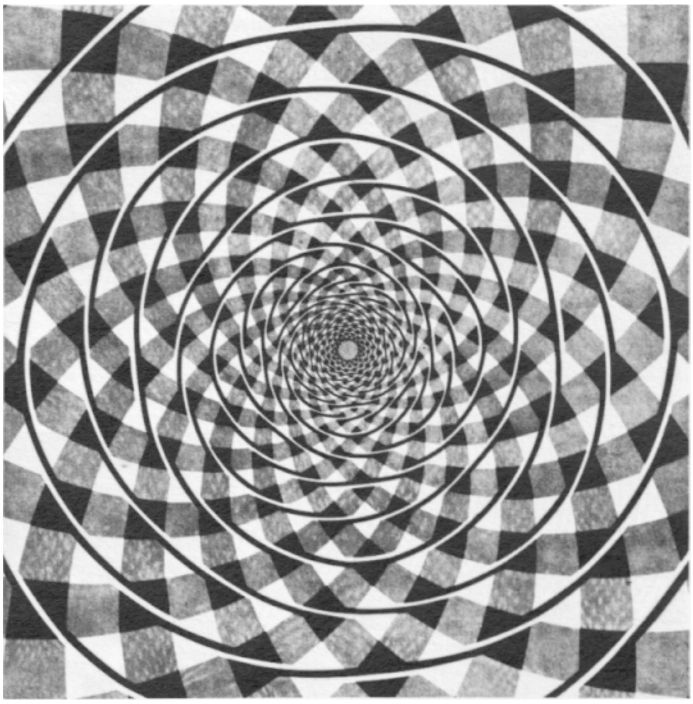
CHAPTER 5

Scales and temperaments: the fivefold way



"A perfect fourth? cries Tom. Whoe'er gave birth
To such a riddle, should stick or fiddle
On his numbskull ring until he sing
A scale of perfect fourths from end to end.
Was ever such a noddy? Why, almost everybody
Knows that not e'en one thing perfect is on earth—
How then can we expect to find a perfect fourth?"

(Musical World, 1863)[1]

5.1. Introduction

We saw in the last chapter that for notes played on conventional instruments, where partials occur at integer multiples of the fundamental frequency, intervals corresponding to frequency ratios expressable as a ratio of small integers are favored as consonant. In this chapter, we shall investigate

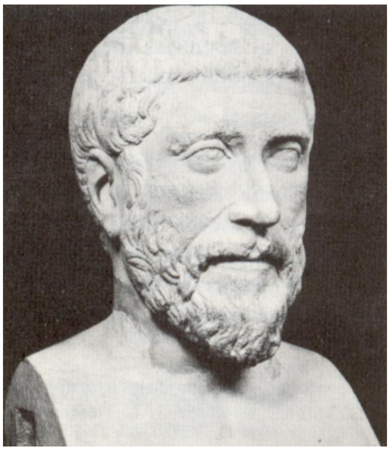
[1]Quoted in Nicolas Slonimsky's *Book of Musical Anecdotes*, reprinted by Schirmer, 1998, p. 299. The picture comes from J. Frazer, *A new visual illusion of direction*, British Journal of Psychology, 1908.

how this gives rise to the scales and temperaments found in the history of western music.

Scales based around the octave are categorized by Barbour [**5**] into five broad groups: Pythagorean, just, meantone, equal, and irregular systems. The title of this chapter refers to this fivefold classification, and to the interval of a perfect fifth as the starting point for the development of scales. We shall try to indicate where these five types of scales come from.

In Chapter 6 we shall discuss further developments in the theory of scales and temperaments, and in particular, we shall study some scales which is not based around the interval of an octave. These are the Bohlen–Pierce scale, and the scales of Wendy Carlos.

5.2. Pythagorean scale



Pythagoras

As we saw in Section 4.2, Pythagoras discovered that the interval of a perfect fifth, corresponding to a frequency ratio of 3:2, is particularly consonant. He concluded from this that a convincing scale could be constructed just by using the ratios 2:1 and 3:2. Greek music scales of the Pythagorean school are built using only these intervals, although other ratios of small integers played a role in classical Greek scales.

So for example, if we use the ratio 3:2 twice, we obtain an interval with ratio 9:4, which is a little over an octave. Reducing by an octave means halving this ratio to give 9:8. Using the ratio 3:2 again will then bring us to 27:16, and so on.

What we now refer to as the Pythagorean scale is the one obtained by tuning a sequence of fifths

fa–do–so–re–la–mi–ti.

This gives the following table of frequency ratios for a major scale:[2]

note	do	re	mi	fa	so	la	ti	do
ratio	1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1

In this system, the two intervals between successive notes are a *major tone* of 9:8 and a *minor semitone* of 256:243 or $2^8{:}3^5$. The semitone is not

[2]A Pythagorean minor scale can be constructed using ratios 32:27 for the minor third, 128:81 for the minor sixth and 16:9 for the minor seventh.

quite half of a tone in this system: two minor semitones give a frequency ratio of 2^{16}:3^{10} rather than 9:8. The Pythagoreans noticed that these were almost equal:

$$2^{16}/3^{10} = 1.10985715\ldots$$
$$9/8 = 1.125$$

In other words, the Pythagorean system is based on the fact that

$$2^{19} \approx 3^{12}, \quad \text{or} \quad 524288 \approx 531441,$$

so that going up 12 fifths and then down 7 octaves brings you back to almost exactly where you started. The fact that this is not quite so gives rise to the *Pythagorean comma* or *ditonic comma*, namely the frequency ratio

$$3^{12}/2^{19} = 1.013643265\ldots$$

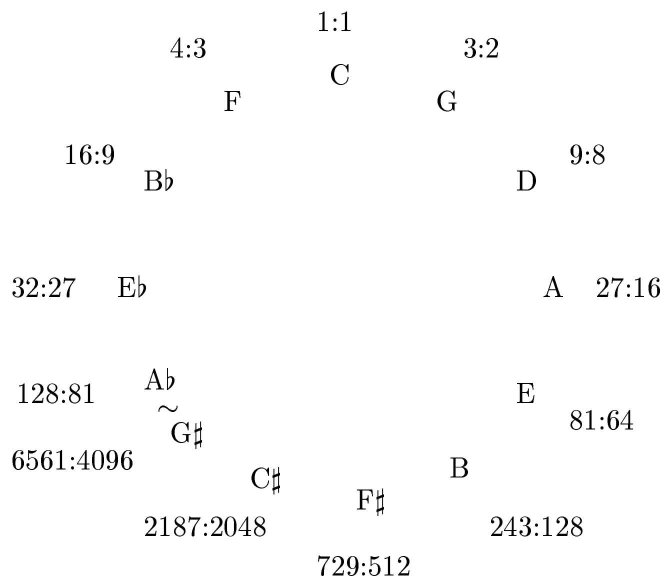
or just slightly more than one nineth of a whole tone.[3]

It seems likely that the Pythagoreans thought of musical intervals as involving the process of continued subtraction or *antanairesis*, which later formed the basis of Euclid's algorithm for finding the greatest common divisor of two integers. A 2:1 octave minus a 3:2 perfect fifth is a 4:3 perfect fourth. A perfect fifth minus a perfect fourth is a 9:8 Pythagorean whole tone. A perfect fourth minus two whole tones is a 256:243 Pythagorean minor semitone. It was called a *diesis* (difference), and was later referred to as a *limma* (remnant). A tone minus a diesis is a 2187:2048 Pythagorean major semitone, called an *apotomē*. An apotomē minus a diesis is a 531441:524288 Pythagorean comma.

5.3. The cycle of fifths

The Pythagorean tuning system can be extended to a twelve tone scale by tuning perfect fifths and octaves, at ratios of 3:2 and 2:1. This corresponds to tuning a "cycle of fifths" as in the following diagram:

[3]Musical intervals are measured logarithmically, so dividing a whole tone by nine really means taking the nineth root of the ratio, see §5.4.

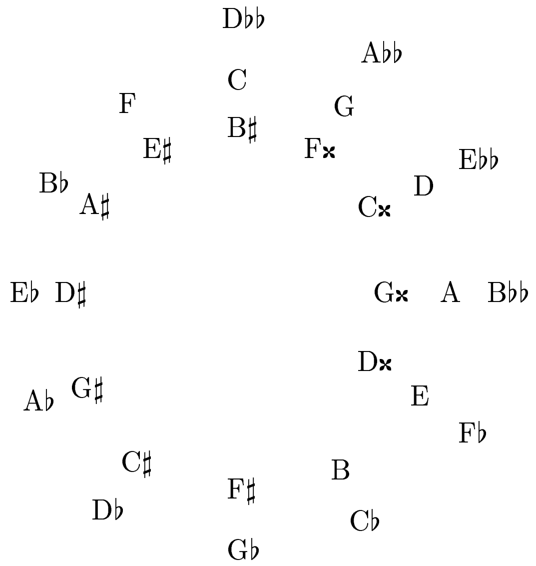


In this picture, the Pythagorean comma appears as the difference between the notes A♭ and G♯, or indeed any other *enharmonic* pair of notes:

$$\frac{6561/4096}{128/81} = \frac{3^{12}}{2^{19}} = \frac{531441}{524288}$$

In these days of equal temperament (see §5.14), we think of A♭ and G♯ as just two different names for the same note, so that there is really a *circle* of fifths. Other notes also have several names, for example the notes C and B♯, or the notes E♭♭, D and C𝄪.[4] In each case, the notes are said to be enharmonic, and in the Pythagorean system that means a difference of exactly one Pythagorean comma. So the Pythagorean system does not so much have a circle of fifths, more a sort of *spiral of fifths.*

[4] The symbol 𝄪 is used in music instead of ♯♯ to denote a double sharp.



So for example, going clockwise one complete revolution takes us from the note C to B♯, one Pythagorean comma higher. Going round the other way would take us to D♭♭, one Pythagorean comma lower. We shall see in Section §6.2 that the Pythagorean spiral never joins up. In other words, no two notes of this spiral are equal. The twelfth note is reasonably close, the 53rd is closer, and the 665th is very close indeed.

Exercises

1. What is the name of the note

(a) one Pythagorean comma lower than F,

(b) two Pythagorean commas higher than B,

(c) two Pythagorean commas lower than B?

Further listening: (See Appendix R)

Guillaume de Machaut, *Messe de Notre Dame*, Hilliard Ensemble, sung in Pythagorean intonation.

5.4. Cents

We should now explain the system of *cents*, first introduced by Alexander Ellis around 1875, for measuring frequency ratios. This is the system most often employed in the modern literature. This is a logarithmic scale[5] in which there are 1200 cents to the octave. Each whole tone on the modern

[5] See Appendix L for more on logarithms.

equal tempered scale (described below) is 200 cents, and each semitone is 100 cents. To convert from a frequency ratio of r:1 to cents, the value in cents is

$$1200\log_2(r) = 1200\ln(r)/\ln(2).$$

To convert an interval of n cents to a frequency ratio, the formula is

$$2^{\frac{n}{1200}} : 1.$$

In cents, the Pythagorean scale in the key of C major comes out as follows:

note	C	D	E	F	G	A	B	C
ratio	1:1	9:8	81:64	4:3	3:2	27:16	243:128	2:1
cents	0.000	203.910	407.820	498.045	701.955	905.865	1109.775	1200.000

In these notes, we shall usually give our scales in the key of C, and assign the note C a value of 0 cents. Everything else is measured in cents above the note C.

In France, rather than measuring intervals in cents, they use as their basic unit the *savart*, named after its proponent, the French physicist Félix Savart (1791–1841). In this system, a ratio of 10:1 is assigned a value of 1000 savarts. So a 2:1 octave is

$$1000\log_{10}(2) \approx 301.030 \text{ savarts.}$$

One savart corresponds to a frequency ratio of $10^{\frac{1}{1000}}$:1, and is equal to

$$\frac{1200}{1000\log_{10}(2)} = \frac{6}{5\log_{10}(2)} \approx 3.98631 \text{ cents.}$$

Exercises

1. Show that to three decimal places, the Pythagorean comma is equal to 23.460 cents. What is it in savarts?

2. Convert the frequency ratios for the vibrational modes of a drum, given in §3.5, into cents above the fundamental.

3. Assigning C the value of 0 cents, what is the value of the note E♭♭ in the Pythagorean scale?

5.5. Just intonation

Just intonation refers to any tuning system that uses small, whole numbered ratios between the frequencies in a scale. This is the natural way for the ear to hear harmony, and it's the foundation of classical music theory. The dominant Western tuning system - equal temperament - is merely a 200 year old compromise that made it easier to build mechanical keyboards. Equal temperament is a lot easier to use than JI, but I find it lacks expressiveness. It sounds dead and lifeless to me. As soon as I began working microtonally, I felt like I moved from black & white into color. I found that certain combinations of intervals moved me in a

> *deep physical way. Everything became clearer for me, more visceral and expressive. The trade-off is that I had to be a lot more careful with my compositions, for while I had many more interesting consonant intervals to choose from, I also had new kinds of dissonances to avoid. Just intonation also opened me up to a greater appreciation of non-Western music, which has clearly had a large impact on my music.*

Robert Rich (synthesist)
from his FAQ at www.amoeba.com

After the octave and the fifth, the next most interesting ratio is 4:3. If we follow a perfect fifth (ratio 3:2) by the ratio 4:3, we obtain a ratio of 4:2 or 2:1, which is an octave. So 4:3 is an octave minus a perfect fifth, or a perfect fourth. So this gives us nothing new. The next new interval is given by the ratio 5:4, which is the fifth harmonic brought down two octaves.

If we continue this way, we find that the series of harmonics of a note can be used to construct scales consisting of notes that are for the most part related by small integer ratios. Given the fundamental role of the octave, it is natural to take the harmonics of a note and move them down a number of octaves to place them all in the same octave. In this case, the ratios we obtain are:

1:1 for the first, second, fourth, eighth, etc. harmonic,

3:2 for the third, sixth, twelfth, etc. harmonic,

5:4 for the fifth, tenth, etc. harmonic,

7:4 for the seventh, fourteenth, etc. harmonic,

and so on.

As we have already indicated, the ratio of 3:2 (or 6:4) is a perfect fifth. The ratio of 5:4 is a more consonant major third than the Pythagorean one, since it is a ratio of smaller integers. So we now have a *just major triad* (do–mi–so) with frequency ratios 4:5:6. Most scales in the world incorporate the major triad in some form. In western music it is regarded as the fundamental building block on which chords and scales are built. Scales in which the frequency ratio 5:4 are included were first developed by Didymus in the first century B.C. and Ptolemy in the second century A.D. The difference between the Pythagorean major third 81:64 and the Ptolemy–Didymus major third 5:4 is a ratio of 81:80. This interval is variously called the *syntonic comma*, *comma of Didymus*, *Ptolemaic comma*, or *ordinary comma*. When we use the word *comma* without further qualification, we shall always be referring to the syntonic comma.

Just intonation in its most limited sense usually refers to the scales in which each of the major triads I, IV and V (i.e., C–E–G, F–A–C and G–B–D) is taken to have frequency ratios 4:5:6. Thus we obtain the following table of ratios for a just major scale:

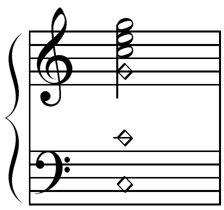
note	do	re	mi	fa	so	la	ti	do
ratio	1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1
cents	0.000	203.910	386.314	498.045	701.955	884.359	1088.269	1200.000

The *just major third* is therefore the name for the interval (do–mi) with ratio 5:4, and the *just major sixth* is the name for the interval (do–la) with ratio 5:3. The complementary intervals (mi–do) of 8:5 and (la–do) of 6:5 are called the *just minor sixth* and the *just minor third*.

The differences between various versions of just intonation mostly involve how to fill in the remaining notes of a twelve tone scale. In order to qualify as just intonation, each of these notes must differ by a whole number of commas from the Pythagorean value. In this context, the comma may be thought of as the result of going up four perfect fifths and then down two octaves and a just major third. In some versions of just intonation, a few of the notes of the above basic scale have also been altered by a comma.

5.6. Major and minor

In the last section, we saw that the basic building block of western music is the major triad, which in just intonation is built up out of the fourth, fifth and sixth notes in the harmonic series.

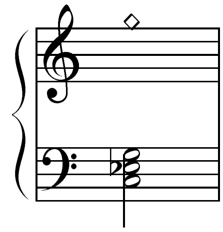


4:5:6

The minor triad is built by reversing the order of the two intervals, to obtain a chord of the form C–E♭–G. The ratios are 5:6 for the C–E♭ and 4:5 for the E♭–G. It seems futile to try to understand these as the harmonics of a common fundamental, because we would have to express the ratios as 10:12:15, making the fundamental 1/10 of the frequency of the C. It makes more sense to look at the harmonics of the notes in the triad, and to notice that all three notes have a common harmonic. Namely,

$$6 \times \mathrm{C} = 5 \times \mathrm{E}\flat = 4 \times \mathrm{G}.$$

So if we play a minor triad, if we listen carefully we can pick out this common harmonic, which is a G two octaves higher. For some subtle psychoacoustic reason, it sometimes sounds as though it's just one octave higher. It is probably the high common harmonic which causes us to associate minor chords with sadness.

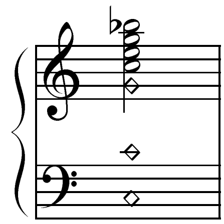


10:12:15

Another point of view regarding the minor triad is to view it as a *modification* of a major triad by slightly lowering the middle note to change the flavor. Music theory is full of modified chords, usually meaning that one of the notes in the chord has been raised or lowered by a semitone.

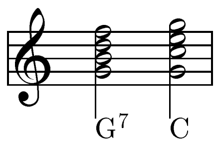
5.7. The dominant seventh

If we go as far as the seventh harmonic, we obtain a chord with ratios 4:5:6:7. This can be thought of as C–E–G–B♭, with a 7:4 B♭.



4:5:6:7

There is a closely related chord called the *dominant seventh* chord, in which the B♭ is the Pythagorean minor seventh, 16:9 higher than the C instead of 7:4. If we start this chord on G (3:2 above C) instead of C, we will obtain a chord G–B–D–F, and the F will be 4:3 above C. This chord has a strong tendency to resolve to C major, whereas the 4:5:6:7 version feels a lot more stable.



We shall have more to say about the seventh harmonic in §6.9.

Further Reading:

Martin Vogel, *Die Naturseptime* [**127**].

5.8. Commas and schismas

Recall from §5.2 that the Pythagorean comma is defined to be the difference between twelve perfect fifths and seven octaves, which gives a frequency ratio of 531441:524288, or a difference of about 23.460 cents. Recall also from §5.5 that the word *comma*, used without qualification, refers to the syntonic comma, which is a frequency ratio of 81:80. This is a difference of about 21.506 cents.

So the syntonic comma is very close in value to the Pythagorean comma, and the difference is called the *schisma*. This represents a frequency ratio of

$$\frac{531441/524288}{81/80} = \frac{32805}{32768},$$

or about 1.953 cents.

The *diaschisma*[6] is defined to be one schisma less than the comma, or a frequency ratio of 2048:2025. This may be viewed as the result of going up three octaves, and then down four perfect fifths and two just major thirds.

The *great diesis*[7] is one octave minus three just major thirds, or three syntonic commas minus a Pythagorean comma. This represents a frequency ratio of 128:125 or a difference of 41.059 cents.

The *septimal comma* is the amount by which the seventh harmonic 7:4 is flatter than the Pythagorean minor seventh 16:9. So it represents a ratio of $(16/9)(4/7) = 64/63$ or a difference of 27.264 cents.

Exercises

1. Show that to three decimal places, the (syntonic) comma is equal to 21.506 cents and the schisma is equal to 1.953 cents.

2. (G. B. Benedetti)[8] Show that if all the major thirds and sixths and the perfect fourths and fifths are taken to be just in the following harmonic progression, then the pitch will drift upwards by exactly one comma from the initial G to the final G.

[6]Historically, the Roman theorist Boethius (ca. 480–524 A.D.) attributes to Philolaus of Pythagoras' school a definition of schisma as one half of the Pythagorean comma and the diaschisma for one half of the diesis, but this does not correspond to the common modern usage of the terms.

[7]The word *diesis* in Greek means 'leak' or 'escape', and is based on the technique for playing the aulos, an ancient Greek wind instrument. To raise the pitch of a note on the aulos by a small amount, the finger on the lowest closed hole is raised slightly to allow a small amount of air to escape.

[8]G. B. Benedetti, *Diversarum speculationum*, Turin, 1585, page 282. The example is borrowed from Lindley and Turner-Smith [**69**], page 16.

$(\frac{3}{4} \times \frac{3}{2} \times \frac{3}{5} \times \frac{3}{2} = \frac{81}{80})$

This example was given by Benedetti in 1585 as an argument against Zarlino's[9] assertion (1558) that unaccompanied singers will tend to sing in just intonation. For a further discussion of the syntonic comma in the context of classical harmony, see §5.11.

5.9. Eitz's notation

Eitz[10] devised a system of notation, used in Barbour [**5**], which is convenient for describing scales based around the octave. His method is to start with the Pythagorean definitions of the notes and then put a superscript describing how many commas to adjust by. Each comma multiplies the frequency by a factor of 81/80.

As an example, the Pythagorean E, notated E^{0} in this system, is 81:64 of C, while E^{-1} is decreased by a factor of 81/80 from this value, to give the just ratio of 80:64 or 5:4.

In this notation, the basic scale for just intonation is given by

$$\mathrm{C}^{0} - \mathrm{D}^{0} - \mathrm{E}^{-1} - \mathrm{F}^{0} - \mathrm{G}^{0} - \mathrm{A}^{-1} - \mathrm{B}^{-1} - \mathrm{C}^{0}$$

A common variant of this notation is to use subscripts rather than superscripts, so that the just major third in the key of C is E_{-1} instead of E^{-1}.

An often used graphical device for denoting just scales, which we use here in combination with Eitz's notation, is as follows. The idea is to place notes in a triangular array in such a way that moving to the right increases the note by a 3:2 perfect fifth, moving up and a little to the right increases a note by a 5:4 just major third, and moving down and a little to the right increases a note by a 6:5 just minor third. So a just major 4:5:6 triad is denoted

$$\begin{array}{ccc} & \mathrm{E}^{-1} & \\ \mathrm{C}^{0} & & \mathrm{G}^{0}. \end{array}$$

[9] G. Zarlino, *Istitutione harmoniche*, Venice, 1558.

[10] Carl A. Eitz, *Das mathematisch-reine Tonsystem*, Leipzig, 1891. A similar notation was used earlier by Hauptmann and modified by Helmholtz [**48**].

A *just minor triad* has these intervals reversed:

$$\begin{array}{ccc} C^0 & & G^0 \\ & E\flat^{+1} & \end{array}$$

and the notes of the just major scale form the following array:

$$\begin{array}{ccccccc} & A^{-1} & & E^{-1} & & B^{-1} & \\ F^0 & & C^0 & & G^0 & & D^0 \end{array}$$

This method of forming an array is usually ascribed to Hugo Riemann,[11] although such arrays have been common in German music theory since the eighteenth century to denote key relationships and functional interpretation rather than frequency relationships.

It is sometimes useful to extend Eitz's notation to include other commas. Several different notations appear in the literature, and we choose to use p to denote the Pythagorean comma and z to denote the septimal comma. So for example $G\sharp^{-p}$ is the same note as $A\flat^{0}$, and the interval from C^0 to $B\flat^{-z}$ is a ratio of $\frac{16}{9} \times \frac{63}{64} = \frac{7}{4}$, namely the seventh harmonic.

Exercises

1. Show that in Eitz's notation, the example of §5.8, Exercise 2 looks like:

$$\begin{array}{cccccc} G^0 & D^0 & A^0 & & E^0 & \\ & & & C^{+1} & & G^{+1} \end{array}$$

2. (a) Show that the schisma is equal to the interval between $D\flat\flat^{+1}$ and C^0, and the interval between C^0 and $B\sharp^{-1}$.

(b) Show that the diaschisma is equal to the interval between C^0 and $D\flat\flat^{+2}$.

(c) Give an example to show that a sequence of six overlapping chords in just intonation can result in a drift of one diaschisma.

(d) How many overlapping chords in just intonation are needed in order to achieve a drift of one schisma?

[11]Hugo Riemann, *Ideen zu einer 'Lehre von den Tonvorstellungen,'* Jahrbuch der Musikbibliothek Peters, 1914–1915, page 20; *Grosse Kompositionslehre*, Berlin, W. Spemann, 1902, volume 1, page 479.

5.10. Examples of just scales

Using Eitz's notation, we list the examples of just intonation given in Barbour [**5**] for comparison. The dates and references have also been copied from that work.

Ramis' Monochord
(Bartolomeus Ramis de Pareja, *Musica Practica*, Bologna, 1482)

$$\begin{array}{l} \qquad\qquad\qquad\qquad\quad \mathrm{D}^{-1} \quad \mathrm{A}^{-1} \quad \mathrm{E}^{-1} \quad \mathrm{B}^{-1} \quad \mathrm{F}\sharp^{-1} \quad \mathrm{C}\sharp^{-1} \\ \mathrm{A}\flat^{0} \quad \mathrm{E}\flat^{0} \quad \mathrm{B}\flat^{0} \quad \mathrm{F}^{0} \quad \mathrm{C}^{0} \quad \mathrm{G}^{0} \end{array}$$

Erlangen Monochord
(anonymous German manuscript, second half of fifteenth century)

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathrm{E}^{-1} \quad \mathrm{B}^{-1} \\ \qquad \mathrm{G}\flat^{0} \qquad \mathrm{D}\flat^{0} \quad \mathrm{A}\flat^{0} \quad \mathrm{E}\flat^{0} \quad \mathrm{B}\flat^{0} \quad \mathrm{F}^{0} \quad \mathrm{C}^{0} \quad \mathrm{G}^{0} \\ \mathrm{E}\flat\flat^{+1} \qquad \mathrm{B}\flat\flat^{+1} \end{array}$$

Erlangen Monochord, Revised
The deviations of $\mathrm{E}\flat\flat^{+1}$ from D^{0}, and of $\mathrm{B}\flat\flat^{+1}$ from A^{0} are equal to the schisma, as are the deviations of $\mathrm{G}\flat^{0}$ from $\mathrm{F}\sharp^{-1}$, of $\mathrm{D}\flat^{0}$ from $\mathrm{C}\sharp^{-1}$, and of $\mathrm{A}\flat^{0}$ from $\mathrm{G}\sharp^{-1}$. So Barbour conjectures that the Erlangen monochord was really intended as

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad \mathrm{E}^{-1} \quad \mathrm{B}^{-1} \quad \mathrm{F}\sharp^{-1} \quad \mathrm{C}\sharp^{-1} \quad \mathrm{G}\sharp^{-1} \\ \mathrm{E}\flat^{0} \quad \mathrm{B}\flat^{0} \quad \mathrm{F}^{0} \quad \mathrm{C}^{0} \quad \mathrm{G}^{0} \quad \mathrm{D}^{0} \quad \mathrm{A}^{0} \end{array}$$

Fogliano's Monochord No. 1
(Lodovico Fogliano, *Musica theorica*, Venice, 1529)

$$\begin{array}{l} \qquad\qquad \mathrm{F}\sharp^{-2} \quad \mathrm{C}\sharp^{-2} \quad \mathrm{G}\sharp^{-2} \\ \qquad \mathrm{D}^{-1} \quad \mathrm{A}^{-1} \quad \mathrm{E}^{-1} \quad \mathrm{B}^{-1} \\ \mathrm{B}\flat^{0} \quad \mathrm{F}^{0} \quad \mathrm{C}^{0} \quad \mathrm{G}^{0} \\ \qquad\qquad\qquad \mathrm{E}\flat^{+1} \end{array}$$

Fogliano's Monochord No. 2

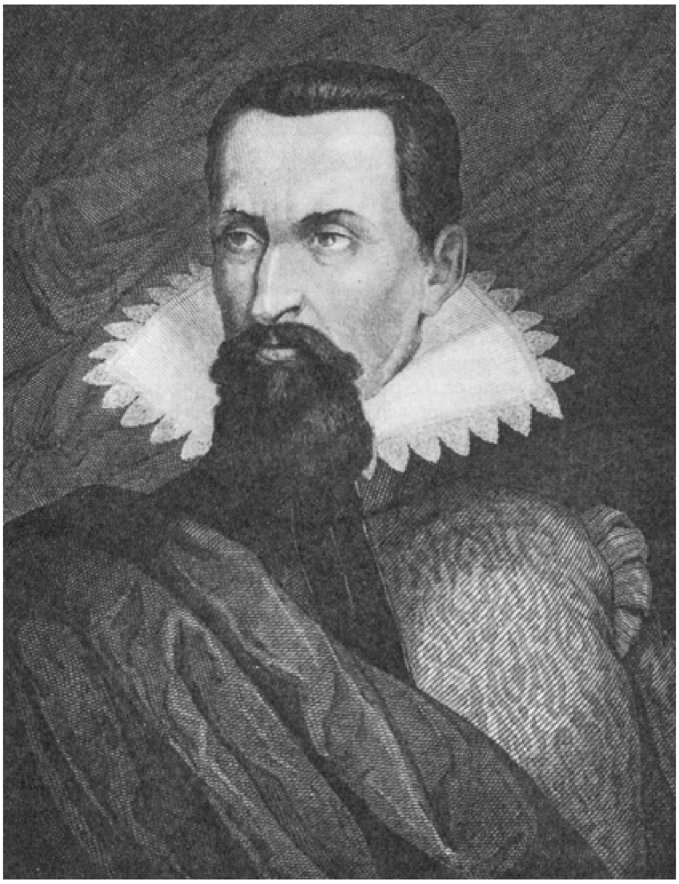
$$\begin{array}{l} \mathrm{F}\sharp^{-2} \quad \mathrm{C}\sharp^{-2} \quad \mathrm{G}\sharp^{-2} \\ \qquad \mathrm{A}^{-1} \quad \mathrm{E}^{-1} \quad \mathrm{B}^{-1} \\ \mathrm{F}^{0} \quad \mathrm{C}^{0} \quad \mathrm{G}^{0} \quad \mathrm{D}^{0} \\ \qquad\qquad \mathrm{E}\flat^{+1} \quad \mathrm{B}\flat^{+1} \end{array}$$

Agricola's Monochord
(Martin Agricola, *De monochordi dimensione*, in *Rudimenta musices*, Wittemberg, 1539)

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad \mathrm{F}\sharp^{-1} \quad \mathrm{C}\sharp^{-1} \quad \mathrm{G}\sharp^{-1} \quad \mathrm{D}\sharp^{-1} \\ \mathrm{B}\flat^{0} \quad \mathrm{F}^{0} \quad \mathrm{C}^{0} \quad \mathrm{G}^{0} \quad \mathrm{D}^{0} \quad \mathrm{A}^{0} \quad \mathrm{E}^{0} \quad \mathrm{B}^{0} \end{array}$$

De Caus's Monochord
(Salomon de Caus, *Les raisons des forces mouvantes avec diverses machines*, Francfort, 1615, Book 3, Problem III)

$$\begin{array}{l} \qquad\qquad \mathrm{F}\sharp^{-2} \quad \mathrm{C}\sharp^{-2} \quad \mathrm{G}\sharp^{-2} \quad \mathrm{D}\sharp^{-2} \\ \qquad \mathrm{D}^{-1} \quad \mathrm{A}^{-1} \quad \mathrm{E}^{-1} \quad \mathrm{B}^{-1} \\ \mathrm{B}\flat^{0} \quad \mathrm{F}^{0} \quad \mathrm{C}^{0} \quad \mathrm{G}^{0} \end{array}$$



Johannes Kepler (1571–1630)

Kepler's Monochord No. 1
(Johannes Kepler, *Harmonices mundi*, Augsburg, 1619)

$$\begin{array}{cccccccccc} & & E^{-1} & & B^{-1} & & F\sharp^{-1} & & C\sharp^{-1} & G\sharp^{-1} \\ F^{0} & C^{0} & & G^{0} & & D^{0} & & A^{0} & & \\ & & E\flat^{+1} & & B\flat^{+1} & & & & & \end{array}$$

(Note: the $G\sharp^{-1}$ is incorrectly labeled $G\sharp^{+1}$ in Barbour, but his numerical value in cents is correct)

Kepler's Monochord No. 2

$$\begin{array}{ccccccccc} & & E^{-1} & & B^{-1} & & F\sharp^{-1} & & C\sharp^{-1} \\ F^{0} & & C^{0} & & G^{0} & & D^{0} & & A^{0} \\ & A\flat^{+1} & & E\flat^{+1} & & B\flat^{+1} & & & \end{array}$$

Mersenne's Spinet Tuning No. 1
(Marin Mersenne, *Harmonie universelle*, Paris, 1636–7)[12]

$$\begin{array}{cccc}
D^{-1} & A^{-1} & E^{-1} & B^{-1} \\
B\flat^{0} & F^{0} & C^{0} & G^{0} \\
G\flat^{+1} & D\flat^{+1} & A\flat^{+1} & E\flat^{+1}
\end{array}$$

Mersenne's Spinet Tuning No. 2

$$\begin{array}{ccccc}
 & F\sharp^{-2} & C\sharp^{-2} & G\sharp^{-2} & D\sharp^{-2} \\
 & A^{-1} & E^{-1} & B^{-1} & \\
B\flat^{0} & F^{0} & C^{0} & G^{0} & D^{0}
\end{array}$$

Mersenne's Lute Tuning No. 1

$$\begin{array}{ccccc}
D^{-1} & A^{-1} & E^{-1} & B^{-1} & \\
F^{0} & C^{0} & G^{0} & & \\
G\flat^{+1} & D\flat^{+1} & A\flat^{+1} & E\flat^{+1} & B\flat^{+1}
\end{array}$$

Mersenne's Lute Tuning No. 2

$$\begin{array}{ccccc}
A^{-1} & E^{-1} & B^{-1} & & \\
F^{0} & C^{0} & G^{0} & D^{0} & \\
G\flat^{+1} & D\flat^{+1} & A\flat^{+1} & E\flat^{+1} & B\flat^{+1}
\end{array}$$

Marpurg's Monochord No. 1
(Friedrich Wilhelm Marpurg, *Versuch über die musikalische Temperatur*, Breslau, 1776)

$$\begin{array}{cccc}
C\sharp^{-2} & G\sharp^{-2} & & \\
A^{-1} & E^{-1} & B^{-1} & F\sharp^{-1} \\
F^{0} & C^{0} & G^{0} & D^{0} \\
E\flat^{+1} & B\flat^{+1} & &
\end{array}$$

[Marpurg's Monochord No. 2 is the same as Kepler's monochord]
Marpurg's Monochord No. 3

$$\begin{array}{cccccc}
C\sharp^{-2} & G\sharp^{-2} & & & & \\
E^{-1} & B^{-1} & F\sharp^{-1} & & & \\
B\flat^{0} & F^{0} & C^{0} & G^{0} & D^{0} & A^{0} \\
E\flat^{+1} & & & & &
\end{array}$$

Marpurg's Monochord No. 4

$$\begin{array}{cccc}
F\sharp^{-2} & C\sharp^{-2} & G\sharp^{-2} & \\
D^{-1} & A^{-1} & E^{-1} & B^{-1} \\
F^{0} & C^{0} & G^{0} & \\
E\flat^{+1} & B\flat^{+1} & &
\end{array}$$

[12] See page 81 for a picture of Mersenne.

Friedrich Wilhelm Marpurg
(1718–1795)

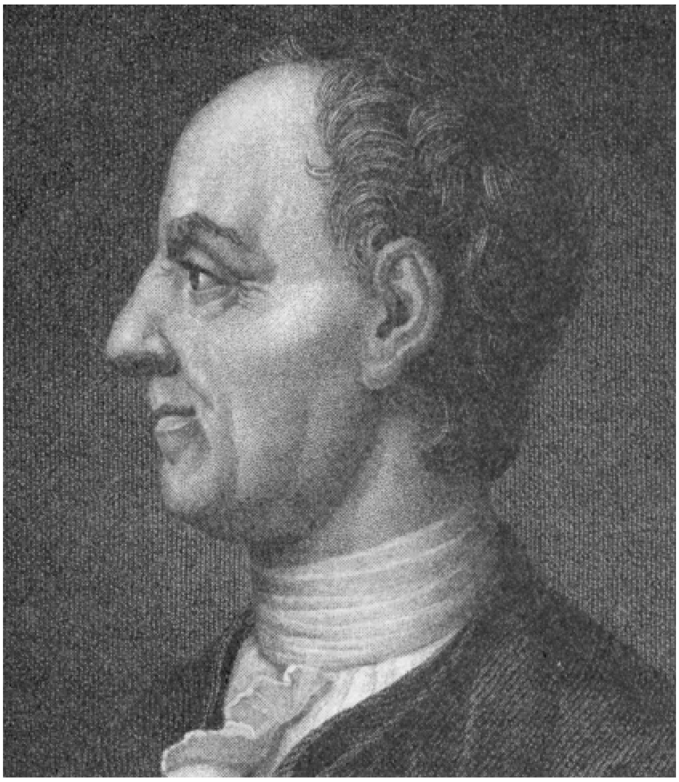
Malcolm's Monochord
(Alexander Malcolm, *A Treatise of Musick*, Edinburgh, 1721)

$$\begin{array}{ccccccccc} & & & A^{-1} & E^{-1} & B^{-1} & F\sharp^{-1} \\ B\flat^{0} & F^{0} & C^{0} & G^{0} & D^{0} \\ D\flat^{+1} & A\flat^{+1} & E\flat^{+1} \end{array}$$

Euler's Monochord
(Leonhard Euler, *Tentamen novæ theoriæ musicæ*, St. Petersburg, 1739)

$$\begin{array}{cccc} C\sharp^{-2} & G\sharp^{-2} & D\sharp^{-2} & A\sharp^{-2} \\ A^{-1} & E^{-1} & B^{-1} & F\sharp^{-1} \\ F^{0} & C^{0} & G^{0} & D^{0} \end{array}$$

Montvallon's Monochord
(André Barrigue de Montvallon, *Nouveau système de musique sur les intervalles des tons et sur les*



Leonhard Euler
(1707–1783)

proportions des accords, Aix, 1742)

$$\begin{array}{cccccccccc} & & A^{-1} & E^{-1} & B^{-1} & F\sharp^{-1} & C\sharp^{-1} & G\sharp^{-1} \\ B\flat^{0} & F^{0} & C^{0} & G^{0} & D^{0} & & & \\ & & E\flat^{+1} & & & & & \end{array}$$

Romieu's Monochord
(Jean Baptiste Romieu, *Mémoire théorique & pratique sur les systèmes tempérés de musique*, Mémoires de l'académie royale des sciences, 1758)

$$\begin{array}{cccccc} & & C\sharp^{-2} & G\sharp^{-2} & & \\ & A^{-1} & E^{-1} & B^{-1} & F\sharp^{-1} & \\ B\flat^{0} & F^{0} & C^{0} & G^{0} & D^{0} & \\ & & E\flat^{+1} & & & \end{array}$$

Kirnberger I
(Johann Phillip Kirnberger, *Construction der gleichschwebenden Temperatur*, Berlin, 1764)

$$\begin{array}{cccccccccccc} & & & & A^{-1} & E^{-1} & B^{-1} & F\sharp^{-1} \\ D\flat^{0} & A\flat^{0} & E\flat^{0} & B\flat^{0} & F^{0} & C^{0} & G^{0} & D^{0} \end{array}$$

Rousseau's Monochord
(Jean Jacques Rousseau, *Dictionnaire de musique*, Paris, 1768)

$$\begin{array}{ccccccccc}
F\sharp^{-2} & & C\sharp^{-2} & & & & & & \\
& A^{-1} & & E^{-1} & & B^{-1} & & & \\
F^{0} & & C^{0} & & G^{0} & & D^{0} & & \\
& A\flat^{+1} & & E\flat^{+1} & & B\flat^{+1} & & &
\end{array}$$

We shall return to the discussion of just intonation in §6.1, where we consider scales built using primes higher than 5. In §6.8, we look at a way of systematizing the discussion by using lattices, and we interpret the above scales as periodicity blocks.

Exercises

1. Choose several of the just scales described in this section, and write down the values of the notes

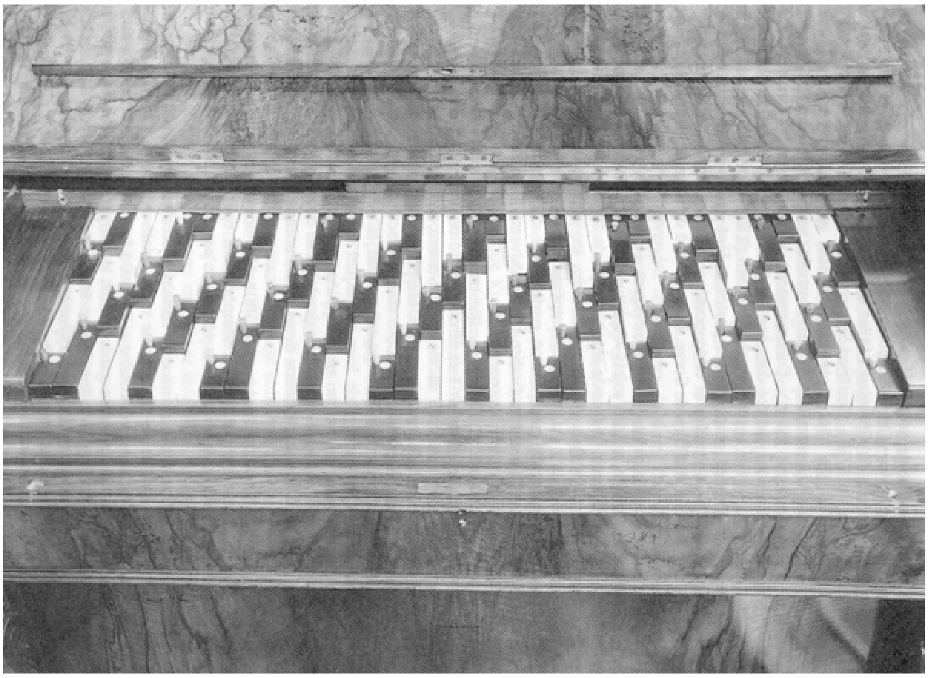
(i) in cents, and

(ii) as frequencies, giving the answers as multiples of the frequency for C.

2. Show that the Pythagorean scale with perfect fifths

$$G\flat^{0} - D\flat^{0} - A\flat^{0} - E\flat^{0} - B\flat^{0} - F^{0} - C^{0} - G^{0} - D^{0} - A^{0} - E^{0} - B^{0},$$

gives good approximations to just major triads on D, A and E, in the form $D^{0} - G\flat^{0} - A^{0}$, $A^{0} - D\flat^{0} - E^{0}$ and $E^{0} - A\flat^{0} - B^{0}$. How far from just are the thirds of these chords (in cents)?



Colin Brown's voice harmonium
(Science Museum, London)

3. The voice harmonium of Colin Brown (1875) is shown above. A plan of a little more than one octave of the keyboard is shown below. Diagonal rows of black keys and white keys alternate, and each black key has a red peg sticking out of its upper left corner, represented by a small circle in the plan. The purpose of this keyboard is to be able to play in a number of different keys in just intonation. Locate examples of the following on this keyboard:

(i) A just major triad.

(ii) A just minor triad.

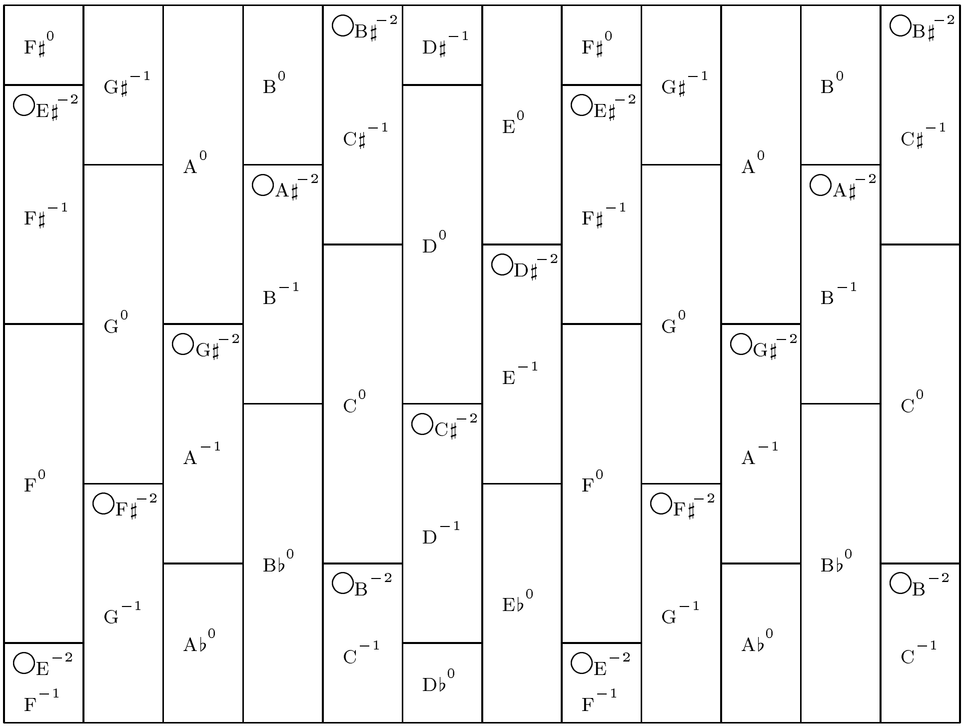
(iii) A just major scale.

(iv) Two notes differing by a syntonic comma.

(v) Two notes differing by a schisma.

(vi) Two notes differing by a diesis.

(vii) Two notes differing by an apotomē.



Keyboard diagram for Colin Brown's voice harmonium

5.11. Classical harmony

The main problem with the just major scale introduced in §5.5 is that certain harmonic progressions which form the basis of classical harmony don't quite work. This is because certain notes in the major scale are being given two different just interpretations, and switching from one to the other is a part of the progression. In this section, we discuss the progressions which form the basis for classical harmony,[13] and find where the problems lie.

We begin with the names of the triads. An upper case roman numeral denotes a major chord based on the given scale degree, whereas a lower case roman numeral denotes a minor chord. So for example the major chords I, IV and V form the basis for the just major scale in §5.5, namely $\mathrm{C}^{0}-\mathrm{E}^{-1}-\mathrm{G}^{0}$, $\mathrm{F}^{0}-\mathrm{A}^{-1}-\mathrm{C}^{0}$ and $\mathrm{G}^{0}-\mathrm{B}^{-1}-\mathrm{D}^{0}$ in the key of C major. The triads $\mathrm{A}^{-1}-\mathrm{C}^{0}-\mathrm{E}^{-1}$ and $\mathrm{E}^{-1}-\mathrm{G}^{0}-\mathrm{B}^{-1}$ are the minor triads vi and iii. The problem comes from the triad on the second note of the scale, $\mathrm{D}^{0}-\mathrm{F}^{0}-\mathrm{A}^{-1}$. If we alter the D^{0} to a D^{-1}, this is a just minor triad, which we would then call ii.

Classical harmony makes use of ii as a minor triad, so maybe we should have used D^{-1} instead of D^{0} in our just major scale. But then the triad V becomes $\mathrm{G}^{0}-\mathrm{B}^{-1}-\mathrm{D}^{-1}$, which doesn't quite work. We shall see that there is no choice of just major scale which makes all the required triads work. To understand this, we discuss classical harmonic practice.

We begin at the end. Most music in the western world imparts a sense of finality through the sequence V–I, or variations of it (V^7–I, vii^0–I).[14] It is not fully understood why V–I imparts such a feeling of finality, but it cannot be denied that it does. A great deal of music just consists of alternate triads V and I.

The progression V–I can stand on its own, or it can be approached in a number of ways. A sequence of fifths forms the basis for the commonest method, so that we can extend to ii–V–I, then to vi–ii–V–I, and even further to iii–vi–ii–V–I, each of these being less common than the previous ones. Here is a chart of the most common harmonic progressions in the music of the western world, in the major mode:

$$[\mathrm{iii}] \to [\mathrm{vi}] \to \begin{bmatrix} \mathrm{IV} \\ \downarrow \\ \mathrm{ii} \end{bmatrix} \begin{matrix} \searrow \\ \nearrow \end{matrix} \begin{bmatrix} \mathrm{vii}^0 \\ \\ \mathrm{V} \end{bmatrix} \begin{matrix} \searrow \\ \nearrow \end{matrix} \mathrm{I}$$

and then either end the piece, or go back from I to any previous triad. Common exceptions are to jump from iii to IV, from IV to I and from V to vi.

Now take a typical progression from the above chart, such as

$$\text{I–vi–ii–V–I},$$

and let us try to interpret this in just intonation. Let us stipulate one simple rule, namely that if a note on the diatonic scale appears in two adjacent triads, it should be given the same just interpretation. So if I is $\mathrm{C}^{0}-\mathrm{E}^{-1}-\mathrm{G}^{0}$ then vi must be interpreted as $\mathrm{A}^{-1}-\mathrm{C}^{0}-\mathrm{E}^{-1}$, since the C and E are in common between the two triads. This means that the ii should be interpreted

[13]The phrase "classical harmony" here is used in its widest sense, to include not only classical, romantic and baroque music, but also most of the rock, jazz and folk music of western culture.

[14]The superscript zero in the notation vii^0 denotes a *diminished triad* with two minor thirds as the intervals. It has nothing to do with the Eitz comma notation.

as D^{-1}– F^{0}– A^{-1}, with the A in common with vi. Then V needs to be interpreted as G^{-1}– B^{-2}– D^{-1} because it has D in commion with ii. Finally, the I at the end is forced to be interpreted as C^{-1}– E^{-2}– G^{-1} since it has G in common with V. We are now one syntonic comma lower than where we started.

To put the same problem in terms of ratios, in the second triad the A is $\frac{5}{3}$ of the frequency of the C, then in the third triad, D is $\frac{2}{3}$ of the frequency of A. In the fourth triad, G is $\frac{4}{3}$ of the frequency of D, and finally in the last triad, C is $\frac{2}{3}$ of the frequency of G. This means that the final C is

$$\frac{5}{3} \times \frac{2}{3} \times \frac{4}{3} \times \frac{2}{3} = \frac{80}{81}$$

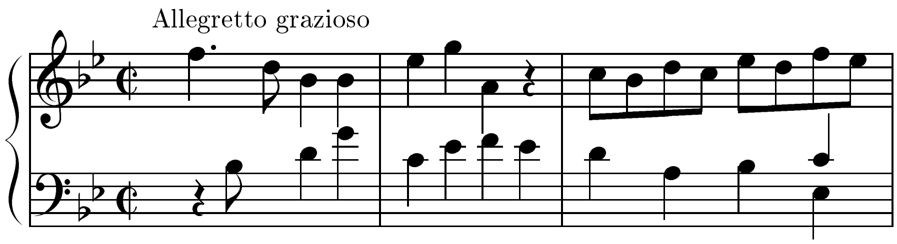
of the frequency of the initial one.

A similar drift downward through a syntonic comma occurs in the sequences

$$\text{I–IV–ii–V–I}$$
$$\text{I–iii–vi–ii–V–I}$$

and so on. Here are some actual musical examples, chosen pretty much at random.

(i) W. A. Mozart, *Sonata* (K. 333), third movement, beginning.



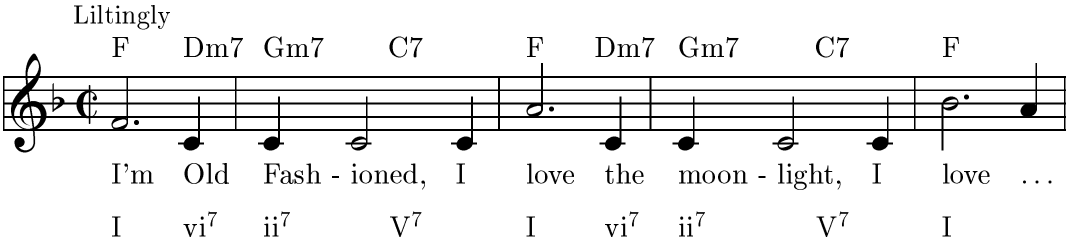
B♭: I vi ii V I

(ii) J. S. Bach, *Partita no. 5, Gigue,* bars 23–24.

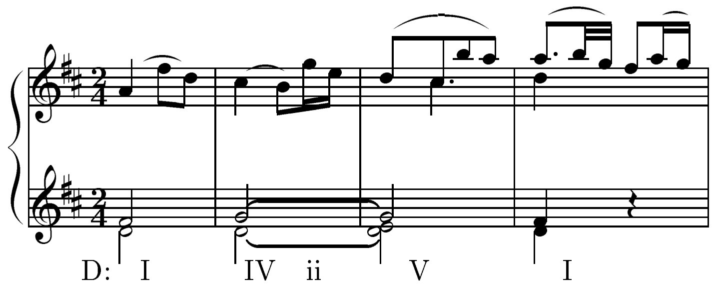
G: I vi ii V I

(iii) *I'm Old Fashioned* (1942).

Music by Jerome Kern, words by Johnny Mercer.

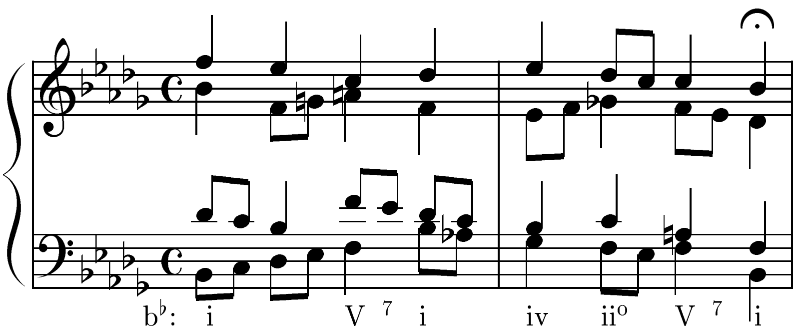


(iv) W. A. Mozart, *Fantasie* (Kv. 397), bars 55–59.



And a minor example:

(v) J. S. Bach, *Jesu, der du meine Seele.*



The meantone scale, which we shall discuss in the next section, solves the problem of the syntonic comma by deviating slightly from the just values of notes in such a way that the comma is spread equally between the four perfect fifths involved, shaving one quarter of a comma from each of them.

Harry Partch discusses this issue at length, towards the end of chapter 11 of [**89**]. He arrives at a different conclusion from the one adopted historically, namely that the progressions above sound fine, played in just intonation in such a way that the second note on the scale is played (in C major) as D^{-1} in ii, and as D^{0} in V. This means that these two versions of the "same" note are played in consecutive triads, but the sense of the harmonic progression is not lost.

5.12. Meantone scale

A *tempered scale* is a scale in which adjustments are made to the Pythagorean or just scale in order to spread around the problem caused by wishing to regard two notes differing by various commas as the same note, as in the example of §5.8, Exercise 2, and the discussion in §5.11.

The meantone scales are the tempered scales formed by making adjustments of a fraction of a (syntonic) comma to the fifths in order to make the major thirds better.

The commonest variant of the meantone scale, sometimes referred to as the classical meantone scale, or quarter-comma meantone scale, is the one in which the major thirds are made in the ratio 5:4 and then the remaining notes are interpolated as equally as possible. So C–D–E are in the ratios $1 : \sqrt{5}/2 : 5/4$, as are F–G–A and G–A–B. This leaves two semitones to decide, and they are made equal. Five tones of ratio $\sqrt{5}/2 : 1$ and two semitones make an octave 2:1, so the ratio for the semitone is

$$\sqrt{2/\left(\sqrt{5}/2\right)^5} : 1 = 8 : 5^{\frac{5}{4}}.$$

The table of ratios is therefore as follows:

note	do	re	mi	fa	so	la	ti	do
ratio	1:1	$\sqrt{5}$:2	5:4	2:$5^{\frac{1}{4}}$	$5^{\frac{1}{4}}$:1	$5^{\frac{3}{4}}$:2	$5^{\frac{5}{4}}$:4	2:1
cents	0.000	193.157	386.314	503.422	696.579	889.735	1082.892	1200.000

The fifths in this scale are no longer perfect.

Another, more enlightening way to describe the classical meantone scale is to temper each fifth by making it narrower than the Pythagorean value by exactly one quarter of a comma, in order for the major thirds to come out right. So working from C, the G is one quarter comma flat from its Pythagorean value, the D is one half comma flat, the A is three quarters of a comma flat, and finally, E is one comma flat from a Pythagorean major third, which makes it exactly equal to the just major third. Continuing in the same direction, this makes the B five quarters of a comma flatter than its Pythagorean value. Correspondingly, the F should be made one quarter comma sharper than the Pythagorean fourth.

Thus in Eitz's notation, the classical meantone scale can be written as

$$\mathrm{C}^{0} - \mathrm{D}^{-\frac{1}{2}} - \mathrm{E}^{-1} - \mathrm{F}^{+\frac{1}{4}} - \mathrm{G}^{-\frac{1}{4}} - \mathrm{A}^{-\frac{3}{4}} - \mathrm{B}^{-\frac{5}{4}} - \mathrm{C}^{0}$$

Writing these notes in the usual array notation, we obtain

$$\begin{array}{ccccccccc}
 & \mathrm{E}^{-1} & & \mathrm{B}^{-\frac{5}{4}} & & & & & \\
\mathrm{C}^{0} & & \mathrm{G}^{-\frac{1}{4}} & & \mathrm{D}^{-\frac{1}{2}} & & \mathrm{A}^{-\frac{3}{4}} & & \mathrm{E}^{-1} \\
 & & & & & \mathrm{F}^{+\frac{1}{4}} & & \mathrm{C}^{0} &
\end{array}$$

The meantone scale can be completed by filling in the remaining notes of a twelve (or more) tone scale according to the same principles. The only question is how far to go in each direction with the quarter comma tempered fifths. Some examples, again taken from Barbour [**5**] follow.

Aaron's Meantone Temperament
(Pietro Aaron, *Toscanello in musica*, Venice, 1523)

$$\mathrm{C}^{0}\ \mathrm{C}\sharp^{-\frac{7}{4}}\ \mathrm{D}^{-\frac{1}{2}}\ \mathrm{E}\flat^{+\frac{3}{4}}\ \mathrm{E}^{-1}\ \mathrm{F}^{+\frac{1}{4}}\ \mathrm{F}\sharp^{-\frac{3}{2}}\ \mathrm{G}^{-\frac{1}{4}}\ \mathrm{A}\flat^{+1}\ \mathrm{A}^{-\frac{3}{4}}\ \mathrm{B}\flat^{+\frac{1}{2}}\ \mathrm{B}^{-\frac{5}{4}}\ \mathrm{C}^{0}$$

Gibelius' Monochord for Meantone Temperament
(Otto Gibelius, *Propositiones mathematico-musicæ*, Münden, 1666)
is the same, but with two extra notes

$$\mathrm{C}^{0}\ \mathrm{C}\sharp^{-\frac{7}{4}}\ \mathrm{D}^{-\frac{1}{2}}\ \mathrm{D}\sharp^{-\frac{9}{4}}\ \mathrm{E}\flat^{+\frac{3}{4}}\ \mathrm{E}^{-1}\ \mathrm{F}^{+\frac{1}{4}}\ \mathrm{F}\sharp^{-\frac{3}{2}}\ \mathrm{G}^{-\frac{1}{4}}\ \mathrm{G}\sharp^{-2}\ \mathrm{A}\flat^{+1}\ \mathrm{A}^{-\frac{3}{4}}\ \mathrm{B}\flat^{+\frac{1}{2}}\ \mathrm{B}^{-\frac{5}{4}}\ \mathrm{C}^{0}$$

These meantone scales are represented in array notation as follows:

$$\begin{array}{ccccccccccc}
 & & & (\mathrm{G}\sharp^{-2}) & & (\mathrm{D}\sharp^{-\frac{9}{4}}) & & & & & \\
 & & \mathrm{E}^{-1} & & \mathrm{B}^{-\frac{5}{4}} & & \mathrm{F}\sharp^{-\frac{3}{2}} & & \mathrm{C}\sharp^{-\frac{7}{4}} & & (\mathrm{G}\sharp^{-2}) \\
 & \mathrm{C}^{0} & & \mathrm{G}^{-\frac{1}{4}} & & \mathrm{D}^{-\frac{1}{2}} & & \mathrm{A}^{-\frac{3}{4}} & & \mathrm{E}^{-1} & \\
\mathrm{A}\flat^{+1} & & \mathrm{E}\flat^{+\frac{3}{4}} & & \mathrm{B}\flat^{+\frac{1}{2}} & & \mathrm{F}^{+\frac{1}{4}} & & \mathrm{C}^{0} & &
\end{array}$$

where the right hand edge is thought of as equal to the left hand edge. Thus the notes can be thought of as lying on a cylinder, with four quarter-comma adjustments taking us once round the cylinder.

So the syntonic comma has been taken care of, and modulations can be made to a reasonable number of keys. The Pythagorean comma has not been taken care of, so that modulation around an entire circle of fifths is still not feasible. Indeed, the difference between the enharmonic notes $\mathrm{A}\flat^{+1}$ and $\mathrm{G}\sharp^{-2}$ is three syntonic commas minus a Pythagorean comma, which is a ratio of 128:125, or a difference of 41.059 cents. This interval, called the great diesis is nearly half a semitone, and is very noticeable to the ear. The imperfect fifth between C♯ and A♭ (or wherever else it may happen to be placed) in the meantone scale is sometimes referred to as the *wolf*[15] interval of the scale.

[15] This has nothing to do with the "wolf" notes on a stringed instrument such as the cello, which has to do with the sympathetic resonance of the body of the instrument.

Although what we have described is the commonest form of meantone scale, there are others formed by taking different divisions of the comma. In general, the α-comma meantone temperament refers to the following temperament:

$$\begin{array}{ccccccccccc}
 & \mathrm{E}^{-4\alpha} & & \mathrm{B}^{-5\alpha} & & \mathrm{F}\sharp^{-6\alpha} & & \mathrm{C}\sharp^{-7\alpha} & & \mathrm{G}\sharp^{-8\alpha} & \\
\mathrm{C}^{0} & & \mathrm{G}^{-\alpha} & & \mathrm{D}^{-2\alpha} & & \mathrm{A}^{-3\alpha} & & \mathrm{E}^{-4\alpha} & & \\
 & \mathrm{E}\flat^{+3\alpha} & & \mathrm{B}\flat^{+2\alpha} & & \mathrm{F}^{+\alpha} & & \mathrm{C}^{0} & & &
\end{array}$$

Without any qualification, the phrase "meantone temperament" refers to the case $\alpha = \frac{1}{4}$. The following names are associated with various values of α:

0	Pythagoras	
$\frac{1}{6}$	Silbermann	Sorge, *Gespräch zwischen einem Musico theoretico*
		und einem Studioso musices, (Lobenstein, 1748), p. 20
$\frac{1}{5}$	Abraham Verheijen,	Simon Stevin, *Van de Spiegeling der Singconst*, c. 1600
	Lemme Rossi	*Sistema musico*, Perugia, 1666, p. 58
$\frac{2}{9}$	Lemme Rossi	*Sistema musico*, Perugia, 1666, p. 64
$\frac{1}{4}$	Aaron/Gibelius/Zarlino/...	
$\frac{2}{7}$	Gioseffo Zarlino	*Istitutioni armoniche*, Venice, 1558
$\frac{1}{3}$	Francisco de Salinas	*De musica libri VII*, Salamanca, 1577

So for example, Zarlino's $\frac{2}{7}$ comma temperament is as follows:

$$\begin{array}{ccccccccccc}
 & \mathrm{E}^{-\frac{8}{7}} & & \mathrm{B}^{-\frac{10}{7}} & & \mathrm{F}\sharp^{-\frac{12}{7}} & & \mathrm{C}\sharp^{-2} & & \mathrm{G}\sharp^{-\frac{16}{7}} & \\
\mathrm{C}^{0} & & \mathrm{G}^{-\frac{2}{7}} & & \mathrm{D}^{-\frac{4}{7}} & & \mathrm{A}^{-\frac{6}{7}} & & \mathrm{E}^{-\frac{8}{7}} & & \\
 & \mathrm{E}\flat^{+\frac{6}{7}} & & \mathrm{B}\flat^{+\frac{4}{7}} & & \mathrm{F}^{+\frac{2}{7}} & & \mathrm{C}^{0} & & &
\end{array}$$

The value $\alpha = 0$ gives Pythagorean intonation, and a value close to $\alpha = \frac{1}{11}$ gives twelve tone equal temperament (see §5.14), so these can (at a pinch) be thought of as extreme forms of meantone. There is a diagram in Appendix J on page 323 which illustrates various meantone scales, and the extent to which the thirds and fifths deviate from their just values.

A useful way of thinking of meantone temperaments is that in order to name a meantone temperament, it is sufficient to name the size of the fifth. We have chosen to name this size as a narrowing of the perfect fifth by α commas. Knowing the size of the fifth, all other intervals are obtained by taking multiples of this size and reducing by octaves. So we say that the fifth *generates* a meantone temperament. In any meantone temperament, every key sounds just like every other key, until the wolf is reached.

Exercises

1. Show that the 1/3 comma meantone scale of Salinas gives pure minor thirds. Calculate the size of the wolf fifth.

2. What fraction of a comma should we use for a meantone system in order to minimize the mean square error of the fifth, the major third and the minor third from their just values?

3. Go to the web site

http://midiworld.com/mw_byrd.htm

and listen to some of John Sankey's MIDI files of keyboard music by William Byrd, sequenced in quarter comma meantone.

4. Charles Lucy is fond of a tuning system which he attributes to John Harrison (1693–1776) in which the fifths are tuned to a ratio of $2^{\frac{1}{2}+\frac{1}{4\pi}} : 1$ and the major thirds $2^{\frac{1}{\pi}} : 1$. Show that this can be considered as a meantone scale in which the fifths are tempered by about $\frac{3}{10}$ of a comma. Charles Lucy's web site can be found at

http://lehua.ilhawaii.net/~lucy/index.html

5. In the meantone scale, the octave is taken to be perfect. Investigate the scale obtained by stretching the octave by $\frac{1}{6}$ of a comma, and shrinking the fifth by $\frac{1}{6}$ of a comma. How many cents away from just are the major third and minor third in this scale? Calculate the values in cents for notes of the major scale in this temperament.

Further listening: (See Appendix R)

The Katahn/Foote recording, *Six degrees of tonality* contains tracks comparing Mozart's *Fantasie* Kv. 397 in equal temperament, meantone, and an irregular temperament of Prelleur.

Edward Parmentier, *Seventeenth Century French Harpsichord Music*, recorded in $\frac{1}{3}$ comma meantone temperament.

Aldert Winkelman, *Works by Mattheson, Couperin and others.* This recording includes pieces by Louis Couperin and Gottlieb Muffat played on a spinet tuned in quarter comma meantone temperament.

Organs tuned in quarter comma meantone temperament are being built even today. The C. B. Fisk organ at Wellesley College, Massachusetts, USA is tuned in quarter comma meantone temperament. See

http://www.wellesley.edu/Music/organ.html

for a more detailed description of this organ. Bernard Lagacé has recorded a CD of music of various composers on this organ.

John Brombaugh apprenticed with the American organ builders Fritz Noack and Charles Fisk between 1964 and 1967, and has built a number of organs in quarter comma meantone temperament. These include the Brombaugh organs in the Duke University Chapel, Oberlin College, Southern College, and the Haga Church in Gothenburg, Sweden.

Another example of a modern organ tuned in meantone temperament is the Hellmuth Wollf organ of Knox College Chapel in Toronto University, Canada.

5.13. Irregular temperaments

The phrases *irregular temperament*, *circulating temperament* and *well tempered scale* all refer to a twelve tone scale in which the notes of the meantone scale have been bent so that the scale works more or less in all twelve possible key signatures. This means that the notes at the extremes of the circle of fifths, near the wolf fifth, have been changed in pitch so as to distribute the wolf between several fifths. The effect is that each of these fifths is more or less acceptable.

Historically, irregular temperaments superseded or lived alongside meantone temperament (§5.12) during the seventeenth century, and were in use for at least two centuries before equal temperament (§5.14) took hold.

Evidence from the 48 Preludes and Fugues of the *Well Tempered Clavier* suggests that rather than being written for meantone temperament, Bach intended a more irregular temperament in which all keys are more or less satisfactorily in tune.[16]

A typical example of such a temperament is Werckmeister's most frequently used temperament. This is usually referred to as Werckmeister III (although Barbour [**5**] refers to it as Werckmeister's Correct Temperament No. 1),[17] which is as follows.

Werckmeister III (Correct Temperament No. 1)
(Andreas Werckmeister, *Musicalische Temperatur* Frankfort and Leipzig, 1691; reprinted by Diapason Press, Utrecht, 1986, with commentary by Rudolph Rasch)

$$\begin{array}{ccccccccccc}
 & \mathrm{E}^{-\frac{3}{4}p} & & \mathrm{B}^{-\frac{3}{4}p} & & \mathrm{F}\sharp^{-1p} & & \mathrm{C}\sharp^{-1p} & & \mathrm{G}\sharp^{-1p} & \\
\mathrm{C}^{0} & & \mathrm{G}^{-\frac{1}{4}p} & & \mathrm{D}^{-\frac{1}{2}p} & & \mathrm{A}^{-\frac{3}{4}p} & & \mathrm{E}^{-\frac{3}{4}p} & & \\
 & \mathrm{E}\flat^{0} & & \mathrm{B}\flat^{0} & & \mathrm{F}^{0} & & \mathrm{C}^{0} & & &
\end{array}$$

In this temperament, the Pythagorean comma (*not* the syntonic comma) is distributed equally on the fifths from C–G–D–A and B–F♯. We use a modified version of Eitz's notation to denote this, in which "p" is used to denote the usage of the Pythagorean comma rather than the syntonic comma. A good way to think of this is to use the approximation discussed in §5.14 which says "$p = \frac{12}{11}$", so that for example $\mathrm{E}^{-\frac{3}{4}p}$ is essentially the same as $\mathrm{E}^{-\frac{9}{11}}$. Note that $\mathrm{A}\flat^{0}$ is equal to $\mathrm{G}\sharp^{-1p}$, so the circle of fifths does join up properly in this temperament. In fact, this was the first temperament to be widely adopted which has this property.

In this and other irregular temperaments, different key signatures have different characteristic sounds, with some keys sounding direct and others

[16]It is a common misconception that Bach intended the *Well Tempered Clavier* to be played in equal temperament. He certainly knew of equal temperament, but did not use it by preference, and it is historically much more likely that the 48 preludes and fugues were intended for an irregular temperament of the kind discussed in this section. (It should be mentioned that there is also evidence that Bach did intend equal temperament, see Rudolf A. Rasch, *Does 'Well-tempered' mean 'Equal-tempered'?*, in Williams (ed.), Bach, Händel, Scarlatti tercentenary Essays, Cambridge University Press, Cambridge, 1985, pp. 293-310.)

[17]*Werckmeister I* usually refers to just intonation, and *Werckmeister II* to classical meantone temperament. *Werckmeister IV* and *V* are described below. There is also a temperament known as *Werckmeister VI*, or "septenarius", which is based on a division of a string into 196 equal parts. This scale gives the ratios 1:1, 196:186, 196:176, 196:165, 196:156, 4:3, 196:139, 196:131, 196:124, 196:117, 196:110, 196:104, 2:1.

more remote. This may account for the modern myth that the same holds in equal temperament.[18]

An interesting example of the use of irregular temperaments in composition is J. S. Bach's Toccata in F♯ minor (BWV 910), bars 109ff, in which essentially the same musical phrase is repeated about twenty times in succession, transposed into different keys. In equal or meantone temperament this could get monotonous, but with an irregular temperament, each phrase would impart a subtly different feeling.

The point of distributing the comma unequally between the twelve fifths is so that in the most commonly used keys, the fifth and major third are very close to just. The price to be paid is that in the more "remote" keys the tuning of the major thirds is somewhat sharp. So for example in Werckmeister III, the thirds on C and F are about four cents sharp, while the thirds on C♯ and F♯ are about 22 cents sharp. Other examples of irregular temperaments with similar intentions include the following, taken from Asselin [**1**], Barbour [**5**] and Devie [**29**].

Mersenne's Improved Meantone Temperament, No. 1
(Marin Mersenne: Cogitata physico-mathematica, Paris, 1644)

$$\begin{array}{ccccccccccc}
 & E^{-1p} & & B^{-\frac54 p} & & F\sharp^{-\frac32 p} & & C\sharp^{-\frac74 p} & & G\sharp^{-2p} \\
C^0 & & G^{-\frac14 p} & & D^{-\frac12 p} & & A^{-\frac34 p} & & E^{-1p} & \\
 & E\flat^{+\frac14 p} & & B\flat^{+\frac14 p} & & F^{+\frac14 p} & & C^0 & &
\end{array}$$

Bendeler's Temperament, No. 1
(P. Bendeler, *Organopoeia*, Frankfurt, 1690; 2nd. ed. Frankfurt & Leipzig, 1739, p. 40)

$$\begin{array}{ccccccccccc}
 & E^{-\frac23 p} & & B^{-\frac23 p} & & F\sharp^{-p} & & C\sharp^{-p} & & G\sharp^{-p} \\
C^0 & & G^{-\frac13 p} & & D^{-\frac23 p} & & A^{-\frac23 p} & & E^{-\frac23 p} & \\
 & E\flat^{0} & & B\flat^{0} & & F^{0} & & C^0 & &
\end{array}$$

[18] If this were really true, then the shift of nearly a semitone in pitch between Mozart's time and our own would have resulted in a permutation of the resulting moods, which seems to be nonsense. Actually, this argument really only applies to keyboard instruments. It is still possible in equal temperament for string and wind instruments to give different characters to different keys. For example, a note on an open string on a violin sounds different in character from a stopped string. Mozart and others have made use of this difference with a technique called *scordatura*, (Italian *scordare*, to mistune) which involves unconventional retuning of stringed instruments. A well known example is his *Sinfonia Concertante*, in which all the strings of the solo viola are tuned a semitone sharp. The orchestra plays in E♭ for a softer sound, and the solo viola plays in D for a more brilliant sound.

A more shocking example (communicated to me by Marcus Linckelmann) is Schubert's *Impromptu No. 3* for piano in G♭ major. The same piece played in G major on a modern piano has a very different feel to it. It is possible that in this case, the mechanics of the fingering are responsible.

Bendeler's Temperament, No. 2
(P. Bendeler, 1690/1739, p. 42)

$$
\begin{array}{cccccccccc}
 & E^{-\frac{2}{3}p} & & B^{-\frac{2}{3}p} & & F\sharp^{-\frac{2}{3}p} & & C\sharp^{-p} & & G\sharp^{-p} \\
C^{0} & & G^{-\frac{1}{3}p} & & D^{-\frac{1}{3}p} & & A^{-\frac{2}{3}p} & & E^{-\frac{2}{3}p} & \\
 & E\flat^{0} & & B\flat^{0} & & F^{0} & & C^{0} & &
\end{array}
$$

Bendeler's Temperament, No. 3
(P. Bendeler, 1690/1739, p. 42)

$$
\begin{array}{cccccccccc}
 & E^{-\frac{1}{2}p} & & B^{-\frac{3}{4}p} & & F\sharp^{-\frac{3}{4}p} & & C\sharp^{-\frac{3}{4}p} & & G\sharp^{-\frac{3}{4}p} \\
C^{0} & & G^{-\frac{1}{4}p} & & D^{-\frac{1}{2}p} & & A^{-\frac{1}{2}p} & & E^{-\frac{1}{2}p} & \\
 & E\flat^{0} & & B\flat^{0} & & F^{0} & & C^{0} & &
\end{array}
$$

Werckmeister III (Correct Temperament No. 1) See page 141.

Werckmeister IV (Correct Temperament No. 2)
(Andreas Werckmeister, 1691; the least satisfactory of Werckmeister's temperaments)

$$
\begin{array}{cccccccccc}
 & E^{-\frac{2}{3}p} & & B^{-1p} & & F\sharp^{-1p} & & C\sharp^{-\frac{4}{3}p} & & G\sharp^{-\frac{4}{3}p} \\
C^{0} & & G^{-\frac{1}{3}p} & & D^{-\frac{1}{3}p} & & A^{-\frac{2}{3}p} & & E^{-\frac{2}{3}p} & \\
 & E\flat^{0} & & B\flat^{+\frac{1}{3}p} & & F^{0} & & C^{0} & &
\end{array}
$$

Werckmeister V (Correct Temperament No. 3)
(Andreas Werckmeister, 1691)

$$
\begin{array}{cccccccccc}
 & E^{-\frac{1}{2}p} & & B^{-\frac{1}{2}p} & & F\sharp^{-\frac{1}{2}p} & & C\sharp^{-\frac{3}{4}p} & & G\sharp^{-1p} \\
C^{0} & & G^{0} & & D^{0} & & A^{-\frac{1}{4}p} & & E^{-\frac{1}{2}p} & \\
 & E\flat^{+\frac{1}{4}p} & & B\flat^{+\frac{1}{4}p} & & F^{+\frac{1}{4}p} & & C^{0} & &
\end{array}
$$

Neidhardt's Circulating Temperament, No. 1 "für ein Dorf" (for a village)
(Johann Georg Neidhardt, *Sectio canonis harmonici*, Königsberg, 1724, 16–18)

$$
\begin{array}{cccccccccc}
 & E^{-\frac{2}{3}p} & & B^{-\frac{3}{4}p} & & F\sharp^{-\frac{5}{6}p} & & C\sharp^{-\frac{5}{6}p} & & G\sharp^{-\frac{5}{6}p} \\
C^{0} & & G^{-\frac{1}{6}p} & & D^{-\frac{1}{3}p} & & A^{-\frac{1}{2}p} & & E^{-\frac{2}{3}p} & \\
 & E\flat^{0} & & B\flat^{0} & & F^{0} & & C^{0} & &
\end{array}
$$

Neidhardt's Circulating Temperament, No. 2 "für ein kleine Stadt" (for a small town)
(Johann Georg Neidhardt, 1724)[19]

$$
\begin{array}{cccccccccc}
 & E^{-\frac{7}{12}p} & & B^{-\frac{7}{12}p} & & F\sharp^{-\frac{2}{3}p} & & C\sharp^{-\frac{3}{4}p} & & G\sharp^{-\frac{5}{6}p} \\
C^{0} & & G^{-\frac{1}{6}p} & & D^{-\frac{1}{3}p} & & A^{-\frac{1}{2}p} & & E^{-\frac{2}{3}p} & \\
 & E\flat^{+\frac{1}{6}p} & & B\flat^{+\frac{1}{6}p} & & F^{+\frac{1}{12}p} & & C^{0} & &
\end{array}
$$

Neidhardt's Circulating Temperament, No. 3 "für eine grosse Stadt" (for a large town)
(Johann Georg Neidhardt, 1724)

$$
\begin{array}{cccccccccc}
 & E^{-\frac{7}{12}p} & & B^{-\frac{7}{12}p} & & F\sharp^{-\frac{2}{3}p} & & C\sharp^{-\frac{3}{4}p} & & G\sharp^{-\frac{5}{6}p} \\
C^{0} & & G^{-\frac{1}{6}p} & & D^{-\frac{1}{3}p} & & A^{-\frac{1}{2}p} & & E^{-\frac{2}{3}p} & \\
 & E\flat^{+\frac{1}{6}p} & & B\flat^{+\frac{1}{12}p} & & F^{0} & & C^{0} & &
\end{array}
$$

[19]Barbour has $E^{-\frac{1}{12}p}$, which is incorrect, although he gives the correct value in cents. This seems to be nothing more than a typographical error.

Neidhardt's Circulating Temperament, No. 4 "für den Hof" (for the court)
is the same as twelve tone equal temperament.

Kirnberger II
(Johann Phillip Kirnberger, *Construction der gleichschwebenden Temperatur*,[20] Berlin, 1764)

$$\begin{array}{ccccccccc}
 & & E^{-1} & & B^{-1} & & F\sharp^{-1} & & \\
 & C^{0} & & G^{0} & & D^{0} & & A^{-\frac{1}{2}} & E^{-1} \\
A\flat^{0} & & E\flat^{0} & & B\flat^{0} & & F^{0} & & C^{0} \\
 & & & & & D\flat^{0} & & A\flat^{0} &
\end{array}$$

Kirnberger III
(Johann Phillip Kirnberger, *Die Kunst des reinen Satzes in der Musik* 2nd part, 3rd division, Berlin, 1779)

$$\begin{array}{ccccccccc}
 & & E^{-1} & & B^{-1} & & F\sharp^{-1} & & \\
 & C^{0} & & G^{-\frac{1}{4}} & & D^{-\frac{1}{2}} & & A^{-\frac{3}{4}} & E^{-1} \\
A\flat^{0} & & E\flat^{0} & & B\flat^{0} & & F^{0} & & C^{0} \\
 & & & & & D\flat^{0} & & A\flat^{0} &
\end{array}$$

Marpurg's Temperament I
(Friedrich Wilhelm Marpurg, *Versuch über die musikalische Temperatur*, Breslau, 1776)

$$\begin{array}{cccccccccc}
 & E^{-\frac{1}{3}p} & & B^{-\frac{1}{3}p} & & F\sharp^{-\frac{1}{3}p} & & C\sharp^{-\frac{1}{3}p} & & G\sharp^{-\frac{2}{3}p} \\
C^{0} & & G^{0} & & D^{0} & & A^{0} & & E^{-\frac{1}{3}p} & \\
 & E\flat^{+\frac{1}{3}p} & & B\flat^{+\frac{1}{3}p} & & F^{+\frac{1}{3}p} & & C^{0} & &
\end{array}$$

Barca's $\frac{1}{6}$-comma temperament
(Alessandro Barca, *Introduzione a una nuova teoria di musica, memoria prima* Accademia di scienze, lettere ed arti in Padova. Saggi scientifici e lettari (Padova, 1786), 365–418)

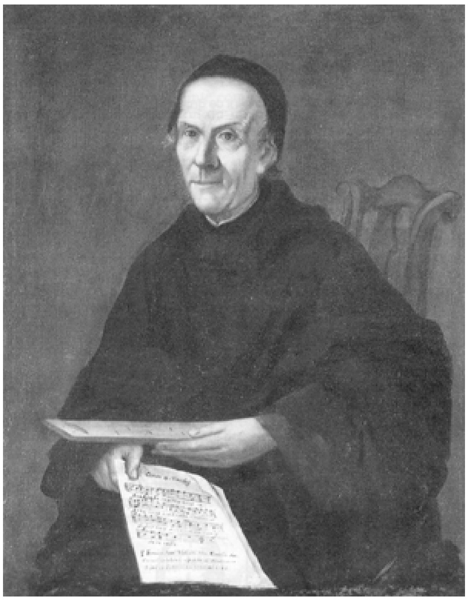
$$\begin{array}{cccccccccc}
 & E^{-\frac{2}{3}} & & B^{-\frac{5}{6}} & & F\sharp^{-1} & & C\sharp^{-1} & & G\sharp^{-1} \\
C^{0} & & G^{-\frac{1}{6}} & & D^{-\frac{1}{3}} & & A^{-\frac{1}{2}} & & E^{-\frac{2}{3}} & \\
 & E\flat^{0} & & B\flat^{0} & & F^{0} & & C^{0} & &
\end{array}$$

Young's Temperament, No. 1
(Thomas Young, *Outlines of experiments and inquiries respecting sound and light* Philosophical Transactions, XC (1800), 106–150)

$$\begin{array}{cccccccccc}
 & E^{-\frac{3}{4}} & & B^{-\frac{5}{6}} & & F\sharp^{-\frac{11}{12}} & & C\sharp^{-\frac{11}{12}} & & G\sharp^{-\frac{11}{12}} \\
C^{0} & & G^{-\frac{3}{16}} & & D^{-\frac{3}{8}} & & A^{-\frac{9}{16}} & & E^{-\frac{3}{4}} & \\
 & E\flat^{+\frac{1}{6}} & & B\flat^{+\frac{1}{6}} & & F^{+\frac{1}{12}} & & C^{0} & &
\end{array}$$

Vallotti and Young $\frac{1}{6}$-comma temperament (Young's Temperament, No. 2)
(Francescantonio Vallotti, *Trattato delle musica moderna*, 1780; Thomas Young, *Outlines of experiments and inquiries respecting sound and light* Philosophical Transactions, XC (1800), 106–150. Below

[20]It is a mistake to deduce from this title that Kirnberger was in favor of equal temperament. In fact, he explicitly states in an article dated 1779 that "die gleichschwebende Temperatur ist schlechterdings ganz verwerflich..." (equal beating temperament is simply completely objectionable). And actually, there's a difference between equal and equal beating—the latter refers to the method of tuning whereby the fifths in a given octave are all made to beat at the same speeds.



Francescantonio Vallotti (1697–1780)

is Young's version of this temperament. In Vallotti's version, the fifths which are narrow by $\frac{1}{6}$ Pythagorean commas are F–C–G–D–A–E–B instead of C–G–D–A–E–B–F♯)

$$\begin{array}{ccccccccccc}
 & \mathrm{E}^{-\frac{2}{3}p} & & \mathrm{B}^{-\frac{5}{6}p} & & \mathrm{F}\sharp^{-1p} & & \mathrm{C}\sharp^{-1p} & & \mathrm{G}\sharp^{-1p} & \\
\mathrm{C}^{0} & & \mathrm{G}^{-\frac{1}{6}p} & & \mathrm{D}^{-\frac{1}{3}p} & & \mathrm{A}^{-\frac{1}{2}p} & & \mathrm{E}^{-\frac{2}{3}p} & & \\
 & \mathrm{E}\flat^{0} & & \mathrm{B}\flat^{0} & & \mathrm{F}^{0} & & \mathrm{C}^{0} & & &
\end{array}$$

The temperament of Vallotti and Young is probably closest to the intentions of J. S. Bach for his well-tempered clavier. According to the researches of Barnes, it is possible that Bach preferred the F♯ to be one sixth of a Pythagorean comma sharper than in this temperament, so that the fifth from B to F♯ is pure. Barnes based his work on a statistical study of prominence of the different major thirds, and the mathematical procedure of Hall[21] for evaluating suitability of temperaments. Other authors, such as Kelletat and Kellner have come to slightly different conclusions, and we will probably never find out who is right. Here are these reconstructions for comparison.

Kelletat's Bach reconstruction (1966)

Herbert Kelletat, *Zur musikalischen Temperatur insbesondere bei J. S. Bach.* Onkel Verlag, Kassel, 1960 and 1980.

$$\begin{array}{ccccccccccc}
 & \mathrm{E}^{-\frac{5}{6}p} & & \mathrm{B}^{-1p} & & \mathrm{F}\sharp^{-1p} & & \mathrm{C}\sharp^{-1p} & & \mathrm{G}\sharp^{-1p} & \\
\mathrm{C}^{0} & & \mathrm{G}^{-\frac{1}{12}p} & & \mathrm{D}^{-\frac{1}{3}p} & & \mathrm{A}^{-\frac{7}{12}p} & & \mathrm{E}^{-\frac{5}{6}p} & & \\
 & \mathrm{E}\flat^{0} & & \mathrm{B}\flat^{0} & & \mathrm{F}^{0} & & \mathrm{C}^{0} & & &
\end{array}$$

[21] D. E. Hall, *The objective measurement of goodness-of-fit for tuning and temperaments*, J. Music Theory **17** (2) (1973), 274–290; *Quantitative evaluation of musical scale tuning*, American J. of Physics **42** (1974), 543–552.

Kellner's Bach reconstruction (1975)

Herbert Anton Kellner, *Eine Rekonstruktion der wohltemperierten Stimmung von Johann Sebastian Bach.* Das Musikinstrument **26** (1977), 34–35; *Was Bach a mathematician?* English Harpsichord Magazine 2/2 April 1978, 32–36; *Comment Bach accordait-il son clavecin?* Flûte à Bec et instruments anciens 13–14, SDIA, Paris 1985.

$$\begin{array}{ccccccccccc}
 & E^{-\frac{4}{5}p} & & B^{-\frac{4}{5}p} & & F\sharp^{-1p} & & C\sharp^{-1p} & & G\sharp^{-1p} & \\
C^{0} & & G^{-\frac{1}{5}p} & & D^{-\frac{2}{5}p} & & A^{-\frac{3}{5}p} & & E^{-\frac{4}{5}p} & & \\
 & E\flat^{0} & & B\flat^{0} & & F^{0} & & C^{0} & & &
\end{array}$$

Barnes' Bach reconstruction (1979)

John Barnes, *Bach's Keyboard Temperament*, Early Music **7** (2) (1979), 236–249.

$$\begin{array}{ccccccccccc}
 & E^{-\frac{2}{3}p} & & B^{-\frac{5}{6}p} & & F\sharp^{-\frac{5}{6}p} & & C\sharp^{-1p} & & G\sharp^{-1p} & \\
C^{0} & & G^{-\frac{1}{6}p} & & D^{-\frac{1}{3}p} & & A^{-\frac{1}{2}p} & & E^{-\frac{2}{3}p} & & \\
 & E\flat^{0} & & B\flat^{0} & & F^{0} & & C^{0} & & &
\end{array}$$

Exercises

1. Take the information on various temperaments given in this section, and work out a table of values in cents for the notes of the scale.

2. If you have a synthesizer where each note of the scale can be retuned separately, retune it to some of the temperaments given in this section, using your answers to Exercise 1. Sequence some harpsichord music and play it through your synthesizer using these temperaments, and compare the results.

Further listening: (See Appendix R)

Johann Sebastian Bach, *The Complete Organ Music*, Volumes 6 and 8, recorded by Hans Fagius, using Neidhardt's Circulating Temperament No. 3 "für eine grosse Stadt" (for a large town).

The Katahn/Foote recording, *Six degrees of tonality* contains tracks comparing Mozart's *Fantasie* Kv. 397 in equal temperament, meantone, and an irregular temperament of Prelleur.

Johann Gottfried Walther, *Organ Works*, Volumes 1 and 2, played by Craig Cramer on the organ of St. Bonifacius, Tröchtelborn, Germany. This organ was restored in Kellner's reconstruction of Bach's temperament.

Aldert Winkelman, *Works by Mattheson, Couperin, and others.* The pieces by Johann Mattheson, François Couperin, Johann Jakob Froberger, Joannes de Gruytters and Jacques Duphly are played on a harpsichord tuned to Werckmeister III.

5.14. Equal temperament

Music is a science which should have definite rules; these rules should be drawn from an evident principle; and this principle cannot really be known to us without the aid of mathematics. Notwithstanding all the experience I may have acquired in music from being associated with it for so long, I must confess that only with the aid of mathematics *did my ideas become clear and did light replace a certain obscurity of which I was unaware before.*

Rameau [**99**], 1722.[22]

Each of the scales described in the previous sections has its advantages and disadvantages, but the one disadvantage of most of them is that they are designed to make one particular key signature or a few adjacent key signatures as good as possible, and leave the remaining ones to look after themselves.

Twelve tone equal temperament is a natural endpoint of these compromises. This is the scale that results when all twelve semitones are taken to have equal ratios. Since an octave is a ratio of 2:1, the ratios for the equal tempered scale give all semitones a ratio of $2^{\frac{1}{12}}$:1 and all tones a ratio of $2^{\frac{1}{6}}$:1. So the ratios come out as follows:

note	do	re	mi	fa	so	la	ti	do
ratio	1:1	$2^{\frac{1}{6}}$:1	$2^{\frac{1}{3}}$:1	$2^{\frac{5}{12}}$:1	$2^{\frac{7}{12}}$:1	$2^{\frac{3}{4}}$:1	$2^{\frac{11}{12}}$:1	2:1
cents	0.000	200.000	400.000	500.000	700.000	900.000	1100.000	1200.000

Equal tempered thirds are about 14 cents sharper than perfect thirds, and sound nervous and agitated. As a consequence, the just and meantone scales are more calm temperaments. To my ear, tonal polyphonic music played in meantone temperament has a clarity and sparkle that I do not hear on equal tempered instruments. The irregular temperaments described in the previous section have the property that each key retains its own characteristics and color; keys with few sharps and flats sound similar to meantone, while the ones with more sharps and flats have a more remote feel to them. Equal temperament makes all keys essentially equivalent.

A factor of

$$\left(\frac{81}{80}\right)^{\frac{12}{11}} \approx 1.013644082,$$

or 23.4614068 cents is an extremely good approximation to the Pythagorean comma of

$$\frac{531441}{524288} \approx 1.013643265,$$

or 23.4600104 cents. It follows that equal temperament is almost exactly equal to the $\frac{1}{11}$-comma meantone scale:

$$\begin{array}{ccccccccccc}
 & & \mathrm{E}^{-\frac{4}{11}} & & \mathrm{B}^{-\frac{5}{11}} & & \mathrm{F}\sharp^{-\frac{6}{11}} & & \mathrm{C}\sharp^{-\frac{7}{11}} & & \mathrm{G}\sharp^{-\frac{8}{11}} \\
 & \mathrm{C}^{0} & & \mathrm{G}^{-\frac{1}{11}} & & \mathrm{D}^{-\frac{2}{11}} & & \mathrm{A}^{-\frac{3}{11}} & & \mathrm{E}^{-\frac{4}{11}} & \\
\mathrm{A}\flat^{+\frac{4}{11}} & & \mathrm{E}\flat^{+\frac{3}{11}} & & \mathrm{B}\flat^{+\frac{2}{11}} & & \mathrm{F}^{+\frac{1}{11}} & & \mathrm{C}^{0} & &
\end{array}$$

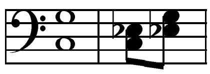
where the difference between $\mathrm{A}\flat^{+\frac{4}{11}}$ and $\mathrm{G}\sharp^{-\frac{8}{11}}$ is 0.0013964 cents.

[22] Page xxxv of the preface, in the Dover edition.

This observation was first made by Kirnberger[23] who used it as the basis for a recipe for tuning keyboard instruments in equal temperament. His recipe was to obtain an interval of an equal tempered fourth by tuning up three perfect fifths and one major third, and then down four perfect fourths. This corresponds to equating the equal tempered F with $E\natural^{-1}$. The disadvantage of this method is clear: in order to obtain one equal tempered interval, one must tune eight intervals by eliminating beats. The fifths and fourths are not so hard, but tuning a major third by eliminating beats is considered difficult. This method of tuning equal temperament was discovered independently by John Farey[24] nearly twenty years later.

Alexander Ellis in his Appendix XX (Section G, Article 11) to Helmholtz [**48**] gives an easier practical rule for tuning in equal temperament. Namely, tune the notes in the octave above middle C by tuning fifths upwards and fourths downwards. Make the fifths perfect and then flatten them (make them more narrow) by one beat per second (cf. §1.7). Make the fourths perfect and then flatten them (make them wider) by three beats every two seconds. The result will be accurate to within two cents on every note. Having tuned one octave using this rule, tuning out beats for octaves allows the entire piano to be tuned.

It is desirable to apply spot checks throughout the piano to ensure that the fifths remain slightly narrow and the fourths slightly wide. Ellis states at the end of Article 11 that there is no way of distinguishing slightly narrow fourths or fifths from slightly wide ones using beats. In fact, there is a method, which was not yet known in 1885, as follows (Jorgensen [**58**], §227).



For the fifth, say C3–G3, compare the intervals C3–E♭3 and E♭3–G3. If the fifth is narrow, as desired, the first interval will beat more frequently than the second. If perfect, the beat frequencies will be equal. If wide, the second interval will beat more frequently than the first.

For the fourth, say G3–C4, compare the intervals C4–E♭4 and G3–E♭4, or compare E♭3–C4 and E♭3–G3. If the fourth is wide, as desired, the first interval will beat more frequently than the second. If perfect, the beat frequencies will be equal. If narrow, the second interval will beat more frequently than the first. This method is based on the observation that in equal temperament, the

[23]Johann Philipp Kirnberger, *Die Kunst des reinen Satzes in der Musik*, 2nd part 3rd division (Berlin, 1779), pp. 197f.

[24]John Farey, *On a new mode of equally tempering the musical scale*, Philosophical Magazine, XXVII (1807), 65–66.

major third is enough wider than a just major third, that gross errors would have to be made in order for it to have ended up narrower and spoil the test.

Exercises

1. Show that taking eleventh powers of the approximation of Kirnberger and Farey described in this section gives the approximation

$$2^{161} \approx 3^{84}\, 5^{12}.$$

The ratio of these two numbers is roughly 1.000008873, and the eleventh root of this is roughly 1.000000806.

2. Use the ideas of §4.6 to construct a spectrum which is close to the usual harmonic spectrum, but in such a way that the twelve tone equal tempered scale has consonant major thirds and fifths, as well as consonant seventh harmonics.

3. Calculate the accuracy of the method of Alexander Ellis for tuning equal temperament, described in this section.

4. Draw up a table of scale degrees in cents for the twelve notes in the Pythagorean, just, meantone and equal scales.

5. (Serge Cordier's equal temperament for piano with perfect fifths) Serge Cordier formalized a technique for piano tuning in the tradition of Pleyel (France). Cordier's recipe is as follows.[25] Make the interval F–C a perfect fifth, and divide it into seven equal semitones. Then use perfect fifths to tune from these eight notes to the entire piano.

Show that this results in octaves which are stretched by one seventh of a Pythagorean comma. This is of the same order of magnitude as the natural stretching of the octaves due to the inharmonicity of physical piano strings. Draw a diagram in Eitz's notation to demonstrate this temperament. This should consist of a horizontal strip with the top and bottom edges identified. Calculate the deviation of major and minor thirds from pure in this temperament.

Further reading:

Ian Stewart, *Another fine math you've got me into...*, W. H. Freeman & Co., 1992. Chapter 15 of this book, *The well tempered calculator*, contains a description of some of the history of practical approximations to equal temperament. Particularly interesting is his description of Strähle's method of 1743.

5.15. Historical remarks

The word *music* ($\mu o \upsilon \sigma \iota \kappa \acute{\eta}$) in ancient Greece had a wider meaning than it does for us, embracing the idea of ratios of integers as the key to understanding both the visible physical universe and the invisible spiritual universe.

It should not be supposed that the Pythagorean scale discussed in §5.2 was the main one used in ancient Greece in the form described there. Rather, this scale is the result of applying the Pythagorean ideal of using only the

[25] Serge Cordier, *L'accordage des instruments à claviers.* Bulletin du Groupe Acoustique Musicale (G. A. M.) **75** (1974), Paris VII; *Piano bien tempéré et justesse orchestrale*, Buchet-Chastel, Paris 1982.

ratios 2:1 and 3:2 to build the intervals. The Pythagorean scale as we have presented it first occurs in Plato's *Timaeus*, and was used in medieval Europe from about the eighth to the fourteenth century A.D.

The *diatonic syntonon* of Ptolemy is the same as the major scale of just intonation, with the exception that the classical Greek octave was usually taken to be made up of two Dorian[26] tetrachords, E–F–G–A and B–C–D–E, as described below, so that C was not the tonal center. It should be pointed out that Ptolemy recorded a long list of Greek diatonic tunings, and there is no reason to believe that he preferred the diatonic sytonic scale to any of the others he recorded.

The point of the Greek tunings was the construction of tetrachords, or sequences of four consecutive notes encompassing a perfect fourth; the ratio of 5:4 seems to have been an incidental consequence rather than representing a recognized consonant major third.

A Greek scale consisted of two tetrachords, either in *conjunction*, which means overlapping (for example two Dorian tetrachords B–C–D–E and E–F–G–A) or in *disjunction*, which means non-overlapping (for example E–F–G–A and B–C–D–E) with a whole tone as the gap. The tetrachords came in three types, called *genera* (plural of *genus*), and the two tetrachords in a scale belong to the same genus. The first genus is the *diatonic genus* in which the lowest interval is a semitone and the two upper ones are tones. The second is the *chromatic genus* in which the lowest two intervals are semitones and the upper one is a tone and a half. The third is the *enharmonic genus* in which the lowest two intervals are quarter tones and the upper one is two tones. The exact values of these intervals varied somewhat according to usage.[27] The interval between the lowest note and the higher of the two movable notes of a chromatic or enharmonic tetrachord is called the *pyknon*, and is always smaller than the remaining interval at the top of the tetrachord.

Little is known about the harmonic content, if any, of European music prior to the decline of the Roman Empire. The music of ancient Greece, for example, survives in a small handful of fragments, and is mostly melodic in nature. There is no evidence of continuity of musical practice from ancient Greece to medieval European music, although the theoretical writings had a great deal of impact.

Harmony, in a primitive form, seems to have first appeared in liturgical plainchant around 800 A.D., in the form of *parallel organum*, or melody in parallel fourths and fifths. Major thirds were not regarded as consonant, and a Pythagorean tuning system works perfectly for such music.

[26]Dorian tetrachords should not be confused with the Dorian mode of medieval church music, which is D–E–F–G–A–B–C–D.

[27]For example, Archytas described tetrachords using the ratios 1:1, 28:27, 32:27, 4:3 (diatonic), 1:1, 28:27, 9:8, 4:3 (chromatic) and 1:1, 28:27, 16:15, 4:3 (enharmonic), in which the primes 2, 3, 5 and 7 appear. Plato, his contemporary, does not allow primes other than 2 and 3, in better keeping with the Pythagorean tradition.

Polyphonic music started developing around the eleventh century A.D. Pythagorean intonation continued to be used for several centuries, and so the consonances in this system were the perfect fourths, fifths and octaves. The major third was still not regarded as a consonant interval, and it was something to be used only in passing.

The earliest known advocates of the 5:4 ratio as a consonant interval are the Englishmen Theinred of Dover (twelfth century) and Walter Odington (fl. 1298–1316),[28] in the context of early English polyphonic music. One of the earliest recorded uses of the major third in harmony is the four part vocal canon *sumer is icumen in*, of English origin, dating from around 1250. But for keyboard music, the question of tuning delayed its acceptance.

The consonant major third traveled from England to the European continent in the fourteenth century, where Guillaume Dufay was one the first composers to use it extensively.

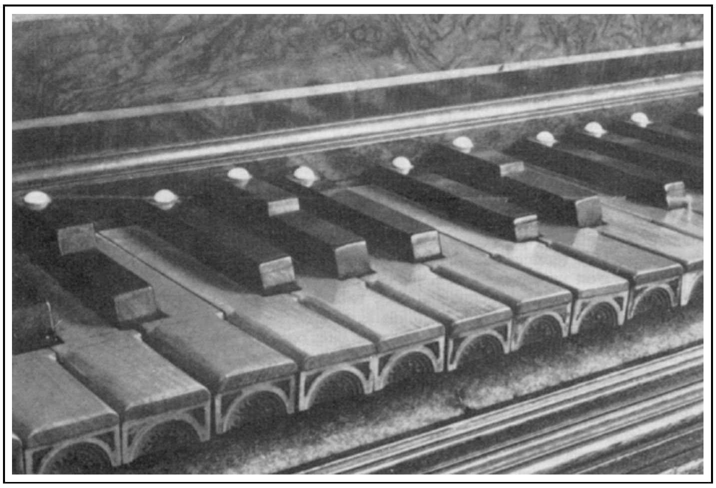
The method for obtaining consonant major thirds in fourteenth and fifteenth century keyboard music is interesting. Starting with a series of Pythagorean fifths

$$\mathrm{G}\flat^{0} - \mathrm{D}\flat^{0} - \mathrm{A}\flat^{0} - \mathrm{E}\flat^{0} - \mathrm{B}\flat^{0} - \mathrm{F}^{0} - \mathrm{C}^{0} - \mathrm{G}^{0} - \mathrm{D}^{0} - \mathrm{A}^{0} - \mathrm{E}^{0} - \mathrm{B}^{0},$$

the triad $\mathrm{D}^{0} - \mathrm{G}\flat^{0} - \mathrm{A}^{0}$ is used as a major triad. A just major triad would be $\mathrm{D}^{0} - \mathrm{F}\sharp^{-1} - \mathrm{A}^{0}$, and the difference between $\mathrm{F}\sharp^{-1}$ and $\mathrm{G}\flat^{0}$ is one schisma, or 1.953 cents. This is much more consonant than the modern equal temperament, in which the major thirds are impure by 13.686 cents. Other major triads available in this system are $\mathrm{A}^{0} - \mathrm{D}\flat^{0} - \mathrm{E}^{0}$ and $\mathrm{E}^{0} - \mathrm{A}\flat^{0} - \mathrm{B}^{0}$, but the system does not include a consonant C – E – G triad.

By the mid to late fifteenth century, especially in Italy, many aspects of the arts were reaching a new level of technical and mathematical precision. Leonardo da Vinci was integrating the visual arts with the sciences in revolutionary ways. In music, the meantone temperament was developed around this time, allowing the use of major and minor triads in a wide range of keys, and allowing harmonic progressions and modulations which had previously not been possible.

[28] The "fl." indicates that these are the years in which he is known to have flourished.



Many keyboard instruments from the sixteenth century have split keys for one or both of G♯/A♭ and D♯/E♭ to extend the range of usable key signatures. This was achieved by splitting the key across the middle, with the back part higher than the front part. The above picture shows the split keys of the Malamini organ in San Petronio, Bologna, Italy.[29]

The practice for music of the sixteenth and seventeenth century was to choose a tonal center and gradually move further away. The furthest reaches were sparsely used, before gradually moving back to the tonal center.

Exact meantone tuning was not achieved in practice before the twentieth century, for lack of accurate prescriptions for tuning intervals. Keyboard instrument tuners tended to color the temperament, so that different keys had slightly different sounds to them. The irregular temperaments of §5.13 took this process further, and to some extent formalized it.

An early advocate for equal temperament for keyboard instruments was Rameau (1730). This helped it gain in popularity, until by the early nineteenth century it was fairly widely used, at least in theory. However, much of Beethoven's piano music is best played with an irregular temperament (see §5.13), and Chopin was reluctant to compose in certain keys (notably D minor) because their characteristics did not suit him. In practice, equal temperament did not really take full hold until the end of the nineteenth century. Nineteenth century piano tuning practice often involved slight deviation from equal temperament in order to preserve, at least to some extent, the individual characteristics of the different keys. In the twentieth century, the dominance of chromaticism and the advent of twelve tone music have pretty much forced the abandonment of unequal temperaments, and piano tuning practice has reflected this.

Equal temperament is an essential ingredient in twentieth century twelve tone music, where combinatorics and chromaticism seem to supersede harmony. Some interesting evidence that harmonic content is irrelevant in

[29]Picture borrowed from page 302 of Devie [**29**].

Schoenberg's music is that the performance version of one of his most popular works, *Pierrot Lunaire*, contained many transcription errors confusing sharps, naturals and flats, until it was reedited for his collected works in the eighties.

The mathematics involved in twelve tone music of the twentieth century is different in nature to most of the mathematics we have described so far. It is more combinatorial in nature, and involves discussions of subsets of the twelve tones, and permutations, of the twelve tones of the chromatic scale. We shall have more to say on this subject in Chapter 9.

Further reading:

1) **Ancient Greek music**

W. D. Anderson, *Music and musicians in ancient Greece*, Cornell University Press, 1994; paperback edition 1997.

Giovanni Comotti, *Music in Greek and Roman culture*, Johns Hopkins University Press, 1989; paperback edition 1991.

John G. Landels, *Music in ancient Greece and Rome*, Routledge, 1999; paperback edition 2001.

M. L. West, *Ancient Greek music*, Oxford University Press, 1992; paperback edition 1994. Chapter 10 of this book reproduces all 51 known fragments of ancient Greek music.

R. P. Winnington-Ingram, *Mode in ancient Greek music*, Cambridge University Press, 1936. Reprinted by Hakkert, Amsterdam, 1968.

2) **Medieval to modern music**

Gustave Reese, *Music in the middle ages*, Norton, 1940, reprinted 1968. Despite the age of this text, it is still regarded as an invaluable source because of the quality of the scholarship. But the reader should bear in mind that much information has come to light since it appeared.

D. J. Grout and C. V. Palisca, *A history of western music*, fifth edition, Norton, 1996. Originally written in the 1950s by Grout, and updated a number of times by Palisca. This is a standard text used in many music history departments.

3) **Nineteenth century intonation**

Jorgensen, *Tuning* [**58**].

4) **Twelve tone music**

Forte, *The structure of atonal music* [**35**].

Perle, *Twelve tone tonality* [**90**].

5) **Translations of original sources**

Andrew Barker, *Greek Musical Writings, Vol. 2: Harmonic and acoustic theory*, Cambridge University Press, 1989. This 581 page book contains translations and commentaries on many of the most important ancient Greek sources, including Aristoxenus' *Elementa Harmonica*, the Euclidean *Sectio Canonis*, Nicomachus' *Enchiridion*, Ptolemy's *Harmonics*, and Aristides Quintilianus' *De Musica*.

Leo Treitler, *Strunk's Source Readings in Music History*, Revised Edition, Norton & Co., 1998. This 1552 page book, originally by Strunk but revised extensively by Treitler, contains translations of historical documents from ancient Greece to the twentieth century. It comes in seven sections, which are available in separate paperbacks.

5.16. The role of the synthesizer

Before the days of digital synthesizers, we had a choice of three different versions of the compromise. The just scales have perfect intervals, but do not allow us to modulate far from the original key, and have problems with syntonic commas interfering in fairly short harmonic sequences. Meantone scales sacrifice a little perfection in the fifths in order to remove the problem of the syntonic comma, but still have a problem with keys far removed from the original key, and with enharmonic modulations. Equal temperament works in all keys equally well, or rather, one might say equally badly. In particular, the equal tempered major third is nervous and agitated.

In these days of digitally synthesized and controlled music, there is very little reason to make do with the equal tempered compromise, because we can retune any note by any amount as we go along. It may still make sense to prefer a meantone scale to a just one on the grounds of interference of the syntonic comma, but it may also make sense to turn the situation around and use the syntonic comma for effect.

It seems that for most users of synthesizers the extra freedom has not had much effect, in the sense that most music involving synthesizers is written using the equal tempered twelve tone scale. A notable exception is Wendy Carlos, who has composed a great deal of music for synthesizers using many different scales. I particularly recommend *Beauty in the Beast*, which has been released on compact disc (SYNCD 200, Audion, 1986, Passport Records, Inc.). For example the fourth track, called *Just Imaginings*, uses a version of just intonation with harmonics all the way up to the nineteenth, and includes some deft modulations. Other tracks use other scales, including Carlos' *alpha* and *beta* scales and the Balinese gamelan *pelog* and *slendro*.

Wendy Carlos' earlier recordings, *Switched on Bach* and *The Well Tempered Synthesizer*, were recorded on a Moog synthesizer fixed in equal temperament. But when *Switched on Bach 2000* came out in 1992, twenty-five years after the original, it made use of a variety of meantone and unequal temperaments. It is not hard to hear from this recording the difference in clarity between these and equal temperament.

Further reading:

Easley Blackwood, *Discovering the microtonal resources of the synthesizer*, Keyboard, May 1982, 26–38.

Benjamin Frederick Denckla, *Dynamic intonation in synthesizer performance*, M.Sc. Thesis, MIT, 1997 (61 pp).

Henry Lowengard, *Computers, digital synthesizers and microtonality*, Pitch 1 (1) (1986), 6–7.

Robert Rich, *Just intonation for MIDI synthesizers*, Electronic Musician, Nov 1986, 32–45.

M. Yunik and G. W. Swift, *Tempered music scales for sound synthesis*, Computer Music Journal 4 (4) (1980), 60–65.