4}{11}}$	Equal major third	§5.14
407.820	81:64	E^0	Pythagorean major third	§5.2
498.045	4:3	F^0	Perfect fourth	§5.2

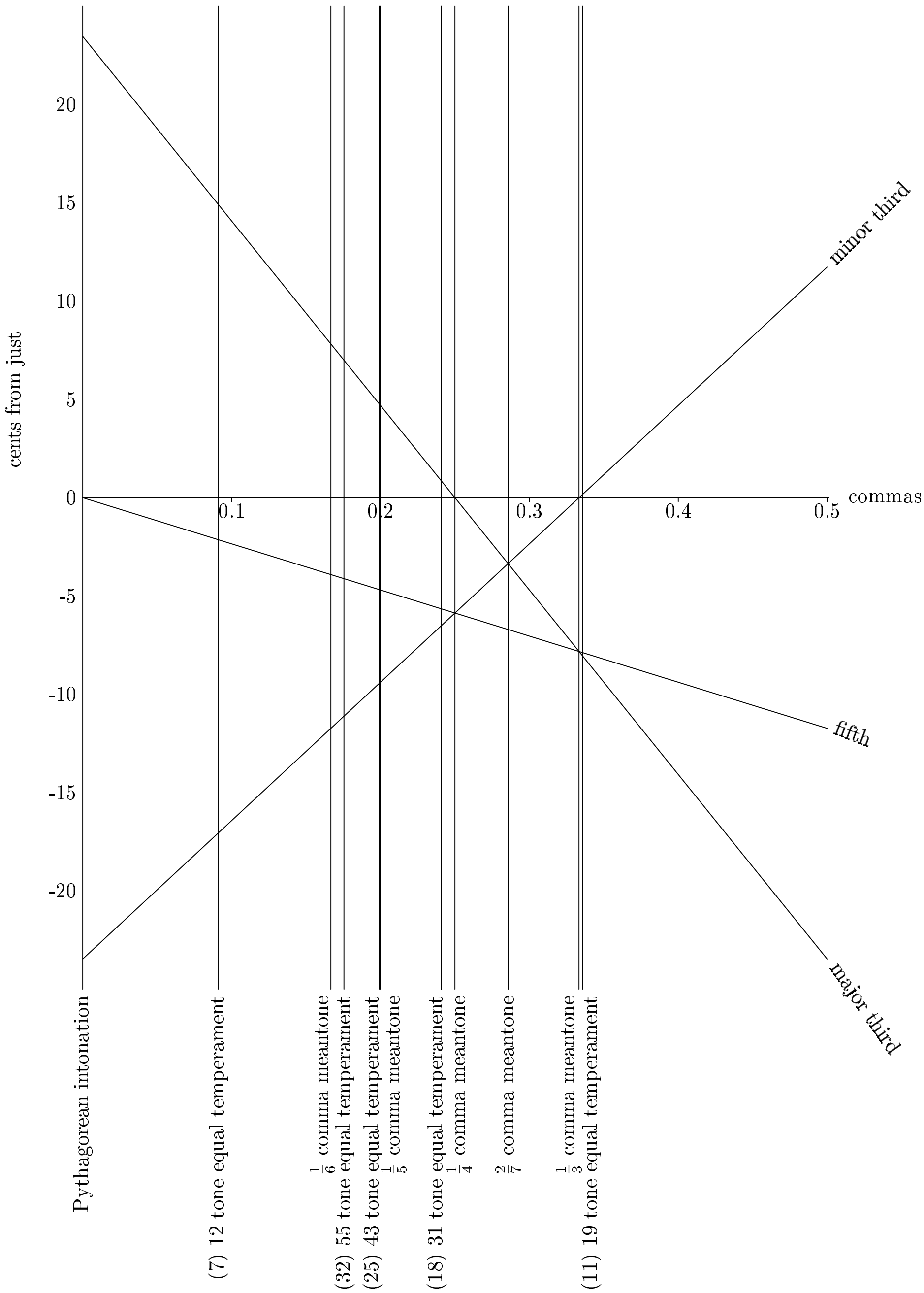
Cents	Interval ratio	Eitz	Name, etc.	Ref
500.000	$2^{\frac{5}{12}}:1$	$\approx \mathrm{F}^{+\frac{1}{11}}$	Equal fourth	§5.14
503.422	$2:5^{\frac{1}{4}}$	$\mathrm{F}^{+\frac{1}{4}}$	Meantone fourth	§5.12
551.318	11:8		Eleventh harmonic	§4.1
600.000	$\sqrt{2}:1$	$\approx \mathrm{F}\sharp^{-\frac{6}{11}}$	Equal tritone	§5.14
611.731	729:512	$\mathrm{F}\sharp^{0}$	Pythagorean tritone	§5.2
696.579	$5^{\frac{1}{4}}:1$	$\mathrm{G}^{-\frac{1}{4}}$	Meantone fifth	§5.12
700.000	$2^{\frac{7}{12}}:1$	$\approx \mathrm{G}^{-\frac{1}{11}}$	Equal fifth	§5.14
701.955	3:2	G^{0}	Just and Pythagorean (perfect) fifth;	§5.2
			Third harmonic	§4.1
792.180	128:81	$\mathrm{A}\flat^{0}$	Pythagorean minor sixth	§5.2
800.000	$2^{\frac{2}{3}}:1$	$\approx \mathrm{A}\flat^{+\frac{4}{11}}$	Equal minor sixth	§5.14
813.687	8:5	$\mathrm{A}\flat^{+1}$	Just minor sixth	§5.5
840.528	13:8		Thirteenth harmonic	§4.1
884.359	5:3	A^{-1}	Just major sixth	§5.5
889.735	$5^{\frac{3}{4}}:2$	$\mathrm{A}^{-\frac{3}{4}}$	Meantone major sixth	§5.12
900.000	$2^{\frac{3}{4}}:1$	$\approx \mathrm{A}^{-\frac{3}{11}}$	Equal major sixth	§5.14
905.865	27:16	A^{0}	Pythagorean major sixth	§5.2
968.826	7:4		Seventh harmonic	§4.1
996.091	16:9	$\mathrm{B}\flat^{0}$	Pythagorean minor seventh	§5.2
1000.000	$2^{\frac{5}{6}}:1$	$\approx \mathrm{B}\flat^{+\frac{2}{11}}$	Equal minor seventh	§5.14
1082.892	$5^{\frac{5}{4}}:4$	$\mathrm{B}^{-\frac{5}{4}}$	Meantone major seventh	§5.12
1088.269	15:8	B^{-1}	Just major seventh;	§5.5
			Fifteenth harmonic	§4.1
1100.000	$2^{\frac{11}{12}}:1$	$\approx \mathrm{B}^{-\frac{5}{11}}$	Equal major seventh	§5.14
1109.775	243:128	B^{0}	Pythagorean major seventh	§5.2
1200.000	2:1	C^{0}	Octave; Second harmonic	§4.1
1466.871	7:3	$\mathrm{A}^{0}_{\mathrm{BP}}$	BP-tenth	§6.7
1901.955	3:1	$\mathrm{C}^{0}_{\mathrm{BP}}$	BP-Tritave	§6.7

APPENDIX J

Just, equal and meantone scales compared

The figure on the next page has its horizontal axis measured in multiples of the (syntonic) comma, and the vertical axis measured in cents. Each vertical line represents a regular scale, generated by its fifth. The size of the fifth in the scale is equal to the Pythagorean fifth (ratio of 3:2, or 701.955 cents) minus the multiple of the comma given by the position along the horizontal axis. The three sloping lines show how far from the just values the fifth, major third and minor third are in these scales. This figure is relevant to Exercise **2** in §6.4.

It is worth noting that if $\frac{1}{11}$ comma meantone were drawn on this diagram, it would be indistinguishable from 12 tone equal temperament; see §5.14.



Regular scales and their deviations from just intonation

APPENDIX L

Logarithms

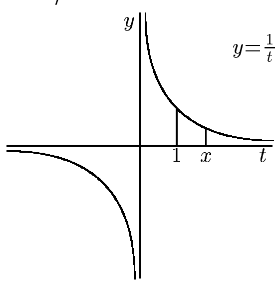
The purpose of this appendix is to give a quick review of the definition and standard properties of logarithms, since they are so important to the theory of scales and temperaments. A commonly used definition of logarithm is that $b = \log_a(c)$ means the same as $a^b = c$.

The main problem in understanding the above definition is understanding what the notation a^b means. If b is rational, this can be explained in terms of multiplication and extraction of roots. But what on earth does 2^π mean? How do we multiply 2 by itself π times? It turns out that logically, the easiest way to develop exponentials and logarithms begins with the logarithm as a definite integral and proceeds in the reverse of the order in which these concepts are usually learned.

The definition of the natural logarithm is

$$\ln(x) = \int_1^x \frac{1}{t}\,dt,$$

which makes sense provided $x > 0$. In other words, $\ln(x)$ is the area under the graph of the function $y = 1/t$ between $t = 1$ and $t = x$.



According to the usual conventions of calculus, if x lies between zero and one, this area is interpreted as negative, while for $x > 1$ it is positive. It is immediately apparent from the definition that

$$\ln(1) = 0.$$

The fundamental theorem of calculus implies that

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Applying the chain rule, if a is a constant then

$$\frac{d}{dx}\ln(ax) = \frac{a}{ax} = \frac{1}{x}.$$

One of the consequences of the mean value theorem is that two functions with the same derivative differ by a constant. We apply this to $\ln(ax)$ and $\ln(x)$, and find out the value of the constant by setting $x = 1$, to get $\ln(ax) - \ln(x) = \ln(a) - \ln(0) = \ln(a)$. If b is another constant, then evaluating at $x = b$ gives

$$\ln(ab) = \ln(a) + \ln(b).$$

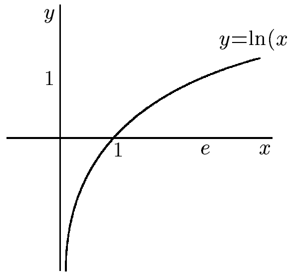
The particular case where $a = 1/b$ gives us

$$\ln(1/b) = -\ln(b).$$

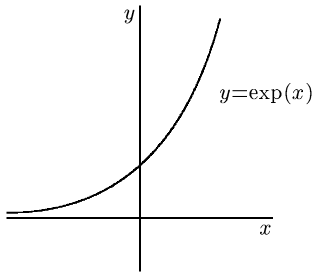
Combining these formulas gives

$$\ln(a/b) = \ln(a) - \ln(b).$$

From these properties and the definition, it easily follows that the logarithm function is monotonically increasing, with domain $(0, \infty)$ and range $(-\infty, \infty)$.



The *exponential function* $\exp(x)$ is defined to be the inverse function of $\ln(x)$. In other words, $y = \exp(x)$ means the same as $x = \ln(y)$.



So the area under the graph of $y = 1/t$ between $t = 1$ and $t = \exp(x)$ is equal to x. The above properties of the logarithm translate into the following properties of the exponential function:

$$\begin{aligned}
&\exp(0) = 1 \\
&\exp(a + b) = \exp(a)\exp(b) \\
&\exp(-b) = 1/\exp(b) \\
&\exp(a - b) = \exp(a)/\exp(b).
\end{aligned}$$

The number e is defined to be $\exp(1)$, and it is an irrational number whose approximate value is 2.71828. The domain of the exponential function is $(-\infty, \infty)$, and its range is $(0, \infty)$.

We define a^b to mean $\exp(b\ln(a))$ $(a > 0)$. So the area under the graph of $y = 1/t$ between $t = 1$ and $t = a^b$ is exactly b times as big as the area between $t = 1$ and $t = a$. If $b = m/n$ is rational, it is not hard to check using the above properties of the exponential and logarithm function that this definition agrees with the more usual one with powers and roots ($a^{m/n}$ is the unique positive number whose nth power equals the mth power of a). But this definition gets us around the problem of trying to understand what it means to multiply a by itself an irrational number of times! Thus for example

$$e^x = \exp(x\ln(e)) = \exp(x)$$

so that the exponential function can be written as e^x. With these definitions, it is easy to prove the usual laws of indices:

$$a^0 = 1, \qquad a^1 = a, \qquad a^{-1} = 1/a, \qquad a^{-b} = 1/a^b, \qquad a^{b+c} = a^b a^c,$$
$$a^{b-c} = a^b/a^c, \qquad a^c b^c = (ab)^c, \qquad (a^b)^c = a^{bc}, \qquad a^{\frac{1}{b}} = \sqrt[b]{a}$$

We define

$$\log_a(b) = \frac{\ln(b)}{\ln(a)} \qquad (a > 0).$$

Thus $c = \log_a(b)$ is equivalent to $c\ln(a) = \ln(b)$, or $\exp(c\ln(a)) = b$, or $a^c = b$. So $c = \log_a(b)$ means that c is the power to which a has to be raised to obtain b. For example, $\log_e(b)$ is the same as $\ln(b)$, the natural logarithm of b.

The scale of cents in music theory is defined in such a way that a frequency ratio of f:1 is represented as an interval of

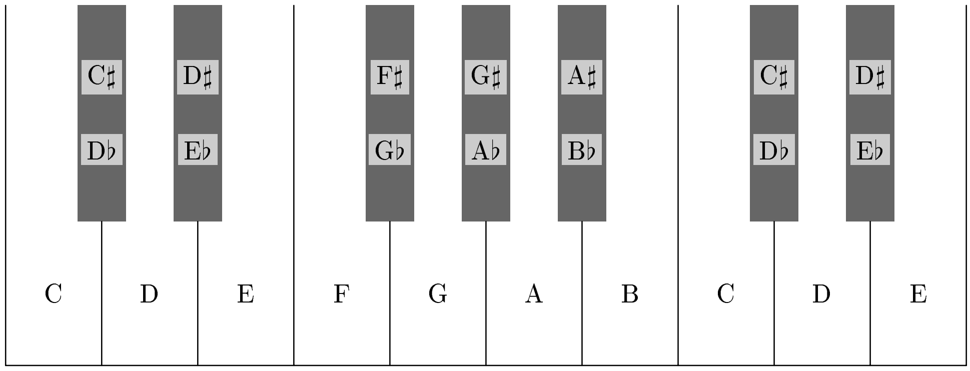
$$1200\log_2(f) \text{ cents} = \frac{1200\ln(f)}{\ln(2)} \text{ cents}.$$

Thus one octave, or a frequency ratio of 2:1, is an interval of 1200 cents. In the 12 tone equal tempered scale, this is divided into 12 equal semitones of 100 cents each. For more details, see §5.4.

APPENDIX M

Music theory

This appendix consists of the background in elementary music theory needed to understand the main text. The emphasis is slightly different than that of a standard music text. We begin with the piano keyboard, as a convenient way to represent the modern scale (see also Appendix F).

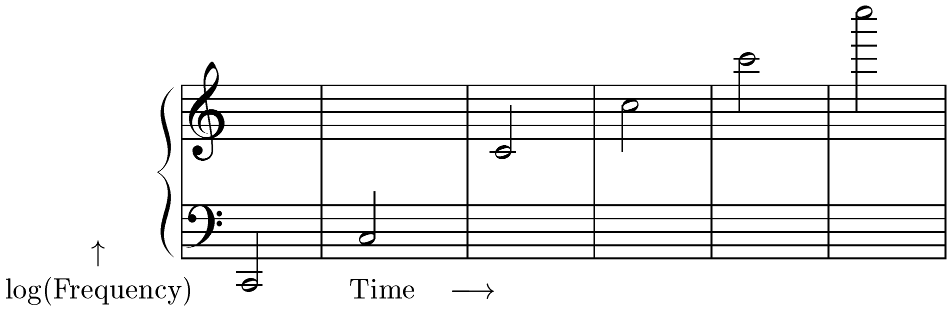


Both the black and the white keys represent notes. This keyboard is periodic in the horizontal direction, in the sense that it repeats after seven white notes and five black notes. The period is one *octave*, which represents doubling the frequency corresponding to the note. The principle of *octave equivalence* says that notes differing by a whole number of octaves are regarded as playing equivalent roles in harmony. In practice, this is almost but not quite completely true.

On a modern keyboard, each of the twelve intervals making up an octave represents the same frequency ratio, called a *semitone*. The name comes from the fact that two semitones make a *tone*. The twelfth power of the semitone's frequency ratio is a factor of 2:1, so a semitone represents a frequency ratio of $2^{\frac{1}{12}}$:1. The arrangement where all the semitones are equal in this way is called *equal temperament*. Frequency is an exponential function of position on the keyboard, and so the keyboard is really a *logarithmic* representation of frequency.

Because of this logarithmic scale, we talk about *adding* intervals when we want to *multiply* the frequency ratios. So when we add a semitone to another semitone, for example, we get a tone with a frequency ratio of $2^{\frac{1}{12}} \times 2^{\frac{1}{12}}$:1 or $2^{\frac{1}{6}}$:1. This transition between additive and multiplicative notation can be a source of great confusion.

Staff notation works in a similar way, except that the logarithmic frequency is represented vertically, and the horizontal direction represents time. So music notation paper can be regarded as graph paper with a linear horizontal time axis and a logarithmic vertical frequency axis.

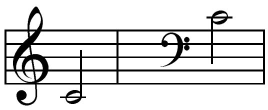


In the above diagram, each note is twice the frequency of the previous one, so they are equally spaced on the logarithmic frequency scale (except for the break between the bass and treble clefs). The gap between adjacent notes is one octave, so the gap between the lowest and highest note is described *additively* as five octaves, representing a *multiplicative* frequency ratio of 2^5:1.

There are two clefs on this diagram. The upper one is called the *treble clef*, with lines representing the notes E, G, B, D, F, beginning with the E two white notes above middle C and working up the lines. The spaces between them represent the notes F, A, C, E between them, so that this takes care of all the white notes between the E above middle C and the F an octave and a semitone above that. The black notes are represented in by using the line or space with the likewise lettered white note with a sharp ($\sharp$) or flat ($\flat$) sign in front.

The lower clef is called the bass clef, with lines representing the notes G, B, D, F, A, with the last note representing the A two white notes below middle C and the first note representing the G an octave and a tone below that.

Middle C itself is represented using a *leger line*, either below the treble clef or above the bass clef.



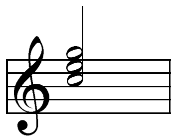
The frequency ratio represented by seven semitones, for example the interval from C to the G above it, is called a *perfect fifth*. Well, actually, this isn't quite true. A perfect fifth is supposed to be a frequency ratio of 3:2, or 1.5:1, whereas seven semitones on our modern equal tempered scale produce a frequency ratio of $2^{\frac{7}{12}}$:1 or roughly 1.4983:1. The perfect fifth is a consonant interval, just as the octave is, for reasons described in Chapter 4. So seven semitones is very close to a consonant interval. It is very difficult to discern the difference between a perfect fifth and an equal tempered fifth except by listening for beats; the difference is about one fiftieth of a semitone.

The perfect fourth represents the interval of 4:3, which is also consonant. The difference between a perfect fourth and the equal tempered fourth of five semitones is exactly the same as the difference between the perfect fifth and the equal tempered fifth, because they are obtained from the corresponding versions of a fifth by subtracting from an octave.

The frequency ratio represented by four semitones, for example the interval from C to the E above it, is called a *major third.* This represents a frequency ratio of $2^{\frac{4}{12}}$:1 or $\sqrt[3]{2}$:1, or roughly 1.25992:1. The *just major third* is defined to be the frequency ratio of 5:4 or 1.25:1. Again it is the just major third which represents the consonant interval, and the major third on our modern equal tempered scale is an approximation to it. The approximation is quite a bit worse than it was for the perfect fifth. The difference between a just major third and an equal tempered major third is quite audible; the difference is about one seventh of a semitone.

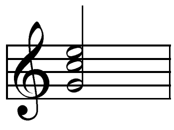
The frequency ratio represented by three semitones, for example the interval from E to the G above it, is called a *minor third.* This represents a frequency ratio of $2^{\frac{3}{12}}$:1 or $\sqrt[4]{2}$:1, or roughly 1.1892:1. The consonant *just minor third* is defined to be the frequency ratio of 6:5 or 1.2:1. The equal tempered minor third again differs from it by about a seventh of a semitone.

A major third plus a minor third makes up a fifth, either in the just/perfect versions or the equal tempered versions. So the intervals C to E (major third) plus E to G (minor third) make C to G (fifth). In the just/perfect versions, this gives ratios 4:5:6 for a *just major triad* C—E—G. We refer to C as the *root* of this chord. The chord is named after its root, so that this is a C major chord.



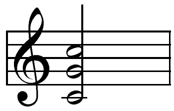
4:5:6

If we used the frequency ratios 3:4:5, it would just give an *inversion* of this chord, which is regarded as a variant form of the C major chord, because of the principle of octave equivalence.



3:4:5

while the frequency ratios 2:3:4 give a much simpler chord with a fifth and an octave.



2:3:4

So the just major triad 4:5:6 is the chord that is basic to the western system of musical harmony. On an equal tempered keyboard, this is approximated with the chord $1 : 2^{\frac{4}{12}} : 2^{\frac{7}{12}}$, which is a good approximation except for the somewhat sharp major third.

The *major scale* is formed by taking three major triads on three notes separated by intervals of a fifth. So for example the scale of C major is formed from the notes of the F major, C major and G major triads. Between them, these account for the white notes on the keyboard, which make up the scale of C major. So in just intonation, the C major scale would have the following frequency ratios.

C	D	E	F	G	A	B	C	D
$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{2}{1}$	$\frac{9}{4}$
4	:	5	:	6	:		(8)	
			4	:	5	:	6	
	(3)	:		4	:	5	:	6

Here, we have made use of 2:1 octaves to transfer ratios between the right and left end of the diagram.

The basic problem with this scale is that the interval from D to A is almost, but not quite equal to a perfect fifth. It is just close enough that it sounds like a nasty, out of tune fifth. It is short of a perfect fifth by a ratio of 81:80. This interval is called a *syntonic comma*. In this text, when we use the word comma without further qualification, it will always mean the syntonic comma. This and other commas are investigated in Section 5.8.

The *meantone* scale addresses this problem by distributing the syntonic comma equally between the four fifths C–G–D–A–E. So in the meantone scale, the fifths are one quarter of a comma smaller than the perfect fifth, and the major thirds are just. In the meantone scale, a number of different keys work well, but the more remote keys do not. For further details, see Section 5.12.

To make all keys work well, the meantone scale must be bent to meet around the back. A number of different versions of this compromise have been used historically, the first ones being due to Werckmeister. Some of these well tempered scales are described in Section 5.13. Meantone and well tempered scales were in common use for about four centuries before equal temperament became widespread in the late nineteenth and early twentieth century.

A *minor triad* is obtained by inverting the order of the intervals in a major triad. So for example the minor triad on the note C consists of C, E♭ and G. In just intonation, the frequency ratios are 5:6 for C–E♭ and 4:5 for E♭–G, so that C–G still makes a perfect fifth. So the ratios are 10:12:15. See §5.6 for a discussion of the role of the minor triad. A *minor scale* can be built out of three minor triads in the same way as we did for the major scale, to give the following frequency ratios.

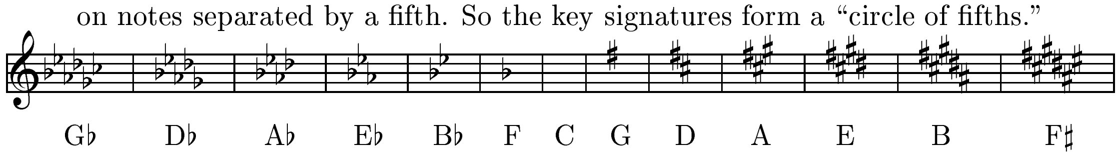
C	D	E♭	F	G	A♭	B♭	C	D
$\frac{1}{1}$	$\frac{9}{8}$	$\frac{6}{5}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{8}{5}$	$\frac{9}{5}$	$\frac{2}{1}$	$\frac{9}{4}$
10	:	12	:	15				
			10	:	12	:	15	
				10	:	12	:	15

This is called the *natural minor* scale. Other forms of the minor scale occur because the sixth and seventh notes can be varied by moving one or both of them up a semitone to their major equivalents.

The concept of *key signature* arises from the following observation. If we look at major scales which start on notes separated by the interval of a fifth, then the two scales have all but one of the notes in common. For example, in C major, the notes are C–D–E–F–G–A–B–C, while in G major, the notes are G–A–B–C–D–E–F♯–G. The only difference, apart from a cyclic rearrangement of the notes, is that F♯ appears instead of F. So to indicate that we are in G major rather than C major, we write a sharp sign on the F at the beginning of each stave.

Similarly, the key of F major uses the notes F–G–A–B♭–C–D–E–F, which only differs from C major in the use of B♭ instead of B.

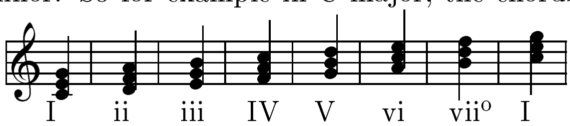
This means that key signatures are regarded as "adjacent" if they begin on notes separated by a fifth. So the key signatures form a "circle of fifths."



In the above sequence of key signatures, the first and last are *enharmonic* versions of the same key. This means that in equal temperament, they are just different ways of writing the same keys, but in other systems such as meantone, the actual pitches may differ.

The notes which occur in a natural minor scale are the same as the notes which occur in the major scale starting three semitones higher. For example, the notes of A minor are A–B–C–D–E–F–G–A. So the same key signature is used for A minor as for C major, and we say that A minor is the *relative minor* of C major.

The note on which a scale starts is called the *tonic*. The word *dominant* refers to the fifth above the tonic. The *roman numeral notation* is a device for naming triads relative to the tonic. So for example the major triad on the dominant is written V. Upper case roman numerals refer to major triads and lower case to minor. So for example in C major, the chords are as follows.



In D major, each chord would be a whole tone higher; so V would refer to the chord of A major instead of G major. So the roman numeral refers to the harmonic function of the chord within the key signature, rather than giving the absolute pitches.

The only triad here which is neither major nor minor is the *diminished* triad on the seventh note of the scale. This is denoted vii$^{\mathrm{o}}$, and consists of two intervals of a minor third with no major thirds.

APPENDIX O

Online papers

Several journals have good selections of papers available online. Access usually requires you to be logged on from an academic establishment which subscribes to the journal in question. Here is a selection of what is available from a typical academic institution.

From http://www.jstor.org you can obtain online copies of papers from the American Mathematical Monthly, a publication which concentrates on undergraduate level mathematics. Papers include the following, in chronological order.

J. M. Barbour, *Synthetic musical scales*, Amer. Math. Monthly 36 (3) (1929), 155–160.

J. M. Barbour, *A sixteenth century Chinese approximation for* π, Amer. Math. Monthly 40 (2) (1933), 69–73.

J. M. Barbour, *Music and ternary continued fractions*, Amer. Math. Monthly 55 (9) (1948), 545–555.

J. B. Rosser, *Generalized ternary continued fractions*, Amer. Math. Monthly 57 (8) (1950), 528–535. This article is a reply to the above article of Barbour.

T. J. Fletcher, *Campanological groups*, Amer. Math. Monthly 63 (9) (1956), 619–626.

J. M. Barbour, *A geometrical approximation to the roots of numbers*, Amer. Math. Monthly 64 (1) (1957), 1–9. This article discusses an eighteenth century geometric method of Strähle for constructing a very good approximation to equal temperament for the frets of a guitar.

Mark Kac, *Can one hear the shape of a drum?* Amer. Math. Monthly 73 (4) (1966), 1–23.

John Rogers and Bary Mitchell, *A problem in mathematics and music*, Amer. Math. Monthly 75 (8) (1968), 871–873.

A. L. Leigh Silver, *Musimatics, or the nun's fiddle*, Amer. Math. Monthly 78 (4) (1971), 351–357.

G. D. Hasley and Edwin Hewitt, *More on the superparticular ratios in music*, Amer. Math. Monthly 79 (10) (1972), 1096–1100.

I. J. Schoenberg, *On the location of the frets on the guitar*, Amer. Math. Monthly 83 (7) (1976), 550–552. Schoenberg was the referee of the 1957 article of Barbour on Strähle's method referred to above, and this article expands on his footnotes to Barbour's article.

David Gale, *Tone perception and decomposition of periodic function*, Amer. Math. Monthly 86 (1) (1979), 36–42.

Murray Schechter, *Tempered scales and continued fractions*, Amer. Math. Monthly 87 (1)

(1980), 40–42.

David L. Reiner, *Enumeration in music theory*, Amer. Math. Monthly 92 (1) (1985), 51–54.

John Clough and Gerald Myerson, *Musical scales and the generalized circle of fifths*, Amer. Math. Monthly 93 (9) (1986), 695–701.

Arthur T. White, *Ringing the cosets*, Amer. Math. Monthly 94 (8) (1987), 721–746.

S. J. Chapman, *Drums that sound the same*, Amer. Math. Monthly 102 (2) (1995), 124–138.

Rachel W. Hall and Krešimir Josić, *The mathematics of musical instruments*, Amer. Math. Monthly 108 (4) (2001), 347–357.

There are occasionally relevant articles in the SIAM[1] journals, also available from http://www.jstor.org. Examples include the following.

A. A. Goldstein, *Optimal temperament*, SIAM Review 19 (3) (1977), 554–562.

A. Inselberg, *Cochlear dynamics: the evolution of a mathematical model*, SIAM Review 20 (2) (1978), 301–351.

Robert Burridge, Jay Kappraff and Christine Mordeshi, *The Sitar string, a vibrating string with a one-sided inelastic constraint*, SIAM J. Appl. Math. 42 (6) (1982), 1231–1251.

M. H. Protter, *Can one hear the shape of a drum? Revisited*, SIAM Review 29 (2) (1987), 185–197.

Tobin A. Driscoll, *Eigenmodes of isospectral drums*, SIAM Review 39 (1) (1997), 1–17.

From http://ojps.aip.org/jasa/ (then hit "browse html" or "search") you can obtain online copies of articles from the Journal of the Acoustical Society of America (JASA) from 1997 to the current issue. Here is a selection of some relevant articles that can be downloaded.

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You can download Julius O. Smith III, *Mathematics of the discrete Fourier transform* (237 pages of lecture notes, pdf or compressed postscript format) from

http://ccrma-www.stanford.edu/~jos/r320/

APPENDIX P

Partial derivatives

Partial derivatives are what happens when we differentiate a function of more than one variable. For example, a geographical map which indicates height above sea level, by some device such as coloration or contours, can be regarded as describing a function $z = f(x, y)$. Here, x and y represent the two coordinates of the map, and z denotes height above sea level. If we move due east, which we take to be the direction of the x axis, then we are keeping y constant and changing x. So the slope in this direction would be the derivative of $z = f(x, y)$ with respect to x, regarding y as a constant. This derivative is denoted $\dfrac{\partial z}{\partial x}$. More formally,

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Similarly, $\dfrac{\partial z}{\partial y}$ is the derivative of z with respect to y, regarding x as a constant. As an example, let $z = x^4 + x^2y - 2y^2$. Then we have $\dfrac{\partial z}{\partial x} = 4x^3 + 2xy$, because x^2y is being regarded as a constant multiple of x^2, and $-2y^2$ is just a constant. Similarly, $\dfrac{\partial z}{\partial y} = x^2 - 4y$, because x^4 is a constant and x^2y is a constant multiple of y.

Second partial derivatives are defined similarly, but we now find that we can mix the variables. As well as $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$, we can now form $\dfrac{\partial^2 z}{\partial x \partial y}$ by taking the partial derivative of $\dfrac{\partial z}{\partial y}$ with respect to x, regarding y as constant, and we can also form $\dfrac{\partial^2 z}{\partial y \partial x}$ by taking partial derivatives in the opposite order. So in the above example, we have

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 + 2y, \qquad \frac{\partial^2 z}{\partial y^2} = -4, \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 2x.$$

In fact, the two mixed partial derivatives agree under some fairly mild hypotheses.

THEOREM P.1. *Suppose that the partial derivatives* $\dfrac{\partial^2 z}{\partial x \partial y}$ *and* $\dfrac{\partial^2 z}{\partial y \partial x}$ *both exist and are both continuous at some point (i.e., for some chosen values of* x *and* y*). Then they are equal at that point.*

PROOF. See any book on elementary analysis; for example, J. C. Burkhill, *A first course in mathematical analysis*, CUP, 1962, theorem 8.3. □

Partial derivatives work in exactly the same way for functions of more variables. So for example if $f(x, y, z) = xy^2 \sin z$ then we have $\dfrac{\partial f}{\partial x} = y^2 \sin z$, $\dfrac{\partial f}{\partial y} = 2xy \sin z$, and $\dfrac{\partial f}{\partial z} = xy^2 \cos z$. For each pair of variables, the two mixed partial derivatives with respect to those variables agree provided they are both continuous.

The chain rule for partial derivatives needs some care. Suppose, by way of example, that z is a function of u, v and w, and that each of u, v and w is a function of x and y. Then z can also be regarded as a function of x and y. A change in the value of x, keeping y constant, will result in a change of all of u, v and w, and each of these changes will result in a change in the value of z. These changes have to be added as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x}.$$

Similarly, we have

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y}.$$

It is essential to keep track of which variables are independent, intermediate, and dependent. In this example, the independent variables are x and y, the intermediate ones are u, v and w, and the dependent variable is z.

A good illustration of the chain rule for partial derivatives is given by the conversion from Cartesian to polar coordinates. If z is a function of x and y then it can also be regarded as a function of r and θ. To convert from polar to Cartesian coordinates, we use $x = r\cos\theta$ and $y = r\sin\theta$, and to convert back we use $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$. Let us convert the quantity

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2},$$

into polar coordinates, assuming that all mixed second partial derivatives are continuous, so that the above theorem applies. This calculation will be needed in §3.5, where we investigate the vibrational modes of the drum. For this purpose, it is actually technically slightly easier to regard x and y as the intermediate variables and r and θ as the independent variables, although it would be quite permissible to interchange their roles. The dependent variable is z. We have

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}. \tag{P.1}$$

To take the second derivative, we do the same again.

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + \sin\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) \\
&= \cos\theta\left(\cos\theta\frac{\partial^2 z}{\partial x^2} + \sin\theta\frac{\partial^2 z}{\partial y \partial x}\right) + \sin\theta\left(\cos\theta\frac{\partial^2 z}{\partial x\partial y} + \sin\theta\frac{\partial^2 z}{\partial y^2}\right) \\
&= \cos^2\theta\frac{\partial^2 z}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 z}{\partial y^2}. \qquad \text{(P.2)}
\end{aligned}$$

Similarly, we have

$$\frac{\partial z}{\partial\theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial\theta} = (-r\sin\theta)\frac{\partial z}{\partial x} + (r\cos\theta)\frac{\partial z}{\partial y},$$

and

$$\begin{aligned}
\frac{\partial^2 z}{\partial\theta^2} &= (-r\sin\theta)\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right) + (-r\cos\theta)\frac{\partial z}{\partial x} \\
&\quad + (r\cos\theta)\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right) + (-r\sin\theta)\frac{\partial z}{\partial y} \\
&= (-r\sin\theta)\left((-r\sin\theta)\frac{\partial^2 z}{\partial x^2} + (r\cos\theta)\frac{\partial^2 z}{\partial y\partial x}\right) + (-r\cos\theta)\frac{\partial z}{\partial x} \\
&\quad + (r\cos\theta)\left((-r\sin\theta)\frac{\partial^2 z}{\partial x\partial y} + (r\cos\theta)\frac{\partial^2 z}{\partial y^2}\right) + (-r\cos\theta)\frac{\partial z}{\partial y} \\
&= r^2\left(\sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 z}{\partial y^2}\right) \\
&\quad - r\left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right). \qquad \text{(P.3)}
\end{aligned}$$

Comparing the formula (P.2) for $\dfrac{\partial^2 z}{\partial r^2}$ with the formula (P.3) for $\dfrac{\partial^2 z}{\partial\theta^2}$, and using the fact that $\sin^2\theta + \cos^2\theta = 1$, we see that

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \frac{1}{r}\left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right).$$

Finally, looking back at equation (P.1) for $\dfrac{\partial z}{\partial r}$, we obtain the formula we were looking for, namely

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}. \qquad \text{(P.4)}$$

APPENDIX R

Recordings

Go to the entry "compact discs" in the index to find the points in the text which refer to these recordings.

Bill Alves, *Terrain of possibilities*, Emf media #2, 2000. Music made with Synclavier and CSound using just intonation.

Johann Sebastian Bach, *The Complete Organ Music*, recorded by Hans Fagius, Volumes 6 and 8, BIS-CD-397/398 (1989) and BIS-CD-443/444 (1989 & 1990). These recordings are played on the reconstructed 1764 Wahlberg organ, Fredrikskyrkan, Karlskrona, Sweden. This organ was reconstructed using the original temperament, which was Neidhardt's Circulating Temperament No. 3 "für eine grosse Stadt" (for a large town).

Clarence Barlow's "OTOdeBLU" is in 17 tone equal temperament, played on two pianos. This piece was composed in celebration of John Pierce's eightieth birthday, and appeared as track 15 on the Computer Music Journal's Sound Anthology CD, 1995, to accompany volumes 15–19 of the journal. The CD can be obtained from MIT press for $15.

Between the Keys, *Microtonal masterpieces of the 20th century*, Newport Classic CD #85526, 1992. This CD contains recordings of Charles Ives' *Three quartertone pieces*, and a piece by Ivan Vyshnegradsky (or Wyschnegradsky) in 72 tone equal temperament. Unfortunately, this CD seems to have gone out of print.

Easley Blackwood has composed a set of microtonal compositions in each of the equally tempered scales from 13 tone to 24 tone, as part of a research project funded by the National Endowment for the Humanities to explore the tonal and modal behavior of these temperaments. He devised notations for each tuning, and his compositions were designed to illustrate chord progressions and practical application of his notations. The results are available on compact disc as Cedille Records CDR 90000 018, Easley Blackwood: *Microtonal Compositions* (1994). Copies of the scores of the works can be obtained from Blackwood Enterprises, 5300 South Shore Drive, Chicago, IL 60615, USA for a nominal cost.

Dietrich Buxtehude, *Orgelwerke*, Volumes 1–7, recorded by Harald Vogel, published by Dabringhaus and Grimm. These works are recorded on a variety of European organs in different temperaments. Extensive details are given in the liner notes.

CD1 Tracks 1–8: Norden – St. Jakobi/Kleine organ in Werckmeister III;

Tracks 9–15: Norden – St. Ludgeri organ in modified $\frac{1}{5}$ Pythagorean comma meantone with $C\sharp^{-\frac{6}{5}p}$, $G\sharp^{-\frac{6}{5}p}$, $B\flat^{+\frac{1}{5}p}$ and $E\flat^{0}$;

CD2 Tracks 1–6: Stade – St. Cosmae organ in modified quarter comma meantone with[1] $C\sharp^{-\frac{3}{2}}$, $G\sharp^{-\frac{3}{2}}$, F^{0}, $B\flat^{0}$, $E\flat^{-\frac{1}{5}}$;

Tracks 7–15: Weener – Georgskirche organ in Werckmeister III;

CD3 Tracks 1–10: Grasberg organ in Neidhardt No. 3;

Tracks 11–14: Damp – Herrenhaus organ in modified meantone with pitches taken from original pipe lengths;

CD4 Tracks 1–8: Noordbroeck organ in Werckmeister III;

Tracks 9–15: Groningen – Aa-Kerk organ in (almost) equal temperament;

CD5 Tracks 1–5: Pilsum organ in modified $\frac{1}{5}$ Pythagorean comma meantone (the same as the Norden – St. Ludgeri organ described above);

Tracks 6–7: Buttforde organ;

Tracks 8–10: Langwarden organ in modified quarter comma meantone with $G\sharp^{-\frac{7}{4}}$, $B\flat^{-\frac{1}{4}}$, $E\flat^{-\frac{1}{4}}$;

Tracks 11–13: Basedow organ in quarter comma meantone;

Tracks 14–15: Groß Eichsen organ in quarter comma meantone;

CD6 Tracks 1–10: Roskilde organ in Neidhardt (no. 3?);

Track 11: Helsingør organ (unspecified temperament);

Tracks 12–15: Torrlösa organ (unspecified temperament);

CD7 Tracks 1–10 modified $\frac{1}{5}$ comma meantone with[2] $C\sharp^{-\frac{6}{5}}$, $G\sharp^{-\frac{6}{5}}$, $B\flat^{+\frac{1}{5}}$ and $E\flat^{\frac{1}{5}-\frac{1}{10}p}$.

William Byrd, *Cantones Sacrae 1575, The Cardinall's Music*, conducted by David Skinner. Track 12, *Diliges Dominum*, exhibits temporal reflectional symmetry, so that it is a perfect palindrome (see §9.1).

Wendy Carlos, *Beauty in the Beast*, Audion, 1986, Passport Records, Inc., SYNCD 200. Tracks 4 and 5 make use of Carlos' just scales described in §6.1.

Wendy Carlos, *Switched-On Bach 2000*, 1992. Telarc CD-80323. Carlos' original "Switched-On Bach" recording was performed on a Moog analog synthesizer, back in the late 1960s. The Moog is only capable of playing in equal temperament. Improvements in technology inspired her to release this new recording, using a variety of temperaments and modern methods of digital synthesis. The temperaments used are $\frac{1}{5}$ and $\frac{1}{4}$ comma meantone, and various circular (irregular) temperaments.

Charles Carpenter has two CDs, titled *Frog à la Pêche* (Caterwaul Records, CAT8221, 1994) and *Splat* (Caterwaul Records, CAT4969, 1996), composed using the Bohlen–Pierce scale, and played in a progressive rock/jazz style. These recordings can be ordered directly from http://www.kspace.com/carpenter for $13.95 each. Although Carpenter does not restrict himself to sounds composed mainly of odd harmonics, his compositions are nonetheless compelling.

Perry Cook (ed.), *Music, congnition and computerized sound. An introduction to psychoacoustics* [**17**] comes with an accompanying CD full of sound examples.

Michael Harrison, *From Ancient Worlds*, for Harmonic Piano, New Albion Records, Inc., 1992. NA 042 CD. The pieces on this recording all make use of his 24 tone just scale, described in §6.1.

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[1]The liner notes are written as though $G\sharp^{-\frac{3}{2}}$ were equal to $A\flat^{-\frac{2}{5}}$, which is not quite true. But the discrepancy is only about 0.2 cents.

[2]The liner notes identify $A\flat^{-\frac{1}{10}p}$ with $G\sharp^{-\frac{6}{5}}$, in accordance with the approximation of Kirnberger and Farey described in §5.14.

Michael Harrison has also just released another CD using his Harmonic Piano, *Revelation*, recorded live in the Lincoln Center in October 2001 and issued in January 2002. In this recording, the harmonic piano is tuned to a just scale using only the primes 2, 3 and 7 (not 5). The 12 notes in the octave have ratios

$$1{:}1,\ 63{:}64,\ 9{:}8,\ 567{:}512,\ 81{:}64,\ 21{:}16,\ 729{:}512,\ 3{:}2,\ 189{:}128,\ 27{:}16,\ 7{:}4,\ 243{:}128,\ (2{:}1).$$

The scale begins on F, and has the peculiarity that ♯ lowers a note by a septimal comma.

Jonathan Harvey, *Mead: Ritual melodies*, Sargasso CD #28029, 1999. Track two on this CD, *Mortuos Plango, Vivos Voco*, makes use of a scale derived from a spectral analysis of the Great Bell of Winchester Cathedral.

Neil Haverstick, *Acoustic stick*, Hapi Skratch, 1998. The pieces on this CD are played on custom made guitars using 19 and 34 tone equal temperament.

In Joseph Haydn's *Sonata 41* in A (Hob. XVI:26), the movement *Menuetto al rovescio* is a perfect palindrome (see §9.1). This piece can be found as track 16 on the Naxos CD number 8.553127, Haydn, *Piano sonatas, Vol. 4*, with Jenõ Jandó at the piano.

A. J. M. Houtsma and T. D. Rossing and W. M. Wagenaars, *Auditory Demonstrations*, Audio CD and accompanying booklet, Philips, 1987. This classic collection of sound examples illustrates a number of acoustic and psychoacoustic phenomena. It can be obtained from the Acoustical Society of America at http://asa.aip.org/discs.html for $26 + shipping.

Ben Johnson, *Music for piano*, played by Phillip Bush, Koch International Classics CD #7369. Pieces for piano in a microtonal just scale.

Enid Katahn, *Beethoven in the Temperaments* (Gasparo GSCD-332, 1997). Katahn plays Beethoven's Sonatas Op. 13, *Pathétique* and Op. 14 Nr. 1 using the Prinz temperament, and Sonatas Op. 27 Nr. 2, *Moonlight* and Op. 53 *Waldstein* in Thomas Young's temperament. The instrument is a modern Steinway concert grand rather than a period instrument. The tuning and liner notes are by Edward Foote.

Enid Katahn and Edward Foote have also brought out a recording, *Six degrees of tonality* (Gasparo GSCD-344, 2000). This begins with Scarlatti's Sonata K. 96 in quarter comma meantone, followed by Mozart's *Fantasie* Kv. 397 in Prelleur temperament, a Haydn sonata in Kirnberger III, a Beethoven sonata in Young temperament, Chopin's *Fantaisie-Impromptu* in DeMorgan temperament, and Grieg's *Glochengeläute* in Coleman 11 temperament. Finally, and in many ways the most interesting part of this recording, the Mozart *Fantasie* is played in quarter comma meantone, Prelleur temperament and equal temperament in succession, which allows a very direct comparison to be made. Unfortunately, the tempi are slightly different, which makes this recording not very useful for a blind test.

Bernard Lagacé has recorded a CD of music of various composers on the C. B. Fisk organ at Wellesley College, Massachusetts, USA, tuned in quarter comma meantone temperament. This recording is available from Titanic Records Ti-207, 1991.

Guillaume de Machaut (1300–1377), *Messe de Notre Dame* and other works. The Hilliard Ensemble, Hyperíon, 1989, CDA66358. This recording is sung in Pythagorean intonation throughout. The mass alternates polyphonic with monophonic sections. The double leading-note cadences at the end of each polyphonic section are particularly striking in Pythagorean intonation. Track 19 of this recording is *Ma fin est mon commencement* (My end is my beginning). This is an example of retrograde canon, meaning that it exhibits temporal reflectional symmetry (see §9.1).

Mathews and Pierce, *Current directions in computer music research* [**74**] comes with a companion CD containing numerous examples; note that track 76 is erroneous, cf. Pierce [**94**], page 257.

Microtonal works, Mode CD #18, contains microtonal works of Joan la Barbara, John Cage, Dean Drummond and Harry Partch.

Edward Parmentier, *Seventeenth Century French Harpsichord Music*, Wildboar, 1985, WLBR 8502. This collection contains pieces by Johann Jakob Froberger, Louis Couperin, Jacques Champion de Chambonnières, and Jean-Henri d'Anglebert. The recording was made using a Keith Hill copy of a 1640 harpsichord by Joannes Couchet, tuned in $\frac{1}{3}$ comma meantone temperament.

Many of Harry Partch's compositions have been rereleased on CD by Composers Recordings Inc., 73 Spring Street, Suite 506, New York, NY 10012-5800. As a starting point, I would recommend *The Bewitched*, CRI CD 7001, originally released on Partch's own label, Gate 5. This piece makes extensive use of his 43 tone just scale, described in §6.1.

A number of Robert Rich's recordings are in some form of just scale. His basic scale is mostly 5-limit with a 7:5 tritone:

1:1, 16:15, 9:8, 6:5, 5:4, 4:3, 7:5, 3:2, 8:5, 5:3, 9:5, 15:8.

This appears throughout the CDs *Numena*, *Geometry*, *Rainforest*, and others. One of the nicest examples of this tuning is *The Raining Room* on the CD *Rainforest*, Hearts of Space HS11014-2. He also uses the 7-limit scale

1:1, 15:14, 9:8, 7:6, 5:4, 4:3, 7:5, 3:2, 14:9, 5:3, 7:4, 15:8.

This appears on *Sagrada Familia* on the CD *Gaudi*, Hearts of Space HS11028-2. See http://www.amoeba.com for a more complete discography of Robert Rich's work.

William Sethares, *Xentonality*, Music in 10-, 17- and 19-tet.

William Sethares, *Tuning, timbre, spectrum, scale* [**119**] comes with a CD full of examples.

Isao Tomita, *Pictures at an Exhibition* (Mussorgsky), BMG 60576-2-RG. This recording was made on analog synthesizers in 1974, and is remarkably sophisticated for that era.

Johann Gottfried Walther, *Organ Works*, Volumes 1 and 2, played by Craig Cramer on the organ of St. Bonifacius, Tröchtelborn, Germany. Naxos CD numbers 8.554316 and 8.554317. This organ was restored in Kellner's reconstruction of Bach's temperament, see §5.13. For more information about the organ (details are not given in the CD liner notes), see http://www.gdo.de/neurest/troechtelborn.html.

Aldert Winkelman, *Works by Mattheson, Couperin, and others.* Clavigram VRS 1735-2. This recording is hard to obtain. The pieces by Johann Mattheson, François Couperin, Johann Jakob Froberger, Joannes de Gruytters and Jacques Duphly are played on a harpsichord tuned to Werckmeister III. The pieces by Louis Couperin and Gottlieb Muffat are played on a spinet tuned in quarter comma meantone.

APPENDIX W

The wave equation

This appendix is a supplement to Section 3.6. Its purpose is to justify the method of separation of variables for the wave equation, and to explain why a drum has "enough" eigenvalues. The account of the solution of the wave equation given here is deliberately much more compressed than the account usually given in books on partial differential equations, to emphasize the shape of the reasoning rather than the more computational aspects usually emphasized. The level of mathematical sophistication needed to follow this appendix is rather greater than for the rest of the book, but it should be accessible to someone who has taken standard undergraduate courses in vector calculus, analysis and linear algebra.

We discuss solutions z of the two dimensional wave equation

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z, \tag{W.1}$$

on a closed, bounded domain Ω. For boundary conditions, we assume that z is identically zero on the boundary S (Dirichlet boundary conditions). Initial conditions are given by specifying the values of z and $\frac{\partial z}{\partial t}$ at $t = 0$.

Throughout this appendix, Ω is a closed, bounded, simply connected domain in $\mathbb{R}^2$ with piecewise twice continuously differentiable boundary S, such that the pieces of the boundary meet at nonzero interior angles. We write $\mathbf{x}$ for the position vector (x, y) on Ω, and $d\mathbf{x}$ for the element $dx\,dy$ of area on Ω. We write $\mathbf{n}$ for the outward normal vector to S, and $d\sigma$ denotes the element of length on S. With this notation, the divergence theorem states that if $f(\mathbf{x})$ is a continuously differentiable function on Ω then

$$\int_S f \,.\, \mathbf{n}\, d\sigma = \int_\Omega \nabla f\, d\mathbf{x}. \tag{W.2}$$

In order to solve the wave equation, we begin with a study of Laplace's equation

$$\nabla^2 \phi = 0$$

on Ω, with Dirichlet boundary conditions. In other words, the value of ϕ is given on the boundary S.

Green's identities

Let Ω be a closed bounded region with boundary S. Suppose that $f(\mathbf{x})$ and $g(\mathbf{x})$ are functions on Ω. Then we have

$$\nabla.(f\nabla g) = f\nabla^2 g + \nabla f\,.\,\nabla g. \tag{W.3}$$

If Ω is a closed bounded region with boundary S, then integrating over Ω and using the divergence theorem (W.2), we get Green's first identity.

THEOREM W.1 (Green's First Identity). *Let $f(\mathbf{x})$ be continuously differentiable, and $g(\mathbf{x})$ be twice continuously differentiable on Ω. Then*

$$\int_S (f\nabla g)\,.\,\mathbf{n}\,d\sigma = \int_\Omega (f\nabla^2 g + \nabla f\,.\,\nabla g)\,d\mathbf{x}. \tag{W.4}$$

Reversing the roles of f and g and subtracting gives Green's second identity.

THEOREM W.2 (Green's Second Identity). *Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be twice continuously differentiable on Ω. Then*

$$\int_S (f\nabla g - g\nabla f)\,.\,\mathbf{n}\,d\sigma = \int_\Omega (f\nabla^2 g - g\nabla^2 f)\,d\mathbf{x}. \tag{W.5}$$

Gauss' formula

We start with the function of two variables $\mathbf{x}$ and $\mathbf{x}'$ in Ω given by $z = \ln|\mathbf{x}-\mathbf{x}'|$. For functions of two variables, it makes sense to apply ∇ with respect to $\mathbf{x}$ keeping $\mathbf{x}'$ constant, or vice versa. These are analogs of partial differentiation. To distinguish between these two options, we write $\nabla_\mathbf{x}$ or $\nabla_{\mathbf{x}'}$.

An easy calculation in terms of coordinates shows that as long as $\mathbf{x} \neq \mathbf{x}'$, we have

$$\nabla_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'| = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2} \tag{W.6}$$

and

$$\nabla^2_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'| = 0. \tag{W.7}$$

For $\mathbf{x} = \mathbf{x}'$, the quantity $\nabla^2_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|$ doesn't make sense, because the logarithm isn't defined. But if we pretend that it is continuously differentiable, and integrate using the divergence theorem (W.2) we get

$$\int_\Omega \nabla^2_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|\,d\mathbf{x}' = \int_S \nabla_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|\,.\,\mathbf{n}'\,d\sigma' = -\int_S \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2}\,.\,\mathbf{n}'\,d\sigma', \tag{W.8}$$

where $\mathbf{n}'$ and σ' are with respect to $\mathbf{x}'$. The shape of the region Ω doesn't matter in this calculation, as long as $\mathbf{x}'$ is in the interior, because of equation (W.7). If we measure using $\mathbf{x}$ as the origin and make the region a unit disk centered at the origin, then the calculation reduces to $\int_S \mathbf{x}'.\mathbf{n}'\,d\sigma'$. But in this case $\mathbf{x}'$ and $\mathbf{n}'$ are unit vectors in the same direction, so $\mathbf{x}'.\mathbf{n}' = 1$. Since the circumference of the unit circle is 2π, the integral gives 2π,

$$\int_S \nabla_{\mathbf{x}'} \ln|\mathbf{x} - \mathbf{x}'| \,.\, \mathbf{n}' \, d\sigma' = 2\pi. \tag{W.9}$$

The interpretation of this calculation is that although $\ln|\mathbf{x} - \mathbf{x}'|$ is not differentiable with respect to $\mathbf{x}'$ at $\mathbf{x}' = \mathbf{x}$, we can think of $\nabla^2_{\mathbf{x}'} \ln|\mathbf{x} - \mathbf{x}'|$ as a distribution, in the sense in which we introduced the term in Section 2.15. We have to replace $\int_{-\infty}^{\infty}$ with $\int_\Omega$, so that the delta function $\delta(\mathbf{x})$ is defined to be zero for $\mathbf{x} \neq \mathbf{0}$, and $\int_\Omega \delta(\mathbf{x})\, d\mathbf{x} = 1$. In terms of this delta function, the above calculation can be expressed as saying that

$$\nabla^2_{\mathbf{x}'} \ln|\mathbf{x} - \mathbf{x}'| = 2\pi\delta(\mathbf{x} - \mathbf{x}'). \tag{W.10}$$

So far, we have assumed that $\mathbf{x}'$ is in the interior of Ω. For a point $\mathbf{x}'$ outside Ω, the integrand in equation (W.8) is zero so the integral is zero. If $\mathbf{x}'$ is on the boundary S, and it is a point where S is continuously differentiable, then instead of a circle, in the above calculation we have to integrate over a semicircle. So the integral is π instead of 2π. At a corner with angle θ, we are integrating over a sector of a circle with angle θ, so the integral is θ. So we define a function $p(\mathbf{x})$ on $\mathbb{R}^2$ by

$$p(\mathbf{x}) = \begin{cases} 2\pi & \text{if } \mathbf{x} \text{ is in the interior of } \Omega, \\ 0 & \text{if } \mathbf{x} \text{ is not in } \Omega, \\ \pi & \text{if } \mathbf{x} \text{ is a continuously differentiable point on } S, \\ \theta & \text{if } \mathbf{x} \text{ is a corner of } S \text{ with interior angle } \theta. \end{cases}$$

Then the extension of equation (W.9) to the plane is **Gauss' formula**

$$\int_S \nabla_{\mathbf{x}'} \ln|\mathbf{x} - \mathbf{x}'| \,.\, \mathbf{n}' \, d\sigma' = p(\mathbf{x}). \tag{W.11}$$

If $f(\mathbf{x})$ is any continuous function on Ω, then we have

$$\int_\Omega f(\mathbf{x}') \nabla^2_{\mathbf{x}'} \ln|\mathbf{x} - \mathbf{x}'| \, d\mathbf{x}' = p(\mathbf{x}) f(\mathbf{x}). \tag{W.12}$$

This is because the integrand is zero except near $\mathbf{x} = \mathbf{x}'$, so $f(\mathbf{x}')$ may as well be replaced by $f(\mathbf{x})$ and taken out of the integral before applying the divergence theorem.

Remark. The above calculation was performed in two dimensions. The corresponding calculation in three dimensions uses the function $1/|\mathbf{x} - \mathbf{x}'|$ instead of $\ln|\mathbf{x} - \mathbf{x}'|$. The unit circle is replaced by the unit sphere, of surface area 4π, and the analog of equation (W.9) is

$$\int_S \nabla_{\mathbf{x}'} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \,.\, \mathbf{n}' \, d\sigma' = 4\pi.$$

The definition of $h(\mathbf{x}, \mathbf{x}')$ and $G(\mathbf{x}, \mathbf{x}')$ below are adjusted accordingly.

Similarly, in n dimensions ($n \geq 3$), the corresponding formula is

$$\int_S \nabla_{\mathbf{x}'} \frac{1}{|\mathbf{x} - \mathbf{x}'|^{n-2}} \,.\, \mathbf{n}' \, d\sigma' = n(n-2)\alpha(n)$$

where $\alpha(n)$ denotes the $(n-1)$-dimensional volume of the surface of the n-dimensional sphere.

Green's functions

Equation (W.10) is an important property of the function $\ln|\mathbf{x}-\mathbf{x}'|$. But the main problem with this function is that it doesn't vanish on the boundary S of Ω. To remedy this, we adjust it as follows. Suppose that we can find a solution $h(\mathbf{x},\mathbf{x}')$ to Laplace's equation

$$\nabla^2_{\mathbf{x}'}h(\mathbf{x},\mathbf{x}')=0 \tag{W.13}$$

on Ω, with boundary conditions

$$h(\mathbf{x},\mathbf{x}')=\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x}'| \tag{W.14}$$

for $\mathbf{x}'$ on S. That is, we insist that $h(\mathbf{x},\mathbf{x}')$ is defined even when $\mathbf{x}=\mathbf{x}'$ (in the interior of Ω). Then the function

$$G(\mathbf{x},\mathbf{x}')=h(\mathbf{x},\mathbf{x}')-\frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x}'|$$

still satisfies

$$\nabla^2_{\mathbf{x}'}G(\mathbf{x},\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}') \tag{W.15}$$

for $\mathbf{x}'$ in the interior of Ω, but it now also satisfies $G(\mathbf{x},\mathbf{x}')=0$ for $\mathbf{x}'$ on S. The function $G(\mathbf{x},\mathbf{x}')$ defined this way is called the Green's function for the Laplace operator ∇^2.

LEMMA W.3. *The Green function, if it exists, satisfies the symmetry relation* $G(\mathbf{x},\mathbf{x}')=G(\mathbf{x}',\mathbf{x})$.

PROOF. Using Lemma W.10, we have

$$\begin{aligned}G(\mathbf{x},\mathbf{x}')&=\int_\Omega G(\mathbf{x},\mathbf{x}'')\delta(\mathbf{x}'-\mathbf{x}'')\,d\mathbf{x}''=\int_\Omega G(\mathbf{x},\mathbf{x}'')\nabla^2_{\mathbf{x}''}G(\mathbf{x}',\mathbf{x}'')\,d\mathbf{x}''\\&=\int_\Omega G(\mathbf{x}',\mathbf{x}'')\nabla^2_{\mathbf{x}''}G(\mathbf{x},\mathbf{x}'')\,d\mathbf{x}''=\int_\Omega G(\mathbf{x},\mathbf{x}'')\delta(\mathbf{x}'-\mathbf{x}'')\,d\mathbf{x}''=G(\mathbf{x}',\mathbf{x}).\end{aligned}$$

□

The construction of the Green's function $G(\mathbf{x},\mathbf{x}')$ depends on solving Laplace's equation (W.13) with boundary conditions (W.14). We do this using Fredholm theory.

Hilbert space

A *Hilbert space* V is a (usually infinite dimensional) complex vector space with inner product $\langle\ ,\ \rangle$ satisfying

(i) $\langle x,\lambda y_1+\mu y_2\rangle=\lambda\langle x,y_1\rangle+\mu\langle x,y_2\rangle$,

(ii) $\langle x,y\rangle=\overline{\langle y,x\rangle}$ (and in particular $\langle x,x\rangle$ is real), and

(iii) $\langle x,x\rangle\geq 0$, and $\langle x,x\rangle=0$ if and only if $x=0$,

(iv) Writing $|x|$ for $\sqrt{\langle x, x\rangle}$, the metric with distance function $|x-y|$ is complete. In other words, every Cauchy sequence has a limit.

For example, if D is a compact domain in $\mathbb{R}^n$ then the space $L^2(D)$ of square integrable functions on D is a Hilbert space, with inner product

$$\langle f, g\rangle = \int_\Omega \bar{f}\, g\, d\mathbf{x}.$$

In this example, the completeness is a standard fact from Lebesgue integration theory. In order to satisfy (iii), we stipulate that two functions are identified if they agree except on a set of measure zero. Of course, this never identifies two continuous functions.

LEMMA W.4 (Schwartz's inequality). *For vectors x and y in Hilbert space, we have $\langle x, y\rangle \leq |x||y|$.*

PROOF. Consider the quantity

$$\langle x-ty, x-ty\rangle = |x|^2 - 2t\langle x, y\rangle + t^2|y|^2 \geq 0.$$

Differentiating with respect to t, we see that this expression is minimized by setting $t = \langle x, y\rangle/|y|^2$. With this value of t, we get

$$|x|^2 - 2\langle x, y\rangle^2/|y|^2 + \langle x, y\rangle^2/|y|^2 \geq 0,$$

or $\langle x, y\rangle^2/|y|^2 \leq |x|^2$. □

Elements x and y satisfying $\langle x, y\rangle = 0$ are said to be *orthogonal.* If W is a subspace of V, we write $W^\perp$ for the subspace consisting of vectors v such that for all $w \in W$ we have $\langle v, w\rangle = 0$. If W is finite dimensional, then any vector v in V can be written in a unique way as $v = w + x$ with w in W and x in $W^\perp$, so that

$$V = W \oplus W^\perp.$$

If $\mathbf{K}$ is a linear operator on V, its *image* is

$$\mathrm{Im}\,(\mathbf{K}) = \{\mathbf{K}v,\ v \in V\}$$

and its *kernel* is

$$\mathrm{Ker}\,(\mathbf{K}) = \{v \in V \mid Kv = 0\}.$$

LEMMA W.5. *If $\mathbf{K}$ and $\mathbf{K}^*$ are adjoint linear operators on V (i.e., for all x and y, $\langle \mathbf{K}^*x, y\rangle = \langle x, \mathbf{K}y\rangle$) and the image of $\mathbf{K}$ is finite dimensional, then*

(i) $V = \mathrm{Im}\,\mathbf{K} \oplus \mathrm{Ker}\,\mathbf{K}^*$, and

(ii) $V = \mathrm{Im}\,\mathbf{K}^* \oplus \mathrm{Ker}\,\mathbf{K}$

are orthogonal direct sum decompositions of V, and

$$\dim \mathrm{Im}\,(\mathbf{K}) = \dim \mathrm{Im}\,(\mathbf{K}^*).$$

Proof. If $\mathbf{K}^*x \in \operatorname{Im}(\mathbf{K}^*)$ and $y \in \operatorname{Ker}(\mathbf{K})$ then

$$\langle \mathbf{K}^*x, y\rangle = \langle x, \mathbf{K}y\rangle = 0$$

so $\operatorname{Im}(\mathbf{K}^*) \perp \operatorname{Ker}(\mathbf{K})$. If $x \in \operatorname{Im}(\mathbf{K}^*) \cap \operatorname{Ker}(\mathbf{K})$ then $\langle x, x\rangle = 0$ and so $x = 0$. Thus

$$\operatorname{Im}(\mathbf{K}^*) \oplus \operatorname{Ker}(\mathbf{K}) \le V. \tag{W.16}$$

so we have

$$\dim \operatorname{Im}(\mathbf{K}) = \dim(V/\operatorname{Ker}(\mathbf{K})) \ge \dim \operatorname{Im}(\mathbf{K}^*), \tag{W.17}$$

with equality if and only if (W.16) is an equality. In particular, it follows that $\operatorname{Im}(\mathbf{K}^*)$ is also finite dimensional. So we may repeat the above argument with the roles of $\mathbf{K}$ and $\mathbf{K}^*$ reversed, so that

$$\operatorname{Im}(\mathbf{K}) \oplus \operatorname{Ker}(\mathbf{K}^*) \le V \tag{W.18}$$

and

$$\dim \operatorname{Im}(\mathbf{K}^*) \ge \dim \operatorname{Im}(\mathbf{K}) \tag{W.19}$$

with equality if and only if (W.18) is an equality. Comparing (W.17) with (W.19), we see that both must be equalities, so (W.16) and (W.18) are equalities. □

Lemma W.6. *If* $\mathbf{K}$ *and* $\mathbf{K}^*$ *are adjoint operators and* $\operatorname{Im}(\mathbf{K})$ *is finite dimensional then*

(i) $V = \operatorname{Im}(\mathbf{I} - \mathbf{K}) \oplus \operatorname{Ker}(\mathbf{I} - \mathbf{K}^*)$ *and*

(ii) $V = \operatorname{Im}(\mathbf{I} - \mathbf{K}^*) \oplus \operatorname{Ker}(\mathbf{I} - \mathbf{K})$

are orthogonal decompositions of V*, and* $\dim \operatorname{Im}(\mathbf{I} - \mathbf{K}) = \dim \operatorname{Im}(\mathbf{I} - \mathbf{K}^*)$ *is finite.*

Proof. By Lemma W.5, $\operatorname{Im}(\mathbf{K}^*)$ is finite dimensional, so $V_1 = \operatorname{Im}(\mathbf{K}) + \operatorname{Im}(\mathbf{K}^*) \le V$ is also finite dimensional. So $V = V_1 \oplus V_2$ where

$$V_2 = V_1^{\perp} = \operatorname{Ker}(\mathbf{K}) \cap \operatorname{Ker}(\mathbf{K}^*).$$

So $\mathbf{I} - \mathbf{K}$ and $\mathbf{I} - \mathbf{K}^*$ send V_1 into V_1 and act as the identity map on V_2. Applying Lemma W.5 with $\mathbf{I} - \mathbf{K}$ instead of $\mathbf{K}$ and V_1 in place of V, we see that V_1 decomposes in the way described in the lemma. Since $\mathbf{I} - \mathbf{K}$ and $\mathbf{I} - \mathbf{K}^*$ act as the identity on V_2, this just contributes another summand to $\operatorname{Im}(\mathbf{I} - \mathbf{K})$ and $\operatorname{Im}(\mathbf{I} - \mathbf{K}^*)$, so the decomposition holds for V. □

The Fredholm alternative

Now let V be the vector space $L^2(D)$ of Lebesgue square integrable functions on a compact domain D in $\mathbb{R}^n$. Suppose that $K(\mathbf{x}, \mathbf{x}')$ is a continuous complex valued function of two variables $\mathbf{x}$ and $\mathbf{x}'$ in D. We are interested in the operator $\mathbf{K}$ on $L^2(D)$ given by

$$\mathbf{K}\psi(\mathbf{x}) = \int_D \psi(\mathbf{x}')K(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}'. \tag{W.20}$$

Such an operator is called a *Fredholm operator*. Its adjoint is given by

$$\mathbf{K}^*\psi(\mathbf{x}) = \int_D \psi(\mathbf{x}')\overline{K(\mathbf{x}',\mathbf{x})}\,d\mathbf{x}'. \tag{W.21}$$

In general, the image of a Fredholm operator is not finite dimensional, so we can't apply Lemma W.6 directly. However, a function of the form $K(\mathbf{x},\mathbf{x}') = g(\mathbf{x})h(\mathbf{x}')$ gives rise to an operator $\mathbf{K}$ with one dimensional image spanned by $g(\mathbf{x})$. Any polynomial function of $\mathbf{x}$ and $\mathbf{x}'$ can be written as a finite sum of monomials, each of which has this form. So if $K(\mathbf{x},\mathbf{x}')$ is a polynomial function, we may apply Lemma W.6.

The Weierstrass approximation theorem states that any continuous function on a compact domain in $\mathbb{R}^n$ may be uniformly approximated by polynomial functions. Applying this to $K(\mathbf{x},\mathbf{x}')$ on $D \times D$, we may write $K = K_1 + K_2$ where K_1 is a polynomial function and K_2 satisfies $B < 1$, where B is defined by

$$B = \iint_D |K_2(\mathbf{x},\mathbf{x}')|^2\,d\mathbf{x}\,d\mathbf{x}'.$$

For any function $\psi(\mathbf{x})$ in $L^2(D)$, Schwartz's inequality (Lemma W.4) implies that

$$|\mathbf{K}_2\psi(\mathbf{x})|^2 \le \langle\psi,\psi\rangle \int_D |K_2(\mathbf{x},\mathbf{x}')|^2\,d\mathbf{x}'.$$

Integrating with respect to $\mathbf{x}$ gives

$$\langle\mathbf{K}_2\psi,\mathbf{K}_2\psi\rangle \le B\langle\psi,\psi\rangle.$$

It follows by comparing with the geometric series

$$1 + B + B^2 + B^3 + \dots$$

that the sequence whose nth term is

$$\sum_{i=0}^{n} \mathbf{K}_2^i\psi$$

forms a Cauchy sequence in $L^2(D)$. Since $L^2(D)$ is complete, it follows that this Cauchy sequence has a limit; in other words, the infinite sum

$$\sum_{i=0}^{\infty} \mathbf{K}_2^i\psi = \psi + \mathbf{K}_2\psi + \mathbf{K}_2^2\psi + \mathbf{K}_2^3\psi + \cdots$$

converges in $L^2(D)$. It is now easy to check that the operator

$$\mathbf{I} + \mathbf{K}_2 + \mathbf{K}_2^2 + \mathbf{K}_2^3 + \dots$$

is an inverse to $\mathbf{I} - \mathbf{K}_2$ on $L^2(D)$. So we write $(\mathbf{I} - \mathbf{K}_2)^{-1}$ for this inverse.

Now we have

$$\mathbf{I} - \mathbf{K} = \mathbf{I} - (\mathbf{K}_1 + \mathbf{K}_2) = (\mathbf{I} - \mathbf{K}_2)(\mathbf{I} - (\mathbf{I} - \mathbf{K}_2)^{-1}\mathbf{K}_1).$$

The operator $(\mathbf{I} - \mathbf{K}_2)^{-1}\mathbf{K}_1$ has finite dimensional image, because $\mathbf{K}_1$ does. So Lemma W.6 enables us to write $L^2(D)$ as a direct sum of the image of

$\mathbf{I}-(\mathbf{I}-\mathbf{K}_2)^{-1}\mathbf{K}_1$ and the kernel of its adjoint. The invertibility of $\mathbf{I}-\mathbf{K}_2$ then gives us the following theorem, which is known as the *Fredholm alternative.*

THEOREM W.7. *With* $\mathbf{K}$ *and* $\mathbf{K}^*$ *defined by equations* (W.20) *and* (W.21)*, the kernels of* $\mathbf{I}-\mathbf{K}$ *and* $\mathbf{I}-\mathbf{K}^*$ *are finite dimensional, and have the same dimension. If this dimension is zero, then* $\mathbf{I}-\mathbf{K}$ *is invertible, so that the equation*

$$\psi - \mathbf{K}\psi = f$$

has a unique solution ψ *for any given element* f *of* $L^2(D)$. □

Solving Laplace's equation

In the section on Green's functions (page 352), we saw that if we can solve Laplace's equation (W.13) with boundary conditions (W.14) then we can construct a Green's function $G(\mathbf{x},\mathbf{x}')$ satisfying equation (W.15) and zero on the boundary S. In this section we use Fredholm theory to solve Laplace's equation

$$\nabla^2\phi(\mathbf{x}) = 0 \tag{W.22}$$

subject to twice continuously differentiable boundary conditions $\phi(\mathbf{x}) = f(\mathbf{x})$ on S.

We begin with uniqueness. We define the *potential energy* of a continuously differentiable function ϕ on Ω by

$$E = \rho c^2 \int_\Omega \nabla\phi \,.\, \nabla\phi \, d\mathbf{x}.$$

So $E \geq 0$, and if $E = 0$ then $\nabla\phi = 0$, so that ϕ is constant. If ϕ_1 and ϕ_2 are solutions of (W.22) satisfying the same boundary conditions, then $\phi = \phi_1 - \phi_2$ satisfies (W.22) and is zero on the boundary. By Green's first identity (W.4) with $f = g = \phi$, we see that we have $E = 0$, so ϕ is constant; since $\phi = 0$ on the boundary, this constant is zero. We conclude that if a solution to Laplace's equation (W.22) with given values on the boundary exists, then it is unique.

The same method can also be used for solutions of Laplace's equation (W.22) for the unbounded region Ω' obtained by removing the interior of Ω from $\mathbb{R}^2$, but we need to be careful about the behavior of ϕ as $\mathbf{x}$ goes off to infinity. The point is that we need to apply Green's first identity (W.4) for a region with a hole, bounded by S and a large circle S' of radius R surrounding Ω, and then let $R \to \infty$. The extra term we get from the second boundary component is $\int_{S'} \phi \nabla\phi \,.\, (\frac{\mathbf{x}}{R})\, d\sigma$, because the unit normal vector is $\mathbf{x}/R$. The length of S' is $2\pi R$, so we need to check that $2\pi R|\phi\nabla\phi \,.\, (\frac{\mathbf{x}}{R})| \to 0$ as $|\mathbf{x}| \to 0$. So we have proved the following theorem.

THEOREM W.8. (i) *If* $\nabla^2\phi = 0$ *has a solution on* Ω *with specified values on* S*, then the solution is unique.*

(ii) *If $\nabla^2\phi = 0$ has a solution on Ω' with specified values on S, and satisfying*

$$\lim_{|\mathbf{x}|\to\infty} |\phi\,\nabla\phi\,.\,\mathbf{x}| = 0$$

then that solution is unique. □

We now examine the question of existence of solutions. To this end, we look for solutions of equation (W.22) of the form

$$\phi(\mathbf{x}) = \int_S \psi(\mathbf{x}')\nabla_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|\,.\,\mathbf{n}'\,d\sigma', \tag{W.23}$$

with ψ a twice continuously differentiable function defined on S.

Any twice continuously differentiable function ψ on S can be extended to a twice continuously differentiable function on Ω, which we also denote by ψ. So we can use Green's first identity (W.4) to write

$$\phi(\mathbf{x}) = \int_\Omega (\psi(\mathbf{x}')\nabla^2_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'| + \nabla\psi(\mathbf{x}')\,.\,\nabla_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}'.$$

By equation (W.12), we have

$$\phi(\mathbf{x}) = p(\mathbf{x})\psi(\mathbf{x}) + \int_\Omega \nabla\psi(\mathbf{x}')\,.\,\nabla_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|\,d\mathbf{x}'. \tag{W.24}$$

In this formula, it can be shown using some elementary estimates that the integral term is continuous as $\mathbf{x}$ crosses the boundary S. It follows that $\phi(\mathbf{x})$ is discontinuous at S, so to solve Laplace's equation (W.22) using ϕ, we should use the limiting value at the boundary. Namely, for $\mathbf{x}_0$ in S and $\mathbf{x}$ in Ω but not in S, we have

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \phi(\mathbf{x}) = 2\pi\psi(\mathbf{x}_0) + \int_\Omega \nabla\psi(\mathbf{x}')\,.\,\nabla_{\mathbf{x}'} \ln|\mathbf{x}_0-\mathbf{x}'|\,d\mathbf{x}',$$

whereas except at the corners, the value of ϕ on S is given by

$$\phi(\mathbf{x}_0) = \pi\psi(\mathbf{x}_0) + \int_\Omega \nabla\psi(\mathbf{x}')\,.\,\nabla_{\mathbf{x}'} \ln|\mathbf{x}_0-\mathbf{x}'|\,d\mathbf{x}'.$$

So we have

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \phi(\mathbf{x}) = \phi(\mathbf{x}_0) + \pi\psi(\mathbf{x}_0).$$

In order to satisfy the boundary condition we want

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \phi(\mathbf{x}) = f(\mathbf{x}_0).$$

So we must solve the equation

$$\phi(\mathbf{x}) + \pi\psi(\mathbf{x}) = f(\mathbf{x}) \tag{W.25}$$

on S. Notice that the value of ψ at corners is irrelevant to the integral (W.23), so we just ignore the anomalous values of ϕ at corners and solve (W.25) for all $\mathbf{x}$ in S.

We rewrite equation (W.25) as

$$\psi(\mathbf{x}) + \frac{1}{\pi}\int_S \psi(\mathbf{x}')\nabla_{\mathbf{x}'}\ln|\mathbf{x}-\mathbf{x}'|.\mathbf{n}'\,d\sigma' = \frac{1}{\pi}f(\mathbf{x}). \tag{W.26}$$

Setting

$$K(\mathbf{x},\mathbf{x}') = -\frac{1}{\pi}\nabla_{\mathbf{x}'}\ln|\mathbf{x}-\mathbf{x}'|.\mathbf{n}' = \frac{(\mathbf{x}-\mathbf{x}').\mathbf{n}'}{\pi|\mathbf{x}-\mathbf{x}'|^2}$$

and $D = S$, we use equation (W.20) to obtain an operator $\mathbf{K}$ on $L^2(S)$ given by

$$\mathbf{K}\psi(\mathbf{x}) = -\frac{1}{\pi}\int_S \psi(\mathbf{x}')\nabla_{\mathbf{x}'}\ln|\mathbf{x}-\mathbf{x}'|.\mathbf{n}'\,d\sigma'.$$

Equation (W.26) then becomes

$$\psi - \mathbf{K}\psi = \frac{1}{\pi}f.$$

Applying Fredholm theory (Theorem W.7), we see that this equation always has a solution provided we can prove that the only solution of the equation

$$\psi - \mathbf{K}\psi = 0$$

is the zero function. So assume that ψ satisfies this equation, and define $\phi(\mathbf{x})$ by equation (W.23). Then $\nabla^2\phi = 0$, and $\phi(\mathbf{x}) \to 0$ as $\mathbf{x}$ approaches the boundary from inside Ω. So by Theorem W.8 (i), we have $\phi(\mathbf{x}) = 0$ for $\mathbf{x}$ in Ω. Similarly, we define $\phi(\mathbf{x})$ by equation (W.23) on Ω'. Then using equation (W.6) we find that $|\phi\,\nabla\phi\,.\,\mathbf{x}| \to 0$ as $R \to \infty$. So by Theorem W.8 (ii), we have $\phi(\mathbf{x}) = 0$ in Ω'. Now it follows from equation (W.24) that for a point $\mathbf{x}_0$ on S which is not a corner,

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}_0\\ \text{in }\Omega}} \phi(\mathbf{x}) - \lim_{\substack{\mathbf{x}\to\mathbf{x}_0\\ \text{in }\Omega'}} \phi(\mathbf{x}) = 2\pi\psi(\mathbf{x}_0).$$

It follows that $\psi(\mathbf{x}_0) = 0$. Since we were only interested in ψ at points which are not corners, this completes the proof that the only solution of $\psi - \mathbf{K}\psi = 0$ is $\psi = 0$. Applying Fredholm theory as mentioned above, this completes the proof of existence of solutions of Laplace's equation.

Conservation of energy

We are now ready to begin proving existence and uniqueness for solutions of the wave equation (W.1). The basic tool for proving uniqueness of solutions is the conservation of energy. We define the *energy* $E(t)$ of a continuously differentiable function z of $\mathbf{x}$ and t to be the quantity

$$E(t) = \rho\int_\Omega \left(\left(\frac{\partial z}{\partial t}\right)^2 + c^2\nabla z.\nabla z\right)d\mathbf{x}. \tag{W.27}$$

The two terms in this integral correspond to kinetic and potential energy respectively. Since $E(t)$ is obtained by integrating a sum of squares, it satisfies $E(t) \geq 0$. Furthermore, $E(t) = 0$ can only occur if the integrand is zero; namely if $\frac{\partial z}{\partial t}$ and ∇z are zero.

Suppose that z satisfies the wave equation (W.1). Differentiating, and using the divergence theorem (W.2), we get

$$\begin{aligned}\frac{dE}{dt} &= \int_\Omega \rho\left(2\frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial t^2} + 2c^2\nabla z \,.\, \frac{\partial \nabla z}{\partial t}\right) d\mathbf{x}\\ &= \int_\Omega \rho\left(2\frac{\partial z}{\partial t}c^2\nabla^2 z + 2c^2\nabla z \,.\, \nabla\frac{\partial z}{\partial t}\right) d\mathbf{x}\\ &= \int_\Omega 2\rho c^2 \nabla.\left(\frac{\partial z}{\partial t}\nabla z\right) d\mathbf{x}\\ &= \int_S 2\rho c^2\left(\frac{\partial z}{\partial t}\nabla z\right).\,\mathbf{n}\,d\sigma.\end{aligned}$$

Since $\frac{\partial z}{\partial t} = 0$ on S, we obtain

$$\frac{dE}{dt} = 0$$

so that E is a constant, independent of t. This is the statement of the conservation of energy for solutions of the wave equation.

Uniqueness of solutions

We now prove the uniqueness theorem for solutions to the wave equation. Suppose that z_1 and z_2 are solutions to the wave equation (W.1) on Ω, with the same initial conditions (i.e., the same values of z and $\frac{\partial z}{\partial t}$ for $t = 0$), and both vanishing on S. Then $z = z_1 - z_2$ satisfies the initial conditions $z = 0$ and $\frac{\partial z}{\partial t} = 0$ at $t = 0$. Equation (W.27) then shows that $E(0) = 0$. Conservation of energy implies that $E(t) = 0$ for all t. So $\frac{\partial z}{\partial t} = 0$ for all t, which implies that z is independent of t. Since it is zero at $t = 0$, we deduce that $z = 0$ for all values of t. Thus z_1 and z_2 are equal. It follows that there is at most one solution to the wave equation (W.1) for a given set of initial conditions for z and $\frac{\partial z}{\partial t}$.

It is less easy to prove existence of solutions. For this, we use the eigenvalue method. This will occupy the rest of the appendix.

Eigenvalues are nonnegative and real

We now prove that the eigenvalues of the Laplace operator ∇^2 are nonnegative and real—even if we allow f to take complex values (for real valued functions, ignore the bars in the proof of the lemma).

LEMMA W.9. *Let Ω be a closed bounded region. If f is a nonzero (complex valued) twice differentiable function satisfying $\nabla^2 f = -\lambda f$ in Ω and $f = 0$ on the boundary S of Ω, then λ is a nonnegative real number.*

PROOF. Let $\bar{f}$ be the complex conjugate of f. Then using Green's first identity (W.4), we have

$$\int_S (\bar{f}\,\nabla f)\,.\,\mathbf{n}\,d\sigma = \int_\Omega \nabla\bar{f}\,.\,\nabla f\,d\mathbf{x} + \int_\Omega \bar{f}(\nabla^2 f)\,d\mathbf{x}$$

$$= \int_\Omega |\nabla f|^2 \, d\mathbf{x} - \lambda \int_\Omega |f|^2 \, d\mathbf{x},$$

Since f is zero on S, the left hand side is zero. Since $\int_\Omega |f|^2 \, d\mathbf{x} > 0$ and $\int_\Omega |\nabla f|^2 \, d\mathbf{x} \geq 0$, this means that

$$\lambda = \frac{\int_\Omega |\nabla f|^2 \, d\mathbf{x}}{\int_\Omega |f|^2 \, d\mathbf{x}} \geq 0$$

so that λ is a nonnegative real number. This expression for λ is called *Rayleigh's quotient.* □

Orthogonality

The relationship between ∇^2 and the inner product for functions on Ω is expressed in the following lemma, which says that ∇^2 is *self-adjoint* with respect to the inner product, for functions vanishing on the boundary.

LEMMA W.10. *For twice continuously differentiable functions f and g on Ω vanishing on the boundary S, we have*

$$\langle f, \nabla^2 g\rangle = \langle \nabla^2 f, g\rangle.$$

PROOF. This follows from Green's second identity (W.5) (replacing f by $\bar{f}$) and the fact that $f(\mathbf{x})$ and $g(\mathbf{x})$ vanish on the boundary S. The left hand side of equation (W.5) is zero, while the right hand side is equal to $\langle f, \nabla^2 g\rangle - \langle \nabla^2 f, g\rangle$. □

This allows us to see easily why the eigenvalues of ∇^2 are real numbers (Lemma W.9). Namely if $\nabla^2 f = -\lambda f$, and $f(\mathbf{x}) = 0$ on the boundary S, then we have

$$\bar{\lambda}\langle f, f\rangle = \langle \lambda f, f\rangle = -\langle \nabla^2 f, f\rangle = -\langle f, \nabla^2 f\rangle = \langle f, \lambda f\rangle = \lambda\langle f, f\rangle.$$

Since $\langle f, f\rangle \neq 0$, we have $\lambda = \bar{\lambda}$. However, positivity is less easy to see from this point of view.

A similar argument shows that eigenfunctions with distinct eigenvalues are orthogonal, as in the following lemma.

LEMMA W.11. *Let f and g be Dirichlet eigenfunctions on Ω with eigenvalues λ and μ respectively. If $\lambda \neq \mu$ Then*

$$\langle f, g\rangle = 0.$$

PROOF. Using the fact that ∇^2 is self-adjoint (see Lemma W.10), we have

$$\lambda\langle f, g\rangle = \langle \nabla^2 f, g\rangle = \langle f, \nabla^2 g\rangle = \mu\langle f, g\rangle,$$

and so $(\lambda - \mu)\langle f, g\rangle = 0$. If $\lambda \neq \mu$, it follows that $\langle f, g\rangle = 0$. □

Inverting ∇^2

The key to understanding the eigenvalues and eigenfunctions of ∇^2 is to find an inverse $\mathbf{K}$ for the operator ∇^2 using Green's functions. The inverse is an integral operator with a wider domain of definition, and whose eigenvalues are the reciprocals of those for ∇^2. The operator $\mathbf{K}$ is an example of a *compact operator*, which is what makes the eigenvalue theory easier.

The construction of the inverse goes as follows. If $f(\mathbf{x})$ satisfies

$$\nabla^2 f(\mathbf{x}) = -\lambda f(\mathbf{x}) \tag{W.28}$$

on Ω and $f(\mathbf{x}) = 0$ on S, then we have

$$\begin{aligned} f(\mathbf{x}) &= \int_\Omega f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}' = \int_\Omega f(\mathbf{x}')\nabla^2 G(\mathbf{x},\mathbf{x}')\,d\mathbf{x}' \\ &= \int_\Omega G(\mathbf{x},\mathbf{x}')\nabla^2 f(\mathbf{x}')\,d\mathbf{x}' = -\lambda \int_\Omega f(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'. \end{aligned}$$

In particular, $f(\mathbf{x}) \neq 0$ implies $\lambda \neq 0$, so zero is not an eigenvalue of ∇^2.

We write $\mathbf{K}$ for the operator defined by

$$\mathbf{K}f(\mathbf{x}) = -\int_\Omega f(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'.$$

Then the above calculation shows that if $f(\mathbf{x})$ satisfies (W.28) then

$$\mathbf{K}f(\mathbf{x}) = \frac{1}{\lambda} f(\mathbf{x}).$$

So $f(\mathbf{x})$ is an eigenfunction of $\mathbf{K}$ with eigenvalue $1/\lambda$. Conversely, if $f(\mathbf{x})$ is an eigenfunction of $\mathbf{K}$ with *nonzero* eigenvalue μ, and f is twice continuously differentiable, then $f(\mathbf{x})$ is also an eigenfunction of ∇^2 with eigenvalue $\lambda = 1/\mu$.

Compact operators

Let V be a Hilbert space. We say that a sequence of elements $x_1, x_2, \dots$ of elements of V is *bounded* if there is some positive constant M such that all the x_i satisfy $|x_i| \leq M$. A continuous operator $\mathbf{K}$ on V is said to be *compact* if, given any bounded sequence $x_1, x_2, \dots$, the images $\mathbf{K}x_1, \mathbf{K}x_2, \dots$ has a convergent subsequence.

Example. If the image of $\mathbf{K}$ is finite dimensional then the Bolzano–Weierstrass theorem implies that $\mathbf{K}$ is compact. More generally, the Fredholm alternative can be expressed in terms of compact operators.

If $\mathbf{K}$ is compact and self-adjoint then there is an upper bound to the values of $\langle \mathbf{K}x, x\rangle$ as x runs over the elements of V satisfying $|x| = 1$. This is because otherwise, there would be a sequence $x_1, x_2, \dots$ such that $\langle \mathbf{K}x_i, x_i\rangle > i$, and then by Schwartz' lemma, $\langle \mathbf{K}x_i, \mathbf{K}x_i\rangle > i^2$, so that there could not exist a convergent subsequence; this would contradict the fact that $\mathbf{K}$ is compact. Writing U for the least upper bound of the values for $\langle \mathbf{K}x, x\rangle$ for $|x| = 1$, we can find a sequence $x_1, x_2, \dots$ of elements with $|x_i| = 1$, such that $\langle \mathbf{K}x_1, x_1\rangle, \langle \mathbf{K}x_2, x_2\rangle, \dots$ converges to U. Using Schwartz' lemma again, we have

$$\begin{aligned}\langle \mathbf{K}x_i - Ux_i, \mathbf{K}x_i - Ux_i\rangle &= \langle \mathbf{K}x_i, \mathbf{K}x_i\rangle - 2U\langle \mathbf{K}x_i, x_i\rangle + U^2 \\ &\le \langle \mathbf{K}x_i, x_i\rangle^2 - 2U\langle \mathbf{K}x_i, x_i\rangle + U^2 \\ &\le 2U^2 - 2U\langle \mathbf{K}x_i, x_i\rangle \\ &= 2U(U - \langle \mathbf{K}x_i, x_i\rangle) \to 0 \quad \text{as} \quad i \to \infty,\end{aligned}$$

and so $\mathbf{K}x_i - Ux_i \to 0$ as $i \to \infty$.

Since $\mathbf{K}$ is compact, we can replace $x_1, x_2, \dots$ by a subsequence with the property that $\mathbf{K}x_1, \mathbf{K}x_2, \dots$ converges. So $Ux_1, Ux_2, \dots$ converges, and provided $U \ne 0$, this implies that $x_1, x_2, \dots$ also converges. Setting $x = \lim_{i\to\infty} x_i$, the continuity of $\mathbf{K}$ implies that $\mathbf{K}x = \lim_{i\to\infty} \mathbf{K}x_i$, so we have

$$\mathbf{K}x = Ux.$$

In other words, x is an eigenvector of $\mathbf{K}$ with eigenvalue U. So if $U \ne 0$ then U is an eigenvalue of $\mathbf{K}$.

Eigenvalue stripping

In the last section, we saw a method for finding an eigenvalue and eigenvector for $\mathbf{K}$. Suppose that we have already found some eigenvalues $\mu_1, \dots, \mu_n$ and corresponding eigenvectors $\psi_1, \dots, \psi_n$ of $\mathbf{K}$, and we wish to find some more. The most convenient method is to form a new operator $\mathbf{K}_n$ whose eigenvalues and eigenvectors are the same as $\mathbf{K}$ except for the removal of the ones we have found. As a preliminary step, we make sure that if there are repeated eigenvalues, then the corresponding eigenvectors are orthogonal. This can be done using the Gram–Schmidt process of linear algebra. Then we define

$$K_n(\mathbf{x}, \mathbf{x}') = K(\mathbf{x}, \mathbf{x}') - \sum_{i=1}^{n} \frac{\psi_i(\mathbf{x})\overline{\psi_i(\mathbf{x}')}}{\mu_i}.$$

Then we define $\mathbf{K}_n$ by

$$\mathbf{K}_n\psi = \int_\Omega K_n(\mathbf{x}, \mathbf{x}')\psi(\mathbf{x}')\, d\mathbf{x}',$$

so that $\mathbf{K}_n$ takes value zero on $\psi_1, \dots, \psi_n$, and takes the same value as $\mathbf{K}$ on any function orthogonal to $\psi_1, \dots, \psi_n$.

To be continued. . .

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12. Pierre Buser and Michel Imbert, *Audition*, MIT Press, 1992. 406 pages, in print. ISBN 0262023318.

13. Murray Campbell and Clive Greated, *The musician's guide to acoustics*, OUP, 1986, reprinted 1998. 613 pages, in print. ISBN 0198165056.

    A well written account of acoustics for the musician, requiring essentially no mathematical background. This book is in print in the UK but not the USA, so try for example www.amazon.co.uk.

14. Peter Castine, *Set theory objects: abstractions for computer-aided analysis and composition of serial and atonal music*, European University Studies, vol. 36, Peter Lang Publishing, 1994. In print. ISBN 3631478976.

15. John Chowning and David Bristow, *FM Theory and applications*, Yamaha Music Foundation, 1986. 195 pages, out of print. ISBN 4636174828.

    This short book came out a couple of years after the Yamaha DX7 became available. It describes FM synthesis using the DX7 for the details of the examples. Note that the graphs for the Bessel functions J_{10} and J_{11} on page 176 have apparently been accidentally interchanged.

16. David Colton, *Partial differential equations, an introduction*, Random House, 1988. 308 pages. ISBN 0394358279.

    This book contains a good treatment of the solution of the wave equation, complete with the background from functional analysis necessary for the proof. The existence of a complete set of eigenfunctions can be found on page 233. A C^2 boundary is assumed, but only in order to solve Laplace's equation with logarithmic boundary conditions, for the construction of Green's functions.

17. Perry R. Cook (ed.), *Music, cognition, and computerized sound. An introduction to psychoacoustics*, MIT Press, 1999. 392 pages, in print. ISBN 0262032562.

This is an excellent collection of essays on various aspects of psychoacoustics, written by some of the leading figures in the area of computer music. It comes with a CD full of sound examples.

Chapter headings: 1. Max Mathews, *The ear and how it works.* 2. Max Mathews, *The auditory brain.* 3. Roger Shepard, *Cognitive psychology and music.* 4. John Pierce, *Sound waves and sine waves.* 5. John Pierce, *Introduction to pitch perception.* 6. Max Mathews, *What is loudness?* 7. Max Mathews, *Introduction to timbre.* 8. John Pierce, *Hearing in time and space.* 9. Perry R. Cook, *Voice physics and neurology.* 10. Roger Shepard, *Stream segregation and ambiguity in audition.* 11. Perry R. Cook, *Formant peaks and spectral valleys.* 12. Perry R. Cook, *Articulation in speech and sound.* 13. Roger Shepard, *Pitch perception and measurement.* 14. John Pierce, *Consonance and scales.* 15. Roger Shepard, *Tonal structure and scales.* 16. Perry R. Cook, *Pitch, periodicity, and noise in the voice.* 17. Daniel J. Levitin, *Memory for musical attributes.* 18. Brent Gillespie, *Haptics.* 19. Brent Gillespie, *Haptics in manipulation.* 20. John Chowning, *Perceptual fusion and auditory perspective.* 21. John Pierce, *Passive nonlinearities in acoustics.* 22. John Pierce, *Storage and reproduction of music.* 23. Daniel J. Levitin, *Experimental design in psychoacoustic research.*

18. Deryck Cooke, *The language of music*, Oxford Univ. Press, 1959, reprinted in paperback, 1990. 289 pages, in print. ISBN 0198161808

This wonderful little book explains how the basic elements of musical expression communicate emotional content, both locally and on a larger scale. Highly recommended to anyone trying to understand how music works. Deryck Cooke is the person who orchestrated Mahler's tenth symphony, starting with Mahler's original draft. Take a listen to the excellent Bournemouth Symphony/Simon Rattle recording.

19. David H. Cope, *New directions in music*, Wm. C. Brown Publishers, Dubuque, Iowa, Fifth edition, 1989. Sixth edition, Waveland Press, 1998. 439 pages, in print. ISBN 0697033422.

An introduction to computers and the avant-garde in twentieth century music. Reads a bit like a scrapbook of ideas, pictures and music.

20. ______, *Computers and musical style*, Oxford University Press, 1991. 246 pages, in print. ISBN 019816274X.

David Cope is well known for his attempts to induce computers to compose music in the style of various famous composers such as Bach and Mozart. Unsurprisingly, the compositions are not an unqualified success, but the account of the process presented in this book is interesting.

21. ______, *Experiments in musical intelligence*, Computer Music and Digital Audio, vol. 12, A-R Editions, Madison, Wisconsin, 1996. 263 pages, in print. ISBN 0895793148/0895793377.

This book is a continuation of the project described in Cope's 1991 book, and comes with a CD-ROM full of examples for the Macintosh platform. I have not seen a copy, but from the review in Computer Music Journal 21 (3) (1997), it seems that the subject has progressed a good deal since [**20**] appeared in 1991. Artificial intelligence is still in a very primitive stage of development, and it will probably take another generation to produce a computational model which convincingly simulates one of the great composers. And then another generation after that, to compose with real originality. I think the real core of the problem is that when a human being composes, a hugely complex world view is invoked, which has taken a lifetime to accumulate. We'll end up teaching a baby computer how to talk before it grows up to be a real composer! But I'm glad that someone of the calibre of Cope is battling with these problems.

22. Lothar Cremer, *The physics of the violin*, MIT Press, 1984. 450 pages, in print. ISBN 0262031027.

    Translation of *Physik der Geige*, S. Hirzel Verlag, Stuttgart, 1981. This book is the standard reference on the physics of the violin. The technical standard is high and the writing is clear. Strongly recommended.

23. Malcolm J. Crocker (ed.), *Handbook of acoustics*, Wiley Interscience, 1998. 1461 pages, large format, in print. ISBN 047125293X.

    This enormous volume consists of 114 chapters by various experts, arranged in parts by subject. The subjects are: I. General linear acoustics, II. Nonlinear acoustics and cavitation, III. Aeroacoustics and atmospheric sound, IV. Underwater sound, V. Ultrasonics, quantum acoustics, and physical effects of sound, VI. Mechanical vibrations and shock, VII. Statistical methods in acoustics, VIII. Noise: its effects and control, IX. Architectural acoustics, X. Acoustic signal processing, XI. Physiological acoustics, XII. Psychological acoustics, XIII. Speech communication, XIV. Music and musical acoustics, XV. Acoustic measurements and instrumentation, XVI. Transducers. Part XIV is particularly relevant, and consists of an introduction by Thomas Rossing; *Stringed instruments: bowed*, by J. Woodhouse; *Woodwind instruments*, by Neville H. Fletcher; *Brass instruments*, by J. M. Bowsher; and *Pianos and other stringed keyboard instruments*, by Gabriel Weinreich.

24. Alain Daniélou, *Sémantique musicale. Essai de psycho-physiologie auditive*, Hermann, Paris, 1967. Reprinted 1978, 131 pages, in print. ISBN 270561334X.

    "Musical semantics. Essay on auditive psycho-physiology." This French book can be obtained from www.amazon.fr, for example.

25. ———, *Music and the power of sound*, Inner Traditions, Rochester, Vermont, 1995, revised from a 1943 publication. 172 pages, in print. ISBN 0892813369.

    This is a book about tuning and scales in different cultures, especially Chinese, Indian and Greek, and their effect on the emotional content of music. The original 1943 version was entitled *Introduction to the study of musical scales*, and published by the India Society, London. This original version has been reprinted by Munshiram Manoharlal Publishers Pvt. Ltd., New Delhi, 1999, 279 pages, in print. ISBN 8121509203.

26. Peter Desain and Henkjan Honig, *Music, mind and machine: Studies in computer music, music cognition, and artificial intelligence (Kennistechnologie)*, Thesis Publishers, 1992. 330 pages, in print. ISBN 9051701497.

27. Diana Deutsch (ed.), *The psychology of music*, Academic Press, 1982; 2nd ed., 1999. 807 pages, in print. ISBN 0122135652 (pbk), 0122135644 (hbk).

    This is an excellent collection of essays on various aspects of the psychology of music, by some of the leading figures in the field. The second edition has been completely revised to reflect recent progress in the subject. It is interesting to compare this collection of essays with Perry Cook's [**17**], which have a slightly different purpose.

    Chapter headings: 1. John R. Pierce, *The nature of musical sound.* 2. Manfred R. Schroeder, *Concert halls: from magic to number theory.* 3. Norman M. Weinberger, *Music and the auditory system.* 4. Rudolf Rasch and Reinier Plomp, *The perception of musical tones.* 5. Jean-Claude Risset and David L. Wessel, *Exploration of timbre by analysis and synthesis.* 6. Johan Sundberg, *The perception of singing.* 7. Edward M. Burns, *Intervals, scales and tuning.* 8. W. Dixon Ward, *Absolute pitch.* 9. Diana Deutsch, *Grouping mechanisms in music.* 10. Diana Deutsch, *The processing of pitch combinations.* 11. Jamshed J. Bharucha, *Neural nets, temporal composites, and tonality.* 12. Eugene Narmour, *Hierarchical expectation and musical style.* 13. Eric F. Clarke, *Rhythm and timing in music.* 14. Alf Gabrielson, *The performance of music.* 15. W. Jay Dowling, *The development of music perception and cognition.* 16. Rosamund Shuter-Dyson, *Musical ability.* 17. Oscar S. M. Marin and David W. Perry, *Neurological aspects of music perception and performance*. 18. Edward C. Carterette and Roger A. Kendall, *Comparative music perception and cognition*.

28. B. Chaitanya Deva, *The music of India: A scientific study*, Munshiram Manoharlal Publishers Pvt. Ltd., 1981. 278 pages, out of print.
29. Dominique Devie, *Le tempérament musical: philosophie, histoire, théorie et practique*, Société de musicologie du Languedoc Béziers, 1990. 540 pages, out of print. ISBN 2905400528.

    "Musical temperament: philosophy, history, theory and practise." This French book is an extensive discussion of scales and temperaments, with a great deal of historical information and philosophical discussion.
30. Charles Dodge and Thomas A. Jerse, *Computer music: synthesis, composition, and performance*, Simon & Schuster, Second ed., 1997. 453 pages, in print. ISBN 0028646827 (pbk), 002873100X (hbk).
31. W. Jay Dowling and Dane L. Harwood, *Music cognition*, Academic Press Series in Cognition and Perception, 1986. 258 pages. ISBN 0122214307.
32. William C. Elmore and Mark A. Heald, *Physics of waves*, McGraw-Hill, 1969. Reprinted by Dover, 1985. 477 pages, in print. ISBN 0486649261.

    This book contains a useful discussion of waves on strings, rods and membranes.
33. Laurent Fichet, *Les théories scientifiques de la musique aux XIX^e et XX^e siècles*, Librairie J. Vrin, 1996. 382 pages, in print. ISBN 2711642844.

    "Nineteenth and twentieth century scientific theories of music." This French book may be obtained from www.amazon.fr, for example.
34. Neville H. Fletcher and Thomas D. Rossing, *The physics of musical instruments*, Springer-Verlag, Berlin/New York, 1991. ISBN 3540941517 (pbk), 3540969470 (hbk).

    This book is at a high technical level, and contains a wealth of interesting material. A difficult read, but worth the effort.
35. Allen Forte, *The structure of atonal music*, Yale Univ. Press, 1973. ISBN 0300021208.

    This book is about 12-tone music, and goes into a great deal of technical detail about the theory of pitch class sets, relations and complexes.
36. Steve De Furia and Joe Scacciaferro, *MIDI programmer's handbook*, M & T Publishing, Inc., 1989.
37. Trudi Hammel Garland and Charity Vaughan Kahn, *Math and music: harmonious connections*, Dale Seymore Publications, 1995. ISBN 0866518290.

    This book is aimed at high school level, and avoids technical material. It looks as though it would make good classroom material at the intended level, and it seems to be the only book on the market with this aim.
38. H. Genevois and Y. Orlarey, *Musique & mathématiques*, Aléas–Grame, 1997. 194 pages, in print. ISBN 2908016834.

    "Music and mathematics." A collection of essays in French on various aspects of the connections between music and mathematics, coming out of the Rencontres Musicales Pluridisciplinaires at Lyons, 1996. This book can be ordered from www.amazon.fr, for example.
39. Ben Gold and Nelson Morgan, *Speech and audio signal processing: processing and perception of speech and music*, Wiley & Sons, 2000. 537 pages, in print. ISBN 0471351547.

    The basic purpose of this book is to understand sound well enough to be able to perform speech recognition, but it contains a lot of material relevant to music recognition and synthesis. By some quirk of international pricing, the price of this book in the UK is about half what it is in the USA, so it may be worth your while checking out UK online bookstores such as amazon.co.uk or the UK branch of bol.com for this one.

40. Heinz Götze and Rudolf Wille (eds.), *Musik und Mathematik. Salzburger Musikgespräch 1984 unter Vorsitz von Herbert von Karajan*, Springer-Verlag, Berlin/New York, 1995. ISBN 3540154078

    "Music and mathematics. Musical dialogue, Salzburg 1984, under the direction of Herbert von Karajan." A collection of essays, mostly in german.

41. Penelope Gouk, *Music, science and natural magic in seventeenth-century England*, Yale University Press, New Haven, 1999. 308 pages, in print. ISBN 0300073836.

42. Karl F. Graff, *Wave motion in elastic solids*, Oxford University Press, 1975. Reprinted by Dover, 1991. ISBN 0486667456.

    This book contains a lot of information about wave motion in strings, bars and plates, relevant to Chapter 3.

43. Niall Griffith and Peter M. Todd (eds.), *Musical networks: parallel distributed perception and performance*, MIT Press, 1999. 350 pages, in print. ISBN 0262071819.

44. Donald E. Hall, *Musical acoustics*, Wadsworth Publishing Company, Belmont, California, 1980. ISBN 0534007589.

    This book has some good chapters on the physics of musical instruments, as well as briefer acounts of room acoustics and of tuning and temperament.

45. R. W. Hamming, *Digital filters*, Prentice Hall, 1989. Reprinted by Dover Publications. 296 pages, in print. ISBN 048665088X

    Hamming is one of the pioneers of twentieth century communications and coding theory. This book on digital filters is a classic.

46. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford University Press, Fifth edition, 1980. 426 pages, in print. ISBN 0198531710.

    This classic contains a good section on the theory of continued fractions, which may be used as a reference for the material presented in §6.2.

47. W. M. Hartmann, *Signals, sound and sensation*, Springer-Verlag, Berlin/New York, 1998. 647 pages, in print. ISBN 1563962837

    This book contains a very nice discussion of psychoacoustics, Fourier theory and digital signal processing, and the relationships between these subjects.

48. Hermann Helmholtz, *Die Lehre von den Tonempfindungen*, Longmans & Co., Fourth German edition, 1877. Translated by Alexander Ellis as *On the sensations of tone*, Dover, 1954 (and reprinted many times). 576 pages, in print. ISBN 0486607534.

    For anyone interested in scales and temperaments, or the history of acoustics and psychoacoustics, this book is an absolute gold mine. The appendices by the translator are also full of fascinating material. Strongly recommended.

49. Michael Hewitt, *The tonal Phoenix; a study of tonal progression through the prime numbers three, five and seven*, Verlag für systematische Musikwissenschaft GmbH, Bonn, 2000. 495 pages, in print. ISBN 3922626963.

    This German book (in English) should be available from www.amazon.de, but it doesn't yet seem to be listed.

50. Douglas R. Hofstadter, *Gödel, Escher, Bach*, Harvester Press, 1979. Reprinted by Basic Books, 1999. 777 pages, in print. ISBN 0465026567.

    A nice popularized account of the connections between mathematical logic, cognitive science, Escher's art and the music of J. S. Bach. A bit too longwinded to make a particularly good read, but fun for the occasional dip.

51. David M. Howard and James Angus, *Acoustics and psychoacoustics*, Focal Press, 1996. 365 pages, in print. ISBN 0240514289.

52. Hua, *Introduction to number theory*, Springer-Verlag, Berlin/New York, 1982. ISBN 3540108181.

This book contains a good section on continued fractions, which may be used as a supplement to §6.2. Be warned that the continued fraction for π given on page 252 of Hua is erronious. The correct continued fraction can be found here on page 162.

53. Stuart M. Isacoff, *Temperament: The idea that solved music's greatest riddle*, Knopf, 2001. 288 pages in small format, in print. ISBN 0375403558.

This is a chatty popularized account of the history of musical temperament. The style is very readable, and the information density is low.

54. Sir James Jeans, *Science & music*, Cambridge Univ. Press, 1937. Reprinted by Dover, 1968. 273 pages, in print. ISBN 0486619648.

Somewhat old fashioned, but still makes an interesting read.

55. Jeffrey Johnson, *Graph theoretical methods of abstract musical transformation*, Greenwood Publishing Group, 1997. 216 pages, in print. ISBN 0313301581.

56. Tom Johnson, *Self-similar melodies*, Editions 75, 75 rue de la Roquette, 75011 Paris, 1996. 291 pages, ring-bound, in print. ISBN 2907200011.

Tom Johnson is a minimalist composer, whose work uses mathematical techniques such as the theory of automata to assist in the compositional process. Copies of this book may be obtained by writing to: Two Eighteen Press, PO Box 218, Village Station, New York, NY 10014, USA.

57. Ian Johnston, *Measured tones: The interplay of physics and music*, Institute of Physics Publishing, Bristol and Philadelphia, 1989. Reprinted 1997. 408 pages, in print. ISBN 0852742363.

This very readable book is about acoustics and the physics of musical instruments, from a historical perspective, and with essentially no equations.

58. Owen H. Jorgensen, *Tuning*, Michigan State University Press, 1991. 798 pages, large format, out of print. ISBN 0870132903.

This enormous book is subtitled: "Containing The Perfection of Eighteenth-Century Temperament, The Lost Art of Nineteenth-Century Temperament, and The Science of Equal Temperament, Complete With Instructions for Aural and Electronic Tuning." It is a mixture of history of tunings and temperaments, and explicit tuning instructions for various temperaments. An interesting thread running through the book is a detailed argument to the effect that equal temperament was not commonplace until the twentieth century.

59. Lawrence E. Kinsler, Austin R. Frey, Alan B. Coppens, and James V. Sanders, *Fundamentals of acoustics*, John Wiley & Sons, Fourth edition, 2000. 548 pages, in print. ISBN 0471847895.

This is an excellent book on acoustics, and deservedly popular. The two original authors of the first (1950) edition were Kinsler and Frey, both now deceased. The book has gone through many print runs and editions. Coppens and Sanders have updated the book and added new material for the fourth edition. This is another book whose price in the UK is about half what it is in the USA, so it may be worth your while checking out UK online bookstores for this one.

60. T. W. Körner, *Fourier analysis*, Cambridge Univ. Press, 1988, reprinted 1990. 591 pages, in print. ISBN 0521389917.

This book makes great reading. There is a fair amount of high level mathematics, but also a number of sections of a more historical or narrative nature, and a wonderful sense of humor pervades the work. The account of the laying of the transatlantic cable in the nineteenth century and the technical problems associated with it is priceless. Several sections are devoted to the life of Fourier. There is also a companion volume entitled *Exercises for Fourier analysis*, ISBN 0521438497, in print.

61. Patricia Kruth and Henry Stobart (eds.), *Sound*, Cambridge Univ. Press, 2000. 235 pages, in print. ISBN 0521572096.

    A nice collection of nontechnical essays on the nature of sound. I particularly like Jonathan Ashmore's contribution. Contents: 1. Philip Peek, *Re-sounding Silences.* 2. Charles Taylor, *The Physics of Sound.* 3. Jonathan Ashmore, *Hearing.* 4. Peter Slater, *Sounds Natural: The Song of Birds.* 5. Peter Ladefoged, *The Sounds of Speech.* 6. Christopher Page, *Ancestral Voices.* 7. Brian Ferneyhough, *Shaping Sound.* 8. Steven Feld, *Sound Worlds.* 9. Michel Chion, *Audio-Vision and Sound.*

62. Albino Lanciani, *Mathématiques et musique. Les Labyrinthes de la phénoménologie*, Éditions Jérôme Millon, Grenoble, 2001. 275 pages, in print. ISBN 2841371131.

    "Mathematics and music. The labyrinths of phenomenology." This French book can be obtained from www.amazon.fr, for example. It is an extended essay based around Bach's *Musical Offering* and mathematical logic, among other subjects. There are some obvious parallels between this book and Hofstadter's [**50**].

63. J. Lattard, *Gammes et tempéraments musicaux*, Masson, Paris, 1988. 130 pages, in print. ISBN 2225812187.

    "Scales and musical temperaments." This French book can be obtained from www.amazon.fr, for example.

64. Marc Leman, *Music and schema theory: cognitive foundations of systematic musicology*, Springer Series on Information Science, vol. 31, Springer-Verlag, Berlin/New York, 1995. In print. ISBN 3540600213.

65. ———, *Music, Gestalt, and computing; studies in cognitive and systematic musicology*, Lecture Notes in Computer Science, vol. 1317, Springer-Verlag, Berlin/New York, 1997. 524 pages, in print. ISBN 3540635262.

    This book of conference proceedings comprises a collection of essays about the interactions between music, psychoacoustics, cognitive science and computer science. There is an accompanying CD of sound examples.

66. Ernő Lendvai, *Symmetries of music*, Kodály Institute, Kecskemét, 1993. 155 pages, in print. ISBN 9637295100.

    This book is a translation of a Hungarian book with the title *Szimmetria a zenében.* It seems to be quite hard to get hold of. I suggest going to the Kodály Institute web site at www.kodaly-inst.hu and emailing them.

67. David Lewin, *Generalized musical intervals and transformations*, Yale University Press, New Haven/London, 1987. ISBN 0300034938.

    This book discusses twelve tone music from a mathematical point of view, using some elementary group theory.

68. Carl E. Linderholm, *Mathematics made difficult*, Wolfe Publishing, Ltd., London, 1971. 207 pages, out of print.

    This book isn't relevant to the subject of the text, but is well worth digging out to pass a happy evening. The humor gets slightly heavy-handed at times, but this is balanced by some priceless moments.

69. Mark Lindley and Ronald Turner-Smith, *Mathematical models of musical scales*, Verlag für systematische Musikwissenschaft GmbH, Bonn, 1993. 308 pages, out of print. ISBN 3922626661.

70. Llewelyn S. Lloyd and Hugh Boyle, *Intervals, scales and temperaments*, Macdonald, London, 1963. 246 pages, out of print.

An extensive discussion of just intonation, meantone and equal temperament.

71. R. Duncan Luce, *Sound and hearing, a conceptual introduction*, Lawrence Erlbaum Associates, Inc., 1993. 322 pages, in print. ISBN 0805813896.

    The book is available with or without the CD of psychoacoustic examples, which is also available separately. Most of these examples are taken from *Auditory Demonstrations*, by Houtsma, Rossing and Wagenaars, see Appendix R.

72. Charles Madden, *Fractals in music—Introductory mathematics for musical analysis*, High Art Press, 1999. ISBN 0967172756.

    This book has a promising title, but both the mathematics and the musical examples could do with some improvement. There is certainly an interesting area here to be investigated, and maybe the real point of the book will be to make us more aware of the possibilities.

73. Max V. Mathews, *The technology of computer music*, MIT Press, 1969. 188 pages, out of print. ISBN 0262130505.

    This book appeared early in the game, and was at one stage a standard reference. Although much of the material is now outdated, it is still worth looking at for its description of the Music V computer music language, one of the antecedants of CSound.

74. Max V. Mathews and John R. Pierce, *Current directions in computer music research*, MIT Press, 1989. Reprinted 1991. 432 pages, in print. ISBN 0262132419.

    A nice collection of articles on computer music, including an article by Pierce describing the Bohlen–Pierce scale. There is a companion CD, see Appendix R.

75. W. A. Mathieu, *Harmonic experience*, Inner Traditions International, Rochester, Vermont, 1997. 563 pages, large format, in print. ISBN 0892815604.

    You would not guess it from the title, but this book is about the conceptual transition from just intonation to equal temperament, and the parallel development of harmonic vocabulary. The writing is down to earth and easy to understand.

76. Guerino Mazzola, *Gruppen und Kategorien in der Musik*, Heldermann-Verlag, Berlin, 1985. 205 pages, out of print. ISBN 3885382105.

    "Groups and categories in music." The next item, by the same author, is much easier to get hold of.

77. ———, *Geometrie der Töne: Elemente der Mathematischen Musiktheorie*, Birkhäuser, 1990. ISBN 3764323531. 364 pages, in print.

    "Geometry of tones: elements of mathematical music theory." This is a book in German about music and mathematics, almost completely disjoint in content from these course notes. The author was a graduate student under the direction of the mathematician Peter Gabriel in Zürich, and the influence is clear. I was rather surprised, for example, to see the appearance of Yoneda's lemma from category theory. This book can be ordered from www.amazon.de, for example.

78. Ernest G. McClain, *The myth of invariance: The origin of the gods, mathematics and music from the Ṛg Veda to Plato*, Nicolas-Hays, Inc., York Beach, Maine, 1976. Paperback edition, 1984. 216 pages, in print. ISBN 0892540125.

    A strange mixture of mysticism and theory of scales and temperaments. If you take this book too seriously, you will go completely insane.

79. Brian C. J. Moore, *Psychology of hearing*, Academic Press, 1997. ISBN 0125056273.

    A standard work on psychoacoustics. Highly recommended.

80. F. Richard Moore, *Elements of computer music*, Prentice Hall, 1990. 560 pages, out of print. ISBN 0132525526.

    A very readable work by an expert in the field. The book is written in terms of the computer music language CMusic, which was a precursor of CSound.

81. Joseph Morgan, *The physical basis of musical sounds*, Robert E. Krieger Publishing Company, Huntington, New York, 1980. 145 pages, in print. ISBN 0882756567.

82. Philip M. Morse and K. Uno Ingard, *Theoretical acoustics*, McGraw Hill, 1968. Reprinted with corrections by Princeton University Press, 1986, ISBN 0691084254 (hbk), 0691024014 (pbk).

 This book is the best textbook on acoustics that I have found, for an audience with a good mathematical background.

83. Bernard Mulgrew, Peter Grant, and John Thompson, *Digital signal processing*, Macmillan Press, 1999. 356 pages, in print. ISBN 0333745310.

 A number of books have recently appeared on the subject of digital signal processing. This is a good readable one.

84. Cornelius Johannes Nederveen, *Acoustical aspects of woodwind instruments*, Northern Illinois Press, 1998. ISBN 0875805779.

85. Erich Neuwirth, *Musical temperaments*, Springer-Verlag, Berlin/New York, 1997. 70 pages, in print. ISBN 3211830405.

 This very slim, overpriced volume explains the basics of scales and temperaments. It comes with a CD-ROM full of examples to go with the text.

86. Harry F. Olson, *Musical engineering*, McGraw Hill, 1952. Revised and enlarged version, Dover, 1967, with new title: *Music, physics and engineering*. ISBN 0486217698.

 This work was a classic in its time, although it is now somewhat outdated.

87. Jack Orbach, *Sound and music*, University press of America, 1999. 409 pages, in print. ISBN 0761813764.

88. Charles A. Padgham, *The well-tempered organ*, Positif Press, Oxford, 1986. ISBN 0906894131.

 This book is hard to get hold of, but has a wealth of information about the usage of temperaments in organs.

89. Harry Partch, *Genesis of a music*, Second edition, enlarged. Da Capo Press, New York, 1974 (hbk), 1979 (pbk). 518 pages, in print. ISBN 030680106X.

 Harry Partch is one of the twentieth century's most innovative experimental composers. This well written book explains the origins of his 43 tone scale, and its applications in his compositions, and puts it into historical context with some unusual insights. The book also contains descriptions and photos of many musical instruments invented and constructed by Partch using this scale.

90. George Perle, *Twelve-tone tonality*, University of California Press, 1977. Second edition, 1996. 256 pages, in print. ISBN 0520033876.

91. Hermann Pfrogner, *Lebendige Tonwelt*, Langen Müller, 1976. 680 pages, out of print. ISBN 3784415776.

 "Living world of tone." This German book contains a discussion of musical scales in India, China, Greece and Arabia, followed by a discussion of the development of western tonality, and then a third section on the music of Arnold Schönberg.

92. Dave Phillips, *Linux music and sound*, Linux Journal Press, 2000. 408 pages, in print. ISBN 1886411344

 This book describes a number of different music and sound programs for the Linux operating system. It comes with a CD-ROM containing the software described in the text, to the extent that it is freely distributable. A book like this quickly becomes out of date, but is nonetheless a useful guide to what is avaiable to the Linux user.

93. James O. Pickles, *An introduction to the physiology of hearing*, Academic Press, London/San Diego, second edition, 1988. Out of print. ISBN 0125547544 (pbk).

94. John Robinson Pierce, *The science of musical sound*, Scientific American Books, 1983; 2nd ed., W. H. Freeman & Co, 1992. 270 pages, in print. ISBN 0716760053.

A classic by an expert in the field. Well worth reading. The second edition has been updated and expanded.

95. Ken C. Pohlmann, *Principals of digital audio*, McGraw-Hill, fourth edition, 2000. 736 pages, in print. ISBN 0071348190.

This is a standard work on digital audio. The fourth edition has been brought completely up to date, with sections on the newest technologies.

96. Giovanni De Poli, Aldo Piccialli, and Curtis Roads (eds.), *Representations of musical signals*, MIT Press, 1991. 494 pages, in print. ISBN 0262041138.

A collection of fourteen essays by various experts in the field. Topics include granular synthesis, wavelets, physical modeling, user interfaces, artificial intelligence and adaptive neural networks.

97. Stephen Travis Pope (ed.), *The well-tempered object: Musical applications of object-oriented software technology*, MIT Press, 1991. 203 pages, in print. ISBN 0262161265.

An edited collection of articles from the Computer Music Journal on applications of object oriented programming to music technology.

98. Daniel R. Raichel, *The science and applications of acoustics*, Amer. Inst. of Physics, 2000. 598 pages, in print. ISBN 0387989072.

A general interdisciplinary textbook on modern acoustics, containing a discussion of musical instruments, as well as music and voice synthesis, and psychoacoustics.

99. Jean-Philippe Rameau, *Traité de l'harmonie*, Ballard, Paris, 1722. Reprinted as "Treatise on Harmony" in English translation by Dover, 1971. 444 pages, in print. ISBN 0486224619.

100. J. W. S. Rayleigh, *The theory of sound (2 vols)*, Second edition, Macmillan, 1896. Dover, 1945. 480/504 pages, in print. ISBN 0486602923/0486602931.

This book revolutionized the field when it came out. It is now mostly of historical interest, because the subject has advanced a great deal during the twentieth century.

101. Joan Reinthaler, *Mathematics and music: some intersections*, Mu Alpha Theta, 1990. 47 pages, out of print. ISBN 0940790084.

This slim volume examines various topics such as the Pythagorean scale, equal temperament, the shape of the grand piano, change ringing and symmetry in music.

102. Geza Révész, *Einführung in die Musikpsychologie*, Amsterdam, 1946. Translated by G. I. C. de Courcy as *Introduction to the psychology of music*, University of Oklahoma Press, 1954, and reprinted by Dover, 2001. 265 pages, in print. ISBN 048641678X.

This book contains an interesting discussion (pages 160–167) of the question of whether mathematicians are more musically gifted than exponents of other special branches and professions. The author gives evidence for a negative answer to this question, in sharp contrast with widely held views on the subject.

103. John S. Rigden, *Physics and the sound of music*, Wiley & Sons, 1977. 286 pages. ISBN 0471024333. Second edition, 1985. 368 pages, in print. ISBN 0471874124.

104. Curtis Roads, *The computer music tutorial*, MIT Press, 1996. 1234 pages, large format, in print. ISBN 0262181584 (hbk), 0262680823 (pbk).

This is a huge work by a renowned expert. It contains an excellent section on various methods of synthesis, but surprisingly, doesn't go far enough with technical aspects of the subject.

105. ———, *Microsound*, MIT Press, 2002. 392 pages, to appear in March 2002. ISBN 0262182157.

This book discusses sound particles and granular synthesis, and comes with a CD full of examples.

106. Curtis Roads, Stephen Travis Pope, Aldo Piccialli, and Giovanni De Poli (eds.), *Musical signal processing*, Swets & Zeitlinger Publishers, 1997. 477 pages, in print. ISBN 9026514824 (hbk), 9026514832 (pbk).

 A collection of articles by various authors, in four sections: I, Foundations of musical signal processing. II, Innovations in musical signal processing. III, Musical signal macrostructures. IV, Composition and musical signal processing.

107. Curtis Roads and John Strawn (eds.), *Foundations of computer music. Selected readings from Computer Music Journal*, MIT Press, 1985. ISBN 0262181142 (hbk), 0262680513 (pbk).

108. Curtis Roads, *The music machine. Selected readings from Computer Music Journal*, MIT Press, 1989. 725 pages. ISBN 0262680785.

109. Juan G. Roederer, *The physics and psychophysics of music*, Springer-Verlag, Berlin/New York, 1995. 219 pages, in print. ISBN 3540943668.

110. Thomas D. Rossing (ed.), *Acoustics of bells*, Van Nostrand Reinhold, 1984. Out of print. ISBN 0442278179.

111. ———, *The science of sound*, Addison-Wesley, Reading, Mass., Second edition, 1990. 686 pages, in print. ISBN 0201157276.

 A very nicely written book by an expert in the field, explaining sound, hearing, musical instruments, acoustics, and electronic music. Highly recommended. I understand that a new edition has just come out (November 2001), ISBN 0805385657.

112. ———, *Science of percussion instruments*, World Scientific, 2000. 208 pages, in print. ISBN 9810241585 (Hbk), 9810241593 (Pbk).

113. Thomas D. Rossing and Neville H. Fletcher (contributor), *Principles of vibration and sound*, Springer-Verlag, Berlin/New York, 1995. 247 pages, in print. ISBN 0387943048.

114. Joseph Rothstein, *MIDI, A comprehensive introduction*, Oxford Univ. Press, 1992. 226 pages, in print. ISBN 0198162936.

 Rothstein is one of the editors of the Computer Music Journal.

115. Heiner Ruland, *Expanding tonal awareness*, Rudolf Steiner Press, London, 1992. 187 pages, out of print. ISBN 1855841703.

 A somewhat ideosynchratic account of the history of scales and temperaments.

116. Joseph Schillinger, *The Schillinger system of musical composition, two volumes*, Carl Fischer, Inc, 1941. Reprinted by Da Capo Press, 1978. 878/? pages, out of print. ISBN 0306775522 and 0306775220.

 Schillinger uses many mathematical concepts in describing his theory of musical composition.

117. Albrecht Schneider, *Tonhöhe, Skala, Klang: Akustiche, tonometrische und psychoakustische Studien auf vergleichender Grundlage*, Verlag für systematische Musikwissenschaft, Bonn, 1997. 597 pages, in print. ISBN 3922626890.

 "Pitch, scale, timbre: Acoustic, tonometric and psychoacoustic studies on comparative foundations." This German book can be ordered from www.amazon.de, for example.

118. Günter Schnitzler, *Musik und Zahl*, Verlag für systematische Musikwissenschaft, 1976. 297 pages, out of print.

 "Music and number." A collection of essays on music and mathematics, in German, by a number of different authors.

119. William A. Sethares, *Tuning, timbre, spectrum, scale*, Springer-Verlag, Berlin/New York, 1998. 345 pages, in print. ISBN 354076173X.

 The basic thesis of this book is the idea, first put forward by John Pierce, that the harmonic spectrum or timbre of an instrument determines the most appropriate scales and temperaments to be used, and therefore we could start with a prescribed scale and design a harmonic spectrum for which it would be appropriate. The book comes with a CD of examples.

120. Ken Steiglitz, *A digital signal processing primer: with applications to digital audio and computer music*, Addison-Wesley, 1996. 314 pages, in print. ISBN 0805316841.

121. Reinhard Steinberg (ed.), *Music and the mind machine. The psychophysiology and psychopathology of the sense of music*, Springer-Verlag, Berlin/New York, 1995. Out of print. ISBN 3540585281.

122. Charles Taylor, *Exploring music: The science and technology of tones and tunes*, Institute of Physics Publishing, Bristol and Philadelphia, 1992. Reprinted 1994. 255 pages, in print. ISBN 0750302135.

123. Stan Tempelaars, *Signal processing, speech and music*, Swets & Zeitlinger Publishers, 1996. 360 pages, in print. ISBN 9026514816.

124. David Temperley, *The cognition of basic musical structures*, MIT Press, 2001. 360 pages, in print. ISBN 0262201348.

125. Martin Vogel, *Die Lehre von den Tonbeziehungen*, Verlag für systematische Musikwissenschaft, 1975. 480 pages, in print. ISBN 3922626092.

    "The study of tonal relationships." This German book is one of the standard works on scales and temperaments. It can be ordered from www.amazon.de, for example.

126. ———, *Anleitung zur harmonischen Analyse und zu reiner Intonation*, Verlag für systematische Musikwissenschaft, 1984. 210 pages, in print. ISBN 3992626246.

    "Introduction to harmonic analysis and just intonation." This book can be ordered from www.amazon.de, for example.

127. ———, *Die Naturseptime*, Verlag für systematische Musikwissenschaft, 1991. 507 pages, out of print. ISBN 3922626610.

    "The natural seventh." The title refers to the seventh harmonic 7:4, as a just musical interval.

128. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, 1922. Reprinted, 1996. 804 pages, in print. ISBN 0521483913.

    This encyclopedic tome contains more than you will ever want to know about Bessel functions. After all these years, it is still the standard work on the subject.

129. Scott R. Wilkinson, *Tuning in: Microtonality in electronic music*, Hal Leonard Books, Milwaukee, 1988. ISBN 0881886335.

    This is a nice short popularized account of the theory of scales and temperaments, designed for synthesists. Beware, though, that the tables have some inaccuracies. This applies especially to the values in cents given in the tables for the Werckmeister III and Vallotti-Young temperaments.

130. Fritz Winckel, *Music, sound and sensation, a modern exposition*, Dover, 1967. 189 pages, in print. ISBN 0486217647.

131. Iannis Xenakis, *Formalized music: Thought and mathematics in composition*, Indiana University Press, Bloomington/London, 1971. ISBN 0253323789. Pendragon revised edition (four new chapters and a new appendix), Pendragon Press, New York, 1992. 387 pages, in print. ISBN 0945193246. Paperback edition, 2001, ISBN 1576470792.

    Xenakis is one of the twentieth century's leading composers and theorists of music using aleatoric, or stochastic processes. This book is about the theory behind these processes. It is hard to read, and the mathematics is suspect in places. Nonetheless, the book is full of interesting ideas.

132. Joseph Yasser, *A theory of evolving tonality*, American Library of Musicology, Inc., 1932. 381 pages, out of print.

This book is a classic work on just intonation. Somebody should reprint it.

133. William A. Yost, *Fundamentals of hearing. An introduction*, Academic Press, San Diego, 1977. 326 pages, in print. ISBN 012772690X.

This is a nice, well written introduction to acoustics and psychoacoustics, written in textbook style, and very accessible.

134. Eberhard Zwicker and H. Fastl, *Psychoacoustics: facts and models*, Springer-Verlag, Berlin/New York, Second edition, 1999. 380 pages, in print. ISBN 3540650636.

Eberhard Zwicker was one of the great names in psychoacoustics. This book, written with his student Hugo Fastl, is an excellent introduction to psychoacoustics, and presents a modern account with a great deal of qualitative and quantitative information. The second edition has been updated by Fastl.

Mobile instrument, Arthur Frick

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