An Overview of Mathematical General Relativity

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Outline

- Lorentzian manifolds
- Einstein's equation
- The Schwarzschild solution
- Initial value formulation & existence theorems
- Singularity theorems
- Mass positivity & Penrose's inequality

Lorentzian manifolds

- A **Lorentzian manifold** is a pair (M, g), where M is a manifold and g is a nondegenerate symmetric 2-tensor with signature $(- + \ldots +)$.

- Example: the analogue of Euclidean space is the so-called **Minkowski spacetime**:

$$M = \mathbb{R}^{n+1}, \quad g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n$$

- Many things are the same as for Riemannian manifolds: for instance, there exists a unique Levi-Civita connection ∇.

- **Many things are different:**

- Vectors v come in 3 types:

  1. **Timelike**: $g(v, v) < 0$

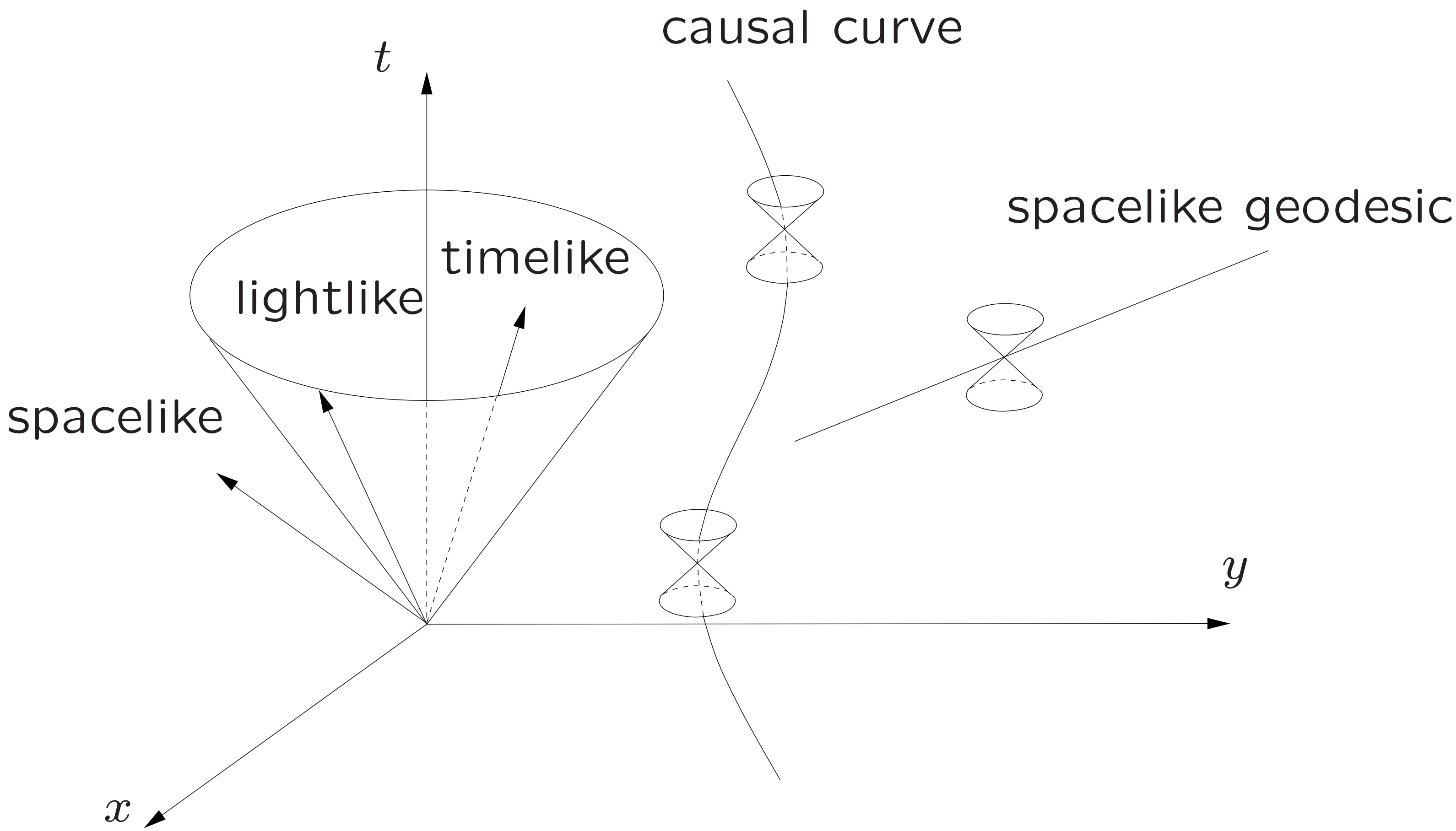
  2. **Lightlike**: $g(v, v) = 0$

  3. **Spacelike**: $g(v, v) > 0$

  (v is **causal** if it is not spacelike).

- A curve $c : I \subset \mathbb{R} \to M$ is **timelike**, **spacelike**, **lightlike** or **causal** if $\dot{c}$ is. (If c is a geodesic then $g(\dot{c}, \dot{c})$ is constant).

(Minkowski spacetime)



- (M, g) is said to be **time orientable** if there exists a nonvanishing timelike vector field v. If M is time orientable then there are two possible choices of time orientation.

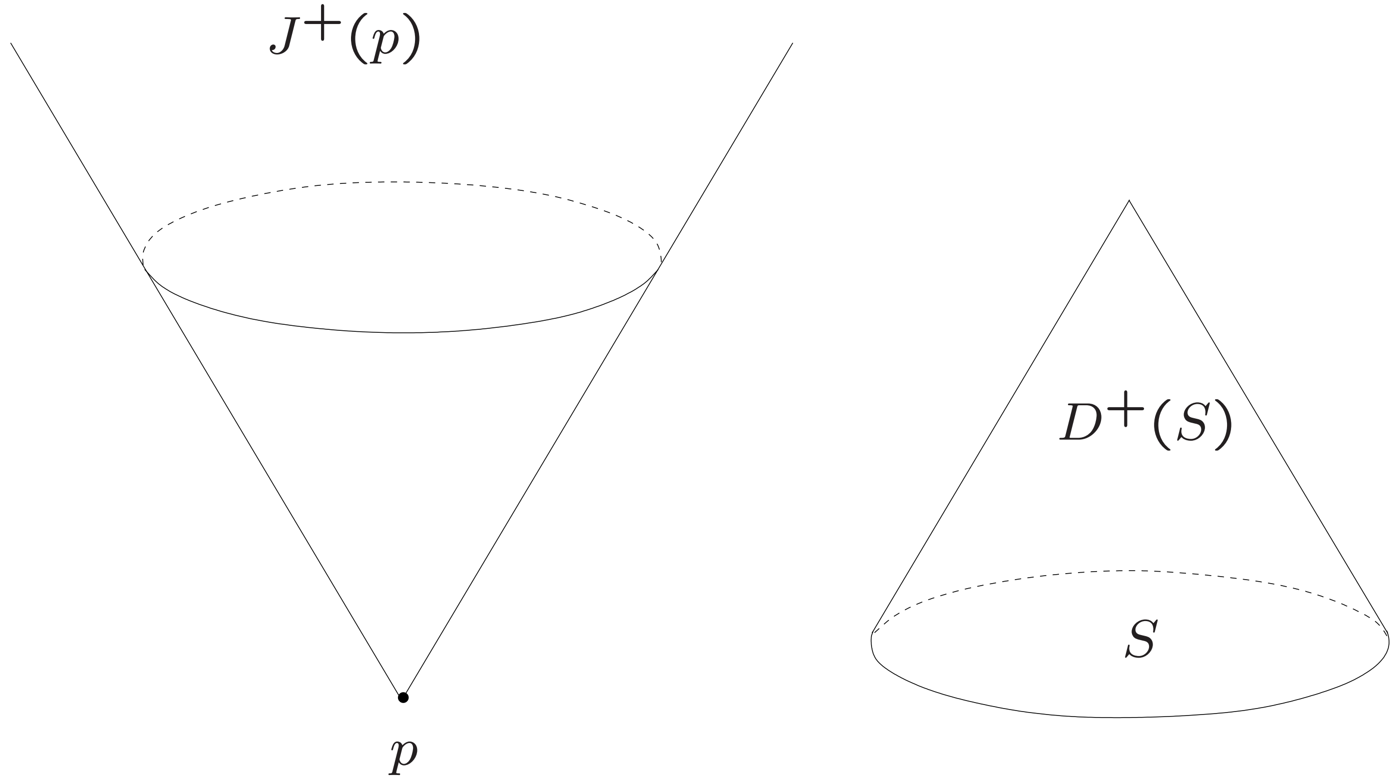
- The **causal future** of a set $S \subset M$ is

$$J^+(S) = \{p \in M \mid \exists \text{ future-directed causal curve } c : [0, 1] \to M \text{ with } c(0) \in S \text{ and } c(1) = p\}$$

- The **future domain of dependence** of a set $S \subset M$ is

$$D^+(S) = \{p \in M \mid \text{every past-directed inextendible causal curve intersects S}\}$$

(Minkowski spacetime)

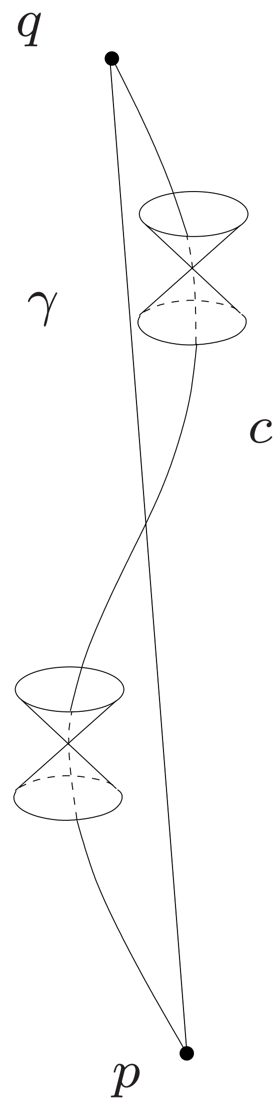


- Not all manifolds admit a Lorentzian structure (e.g. S^{2n}).

- Metric does not provide distance function.

- Geodesics do not minimize length. Actually, timelike geodesics **maximize** length (**proper time**) among causal curves (**twin paradox**).

$$\tau(\gamma) > \tau(c)$$



- **Interpretation:**

  1. $(M, g) =$ spacetime with gravitational field

  2. Points $=$ events

  3. Timelike curves $=$ histories of material point particles

  4. Arclength of timelike curve $=$ (proper) time measured by material point particle

  5. Timelike geodesics $=$ histories of free-falling material point particles

  6. Lightlike geodesics $=$ histories of light rays

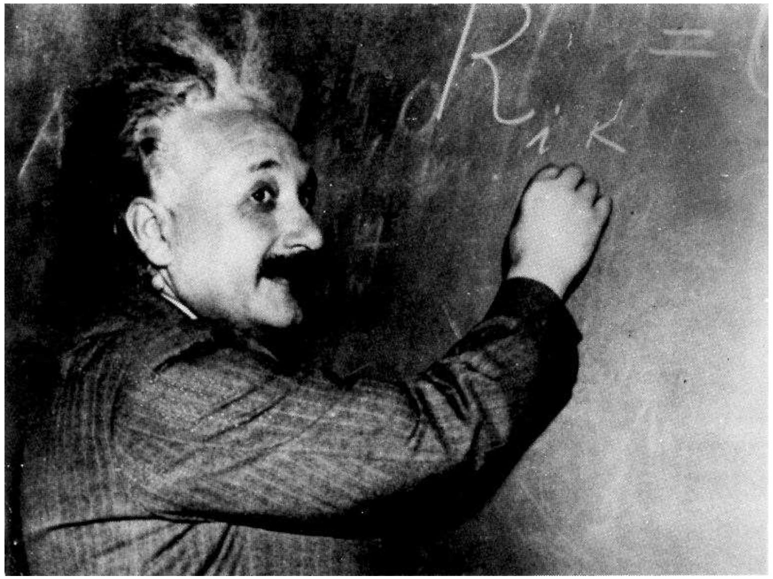
7. Causal future of a point $=$ set of all events which can be influenced by the given event

8. Future domain of dependence of S $=$ set of all events which depend only on what happened at S

9. Twin paradox $=$ free falling observer always measures the longer time

Einstein's equation

- Vacuum Einstein equation (Einstein, 1915): $Ricci = 0$



- Hilbert-Einstein action (Hilbert, 1916): $S = \int_M \mathrm{tr}\, Ricci\; dV_4$

- For a family of metrics $g = g(\lambda)$, $\frac{dg}{d\lambda}$ compactly supported, one has

$$\frac{dS}{d\lambda} = \int_M Einstein \cdot \frac{dg}{d\lambda}\, dV_4$$

where

$$Einstein = Ricci - \frac{1}{2}(\mathrm{tr}\, Ricci)\, g$$

In dimension greater then 2, $Einstein = 0 \Leftrightarrow Ricci = 0$. In dimension 2, $Einstein \equiv 0$ yields Gauss-Bonnet theorem.

The Schwarzschild solution

- If (M, g) satisfies Einstein equation and its isometry group contains $SO(3)$ then it must be (locally isometric to) the **Schwarzschild solution** (Schwarzschild, 1916)

$$g = -\left(1 - \frac{2M}{r}\right) dt \otimes dt + \left(1 - \frac{2M}{r}\right)^{-1} dr \otimes dr$$
$$+ r^2 \left(d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi\right)$$

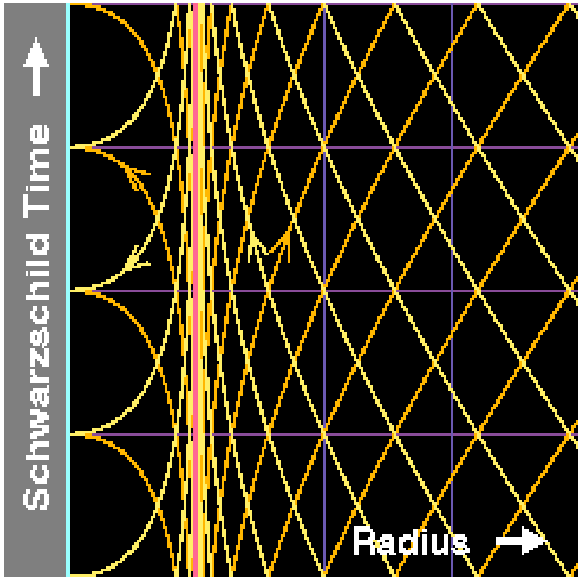
- Interpreted as the gravitational field generated by a point mass M placed at $r = 0$.

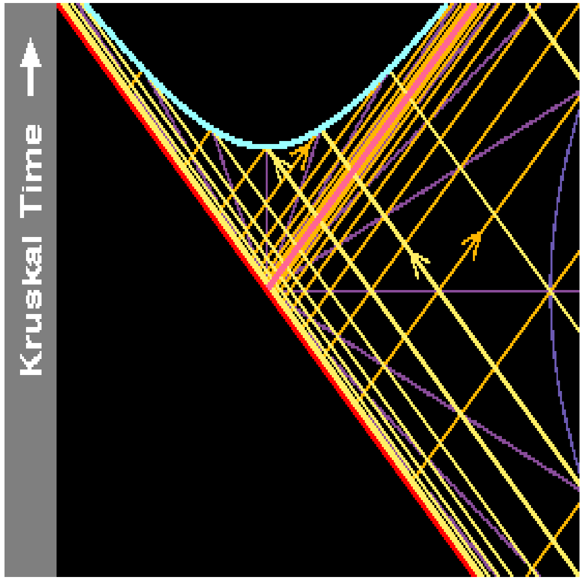
- Curvature invariants diverge at $r = 0$ (**singularity**).

- Isometry group $= \mathbb{R} \times SO(3)$, with Killing vector field $\frac{\partial}{\partial t}$ (**stationary**) and $\frac{\partial}{\partial \varphi}, \ldots$ (**spherically symmetric**).

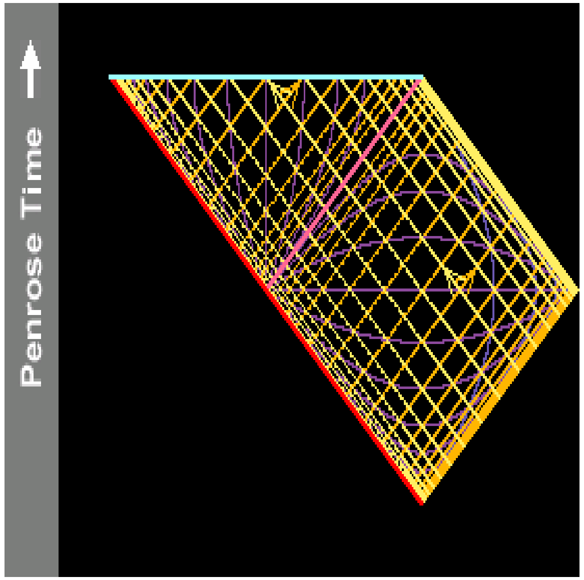
- Clearly something strange is going on at the **horizon** $r = 2M$: in fact the full solution describes a **black hole**. Once you cross the horizon you cannot avoid the singularity more than you can avoid next Monday.

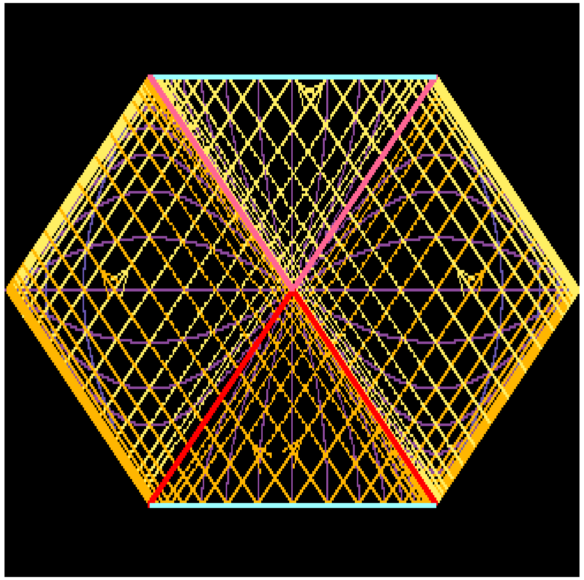
- Here is a slice of constant (θ, φ) in Schwarzschild coordinates (t, r) and in **Kruskal coordinates** (1960):



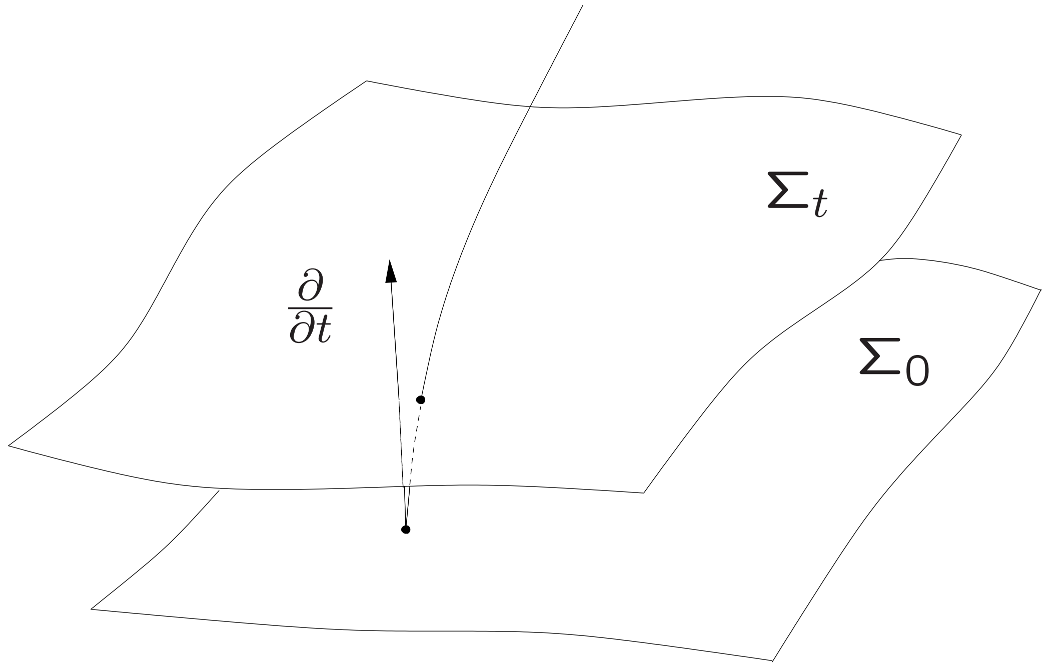


• The same slice as a **Penrose conformal diagram** and the corresponding slice of the **maximal analytical extension**:





Initial value formulation & existence theorems



- Take **spacelike** hypersurface $\Sigma_0 \subset M$ and move it a distance t along normal geodesics; the hypersurface $\Sigma_t \subset M$ thus obtained is still spacelike, and the restriction γ_t of g to Σ_t is a **Riemannian** metric.

- Knowing γ_t is equivalent to knowing $g = -dt \otimes dt + \gamma_t$.

- $K_t = \frac{1}{2}\pounds_{\frac{\partial}{\partial t}}\gamma_t$ is called the **extrinsic curvature**.

- Using the **Gauss-Codazzi relations** one can write Einstein's equation $Ricci = 0$ as

$$\begin{aligned} &\mathrm{tr}\, Ricci_\Sigma + (\mathrm{tr}\, K)^2 - \mathrm{tr}\, K^2 = 0 \\ &\mathrm{div}\, K - d\,(\mathrm{tr}\, K) = 0 \end{aligned}$$

(**constraint equations**, elliptic) plus

$$\pounds_{\frac{\partial}{\partial t}} K = -Ricci_\Sigma + 2K^2 - (\mathrm{tr}\, K)\, K$$

(**evolution equation**, hyperbolic).

- **Ricci flow:** $K = -Ricci_\Sigma$ (parabolic).

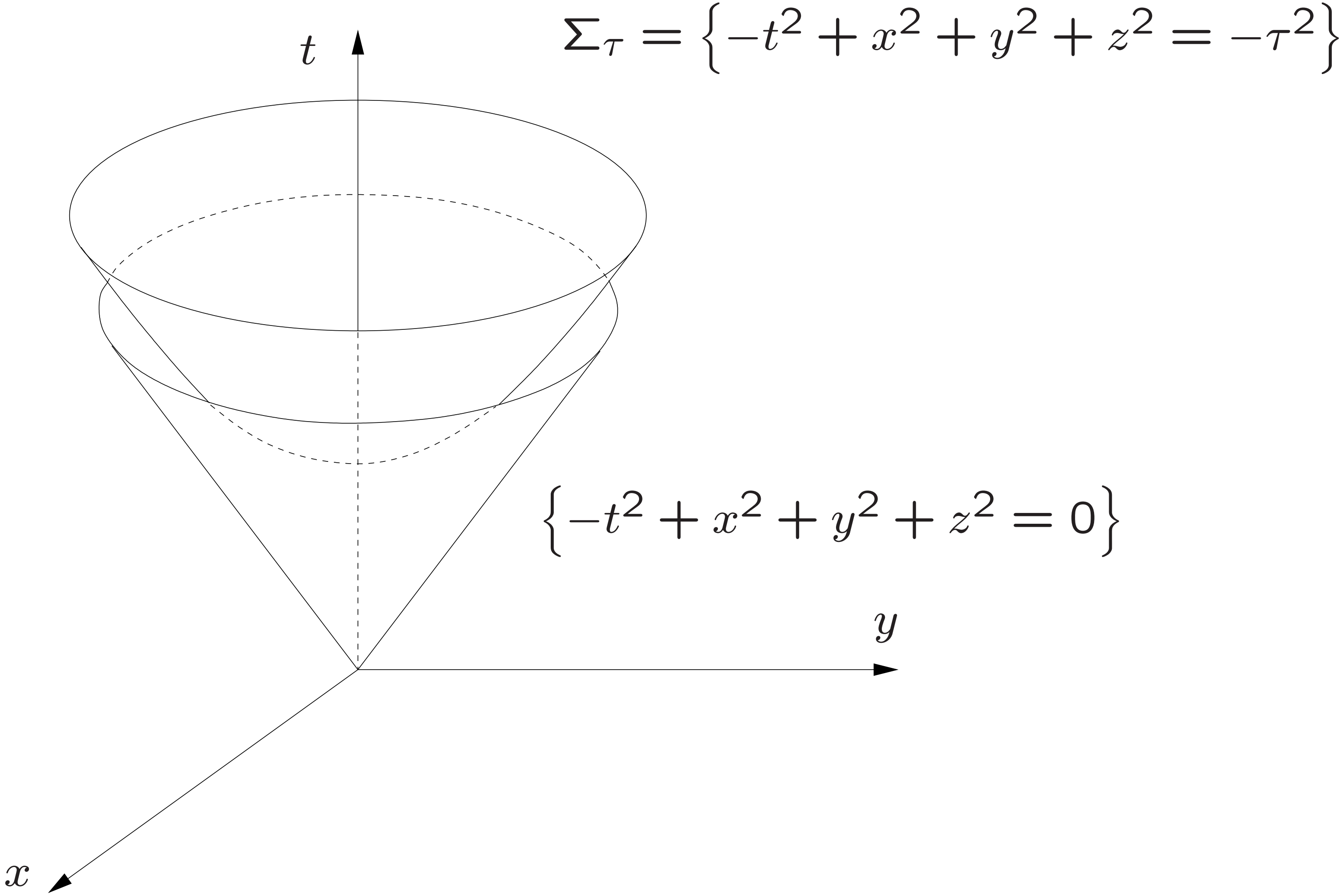
- **Theorem** (Choquet-Bruhat & Geroch, 1969): Given $(\Sigma_0, \gamma_0, K_0)$ satisfying the constraint equations, there exists a unique Lorentzian 4-manifold (M, g), called the **maximal Cauchy development**, satisfying:

1. (M, g) satisfies Einstein's equation;
2. Σ_0 is (diffeomorphic to) a spacelike hypersurface of M and $M = D^+(S) \cup D^-(S)$;
3. γ_0 and K_0 are the induced metric and extrinsic curvature of Σ_0;
4. Every other solution of the above is (isometric to) a subset of (M, g).

Moreover, if two sets of data agree on an open set S then their maximal Cauchy developments agree on $D^+(S) \cup D^-(S)$. Finally, one has continuous dependence of initial data for appropriate topologies.

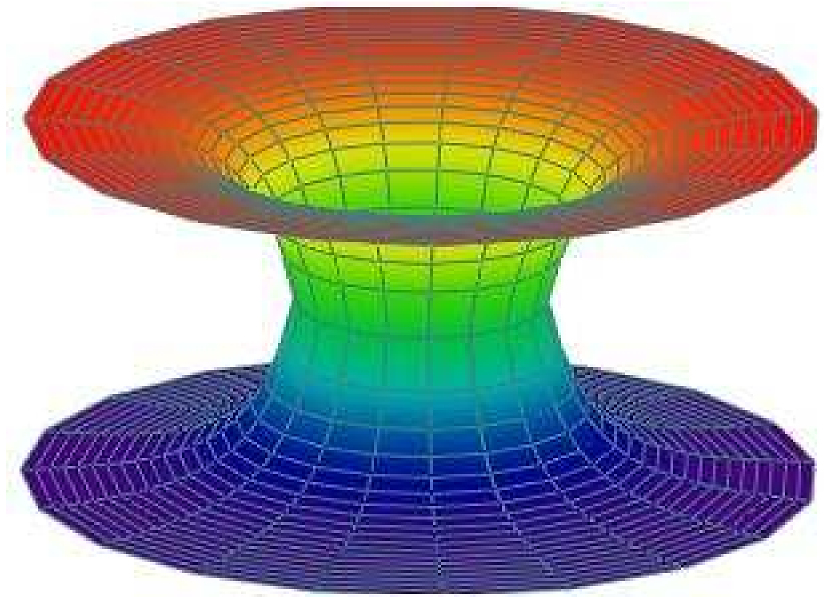
- There are known examples where maximal Cauchy development is **extendible**, i.e., isometric to an open strict subset of a solution of Einstein equation. The **cosmic censorship conjecture** means to exclude this for appropriate initial data.

- Example: $\gamma_t = t^2 \gamma_{\mathbb{H}}$ (**Milne universe**). Singular at $t = 0$ (**big bang**).



$\Sigma_\tau = \left\{ -t^2 + x^2 + y^2 + z^2 = -\tau^2 \right\}$
$\left\{ -t^2 + x^2 + y^2 + z^2 = 0 \right\}$
t
y
x

- Initial data for Schwarzschild: $K = 0$ and $(\Sigma_0, \gamma_0) =$



Singularity theorems

- $\pounds_{\frac{\partial}{\partial t}} dV_3 = (\mathrm{tr}\, K)\, dV_3$ (minimal hypersurfaces: $\mathrm{tr}\, K = 0$).

- Evolution equation $\Rightarrow \frac{\partial}{\partial t}(\mathrm{tr}\, K) + \mathrm{tr}\, K^2 = 0$.

- $(\mathrm{tr}\, K)^2 \leq 3\, \mathrm{tr}\, K^2 \Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{\mathrm{tr}\, K}\right) \geq \frac{1}{3}$.

- Therefore either a **coordinate singularity** (e.g. Milne) or a **geometric singularity** (e.g. Schwarzschild) develops.

- (M, g) is **singular** if it is **not geodesically complete** (sometimes only for causal geodesics).

- **Singularity theorems** (Hawking & Penrose, 1970): A generic solution (M, g) of Einstein's equation is singular if either:

  1. Is spatially compact.

  2. Contains a **trapped surface**.

- **Proof:**

1. Use exponential map to build foliation by future geodesic hyperboloids around point p.

2. Use conditions to prove that $\operatorname{tr} K < -\varepsilon < 0$ at some hyperboloid.

3. Use $\frac{\partial}{\partial t}\left(\frac{1}{\operatorname{tr} K}\right) \geq \frac{1}{3}$ to show that any geodesic must have a **conjugate point** after proper time $\frac{3}{\varepsilon}$.

4. Space $C(p,q)$ of continuous causal curves between p and q with appropriate topology is **compact** and proper time $\tau : C(p,q) \to \mathbb{R}$ is **upper semicontinuous**. Therefore τ must attain a maximum.

5. Maximum must be timelike geodesic with no conjugate points $\Rightarrow$ **cannot extend geodesics past time** $\frac{3}{\varepsilon}$.

Mass positivity & Penrose's inequality

- For simplicity consider only Cauchy data $(\Sigma, \gamma, 0)$ for the so-called **time-symmetric** case. Then constraint equations are just $\mathrm{tr}\, Ricci_{\Sigma} = 0$ (general case with energy conditions: $\mathrm{tr}\, Ricci_{\Sigma} \geq 0$).

- The Riemannian 3-manifold (Σ, γ) is **asymptotically flat** if outside a compact region is diffeomorphic to $\mathbb{R}^3 - \overline{B_R(0)}$ with

$$\gamma = \sum_{i=1}^{3} \left(\delta_{ij} + \mathcal{O}\left(\frac{1}{r}\right)\right) dx^i \otimes dx^j$$

(plus appropriate decay of derivatives).

- From Hamiltonian formulation (Arnowitt, Deser & Misner, 1962) one defines the **ADM mass** of an asymptotically flat Riemannian 3-manifold as

$$m_{ADM} = \frac{1}{16\pi} \lim_{R\to+\infty} \int_{\partial B_R(0)} \sum_{i,j=1}^{3} \left(\frac{\partial \gamma_{ij}}{\partial x^i} - \frac{\partial \gamma_{ii}}{\partial x^j} \right) n^j dV_2$$

- Schwarzschild: $m_{ADM} = M$.

- **Theorem** (Schoen & Yau, 1979): $m_{ADM} \geq 0$, with equality holding exactly for the Euclidean 3-space.

- Proved in the general (non-time-symmetric) case. Greatly simplified proof by Witten (1981).

- An **apparent horizon** is a compact surface $H \subset \Sigma$ such that $\mathrm{tr}\,\kappa = \mathrm{tr}\,(K|_H)$, where κ is the extrinsic curvature of $H \subset \Sigma$. In other words, the expansion of H as it moves out at the speed of light is totally accounted for by the expansion of Σ.

- For the time-symmetric case, an apparent horizon H is simply a **minimal surface**.

- **Theorem** (**Penrose inequality**, Huisken & Ilmanen, 1997): $m_{ADM} \geq \sqrt{\frac{V_2(H)}{16\pi}}$, with equality holding exactly for Schwarzschild.

- **Proof:**

  1. $m_{\text{Hawking}}(H) = \sqrt{\frac{V_2(H)}{16\pi}}\left(1 - \frac{1}{16\pi}\int_H \text{tr}\,\kappa\; dV_2\right)$

  2. Flow H by inverse mean curvature flow, i.e, flow of
  $$\frac{n}{\text{tr}\,\kappa}$$
  (where n is the unit normal vector). **Singularities!**

  3. $\frac{d}{dt}\left(m_{\text{Hawking}}(H(t))\right) \geq 0$

  4. $\lim_{t\to+\infty} m_{\text{Hawking}} = m_{ADM}$

- Also proved for multiple horizons (Bray, 1999), but **not** in the general (non-time-symmetric) case.

What I haven't told you about

- Energy conditions and their role on singularity theorems and mass positivity;
- Black hole uniqueness theorems (Israel, Carter, Hawking & Robinson, 1967 – 1975);
- Nonlinear stability of Minkowski spacetime (Christodoulou & Kleinerman, 1995; Lindblad & Rodnianski, 2003);
- Low regularity solutions (Kleinerman & Rodnianski, 2003).

References

- Books:

1. Hawking & Ellis, *The large scale structure of space-time*, CUP (1973)
2. Wald, *General Relativity*, Chicago (1984)

- Review papers:

1. Andersson, *The global existence problem in General Relativity*, gr-qc/9911032
2. Beig & Chruściel, *Stationary black holes*, gr-qc/0502041
3. Bray & Chruściel, *The Penrose inequality*, gr-qc/0312047
4. Chruściel, *Recent results in mathematical relativity*, gr-qc/0411008
5. Friedrich & Rendall, *The Cauchy problem for the Einstein equations*, gr-qc/0002074
6. Huisken & Ilmanen, *Energy inequalities for isolated systems and hypersurfaces moving by their curvature*