

Relativistic elasticity of rigid rods and strings

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Based on [arXiv:1406.0634](https://arxiv.org/abs/1406.0634)

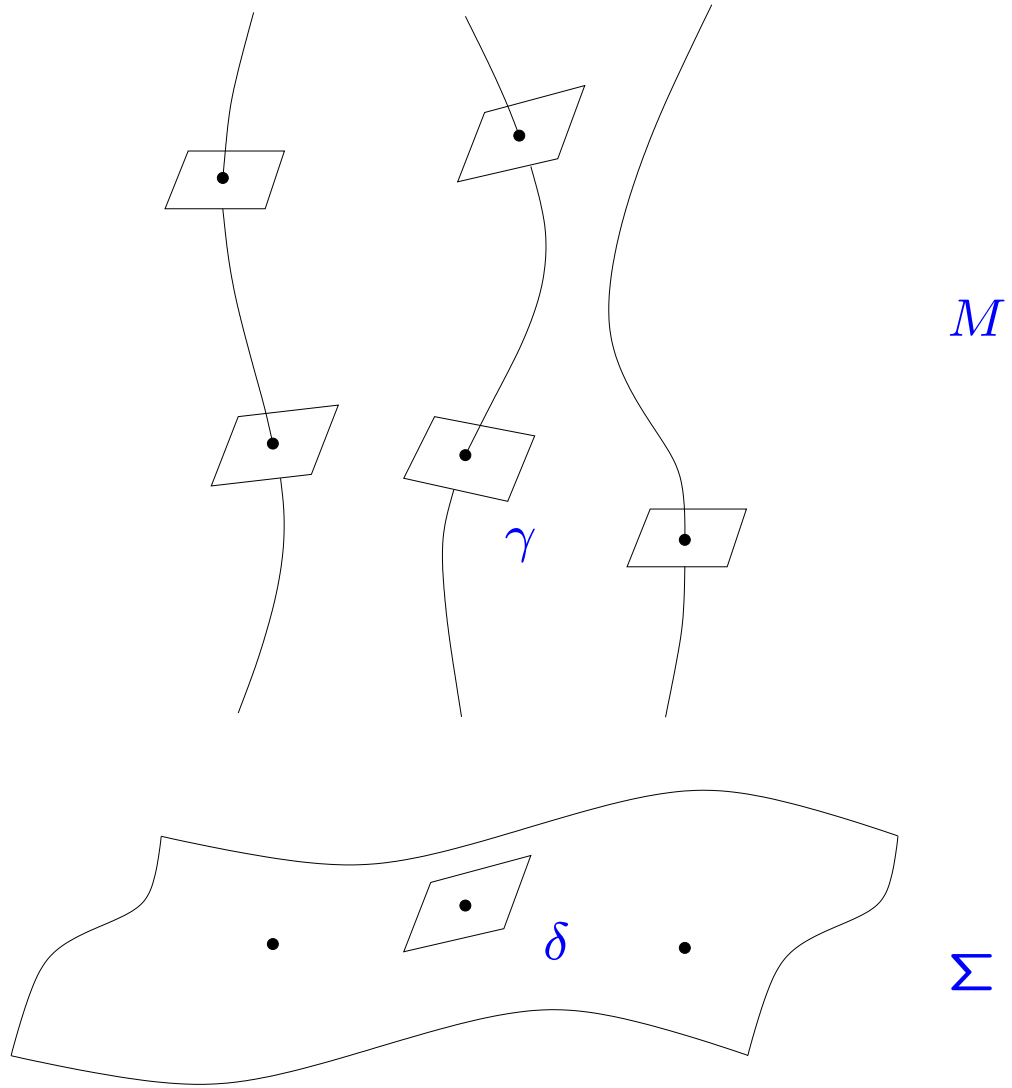
Aveiro, July 2014

Outline

- Relativistic elasticity
- Rigid rods and strings
- Car and garage paradox
- Pushing a rigid rod
- Bell's spaceships paradox
- Fishing in black holes

Relativistic elasticity

- A **continuous medium** in General Relativity is described by:
 - A spacetime (M, g) ;
 - A Riemannian 3-manifold (Σ, δ) (**relaxed configuration**);
 - A projection map $\pi : M \rightarrow \Sigma$ whose level sets are **timelike curves** (the **worldlines** of the medium particles).



- If we choose **local coordinates** $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ on Σ then we can think of π as a set of **three scalar fields**.
- We can complete $(\bar{x}^1, \bar{x}^2, \bar{x}^3)$ into coordinates $(\bar{t}, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ for (M, g) yielding the rest frame of any given worldline:

$$g = -d\bar{t}^2 + \gamma_{ij}d\bar{x}^i d\bar{x}^j \quad (\text{at that worldline}).$$

- Note that

$$\gamma = \gamma_{ij}d\bar{x}^i d\bar{x}^j$$

is a (time-dependent) Riemannian metric on Σ , describing the **local deformations** of the medium along each worldline.

- We can compute the (inverse) metric γ from

$$\gamma^{ij} = g^{\mu\nu} \frac{\partial \bar{x}^i}{\partial x^\mu} \frac{\partial \bar{x}^j}{\partial x^\nu}.$$

- We must choose a **Lagrangian density** \mathcal{L} for the action

$$S = \int_M \mathcal{L} \sqrt{-g} d^4x.$$

- Assume $\mathcal{L} = \mathcal{L}(\bar{x}^i, \gamma^{ij})$. The **energy-momentum** tensor is then

$$T_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L} g_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial \gamma^{ij}} \partial_\mu \bar{x}^i \partial_\nu \bar{x}^j - \mathcal{L} g_{\mu\nu}.$$

- Therefore

$$\mathcal{L} = T_{\bar{0}\bar{0}} = \rho$$

is the **rest energy density**.

- The choice of $\rho = \rho(\bar{x}^i, \gamma^{ij})$ is called the **elastic law**.
- **Isotropic materials**: ρ depends only on (s_1^2, s_2^2, s_3^2) , the eigenvalues of γ_{ij} with respect to δ_{ij} . Note that (s_1, s_2, s_3) are the **stretch factors** along the principal directions.
- Assume that δ_{ij} is the **Kronecker delta**. In particular, we are assuming that the Riemannian 3-manifold (Σ, δ) is **flat**.

- More convenient variables:

$$\lambda_0 = \det(\gamma^{ij}) = \frac{1}{(s_1 s_2 s_3)^2};$$

$$\lambda_1 = \text{tr}(\gamma^{ij}) = \frac{1}{s_1^2} + \frac{1}{s_2^2} + \frac{1}{s_3^2};$$

$$\lambda_2 = \text{tr cof}(\gamma^{ij}) = \frac{1}{(s_1 s_2)^2} + \frac{1}{(s_2 s_3)^2} + \frac{1}{(s_3 s_1)^2}.$$

- Examples:

- **Perfect fluid:** $\rho = \rho(\lambda_0)$, yielding $p = 2\lambda_0 \frac{d\rho}{d\lambda_0} - \rho$.

- **Dust:** $\rho = \rho_0 \sqrt{\lambda_0}$, yielding $p = 0$.

- Rigid fluid: $\rho = \frac{\rho_0}{2}(\lambda_0 + 1)$, yielding $p = \rho - \rho_0$.
- Stiff fluid: $\rho = A\lambda_0$, yielding $p = \rho$.
- John quasi-Hookean materials: $\rho = f(\lambda_0) + g(\lambda_0)\lambda_1$.
- Karlovini-Samuelsson quasi-Hookean materials: $\rho = f(\lambda_0) + g(\lambda_0)\lambda_1\lambda_2$.
- Stiff ultra-rigid equation of state: $\rho = A\lambda_2 + B$.
- Brotas rigid solid: $\rho = \frac{\rho_0}{8}(\lambda_0 + \lambda_1 + \lambda_2 + 1)$.

Rigid rods and strings

- For **one-dimensional** elastic bodies in a two-dimensional space-time (M, g) there is **no difference** between solids and fluids.
- **Caution:** these are **not** the strings of string theory – they have **internal structure**.
- The Lagrangian depends only on $\lambda_0 = \gamma^{11} = \partial_\alpha \bar{x} \partial^\alpha \bar{x}$.
- For a **rigid elastic body** (speed of sound = speed of light) we have $\rho = \frac{\rho_0}{2}(\lambda_0 + 1)$, yielding

$$T_{\mu\nu} = \rho_0 \left(\partial_\mu \bar{x} \partial_\nu \bar{x} - \frac{1}{2} \partial_\alpha \bar{x} \partial^\alpha \bar{x} g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \right).$$

- This is (essentially) just the energy-momentum tensor for a **massless scalar field**. So the equation of motion is just the **wave equation**:

$$\nabla^\mu T_{\mu\nu} = 0 \Leftrightarrow \square \bar{x} = 0.$$

- We can always find a **conjugated harmonic coordinate \bar{t}** such that

$$g = s^2 (-d\bar{t}^2 + d\bar{x}^2).$$

This provides an an interesting interpretation for **conformal coordinates** in two-dimensional spacetimes.

- **Static spacetimes:**

$$g = -e^{2\phi(x)} dt^2 + dx^2 = e^{2\phi(\bar{x})} (-dt^2 + d\bar{x}^2)$$

(e.g. hanging strings in the Schwarzschild spacetime).

- **Cosmological spacetimes:**

$$g = -dt^2 + a^2(t) dx^2 = a^2(\bar{t}) (-d\bar{t}^2 + dx^2)$$

(stretch factor equals the cosmological radius a).

Car and garage paradox

- Hitting a wall: before the collision

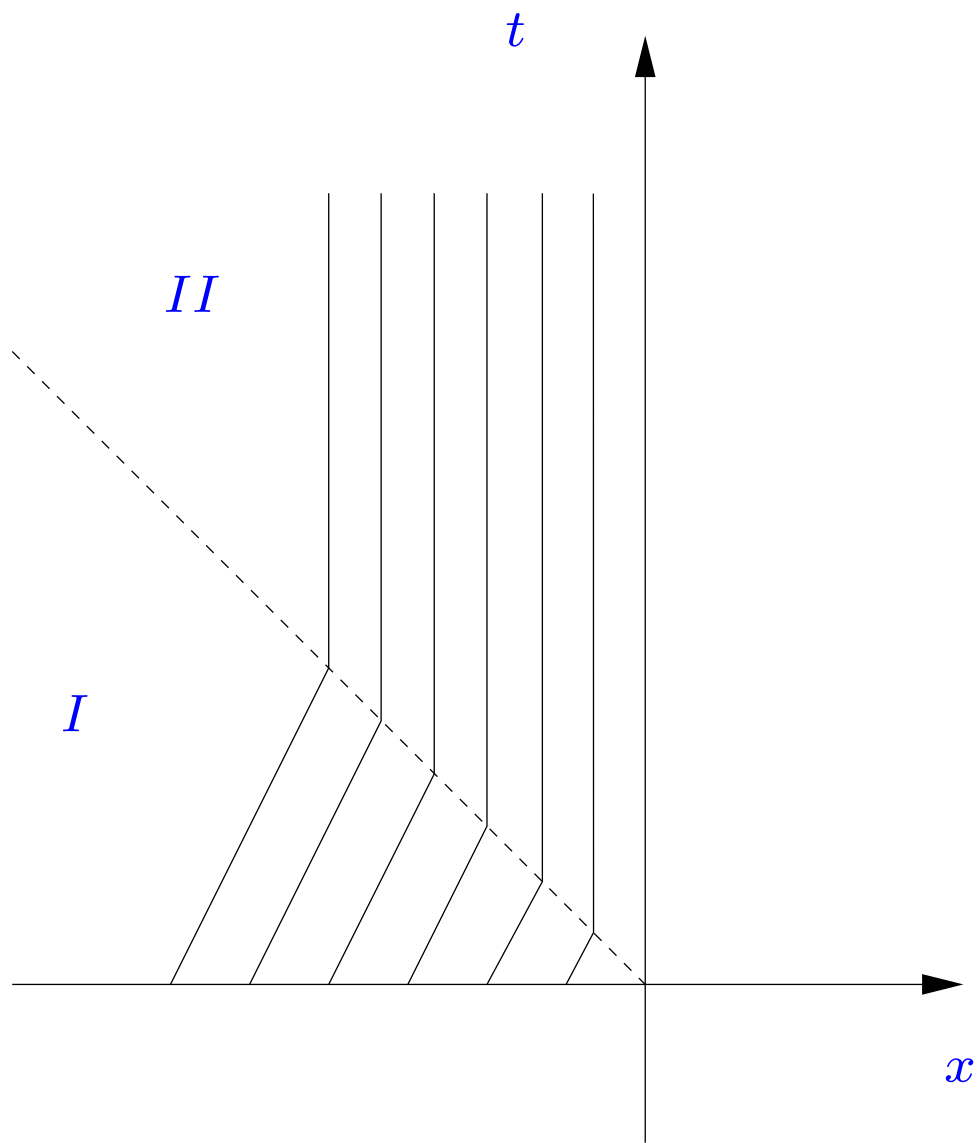
$$\bar{x} = \gamma(x - vt)$$

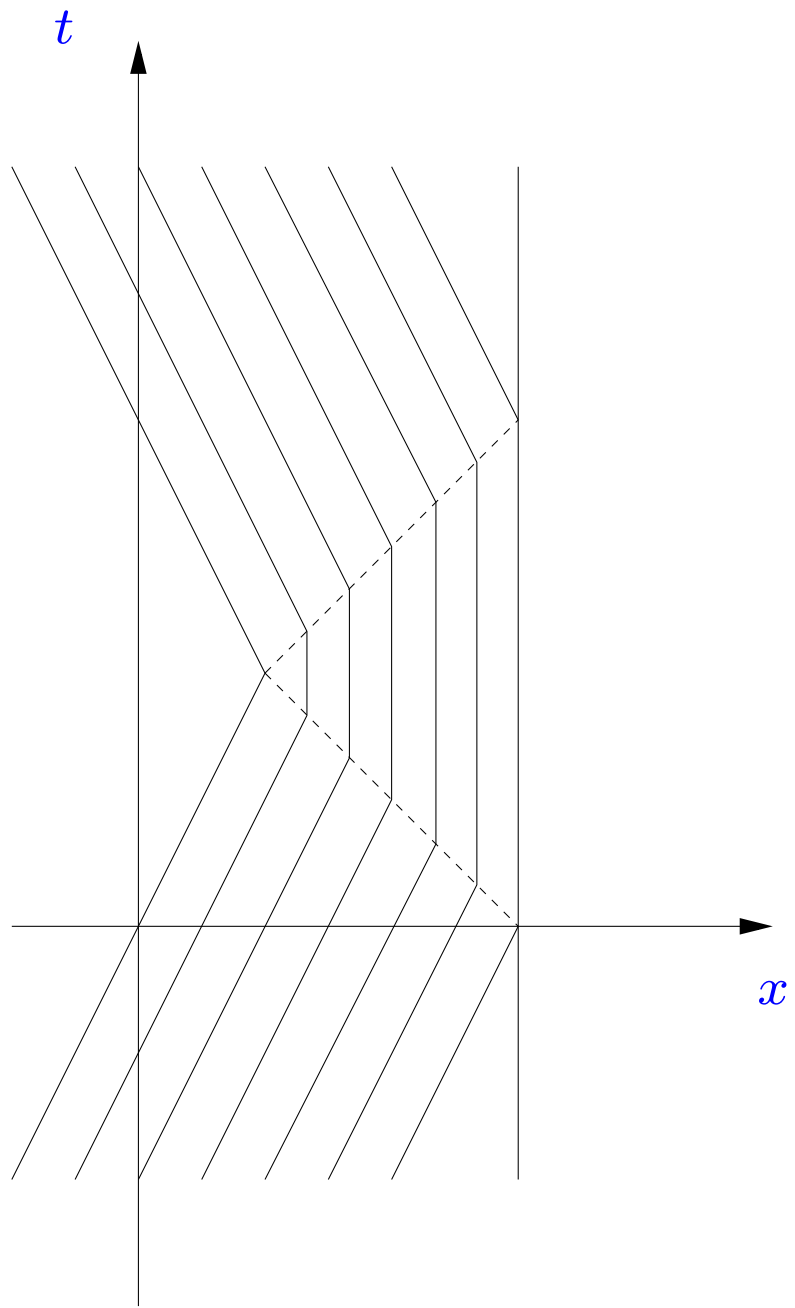
and so we have the **initial-boundary value problem**

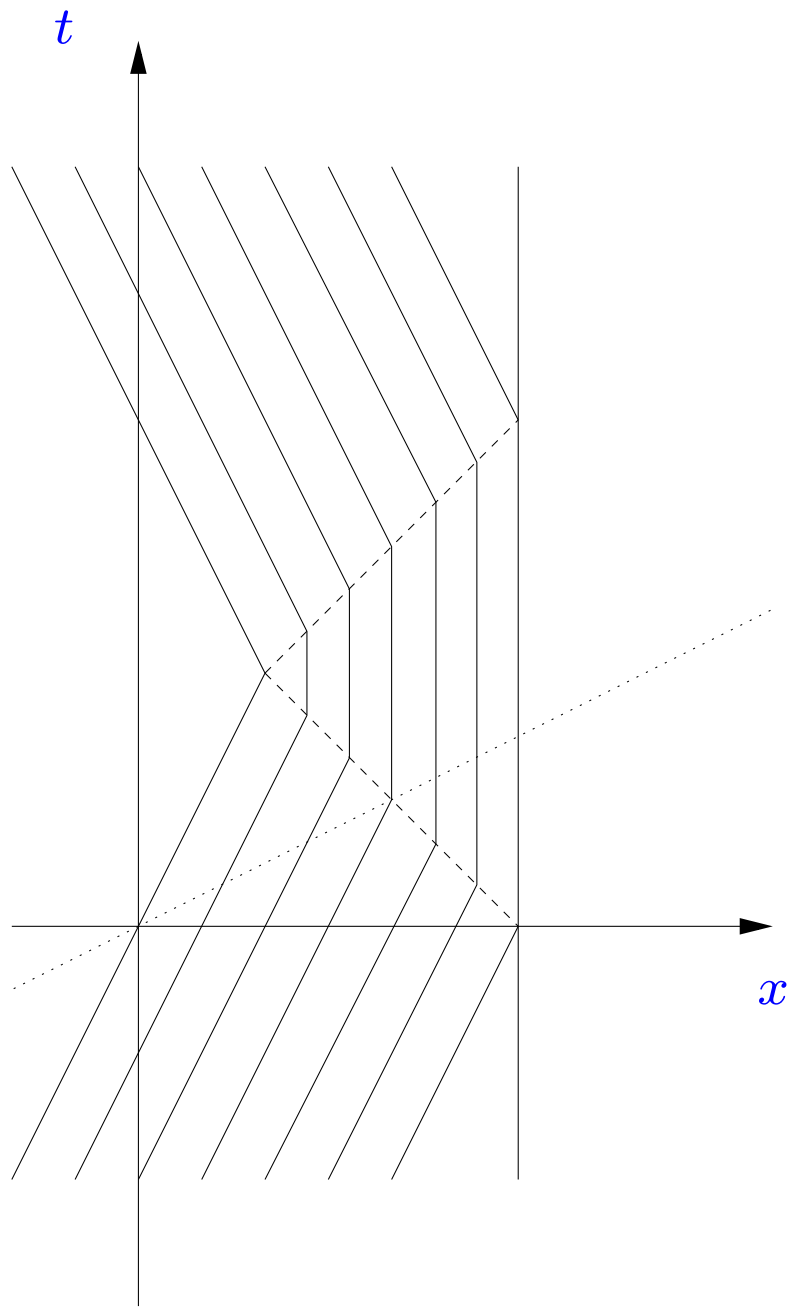
$$\begin{cases} \square \bar{x} = 0 & (t > 0, x < 0) \\ \bar{x}(0, x) = \gamma x & (x < 0) \\ \frac{\partial \bar{x}}{\partial t}(0, x) = -v\gamma & (x < 0) \\ \bar{x}(t, 0) = 0 & (t > 0) \end{cases}$$

- Solution (on any conformal coordinate system):

$$\bar{x}(t, x) = f(x - t) + g(x + t).$$

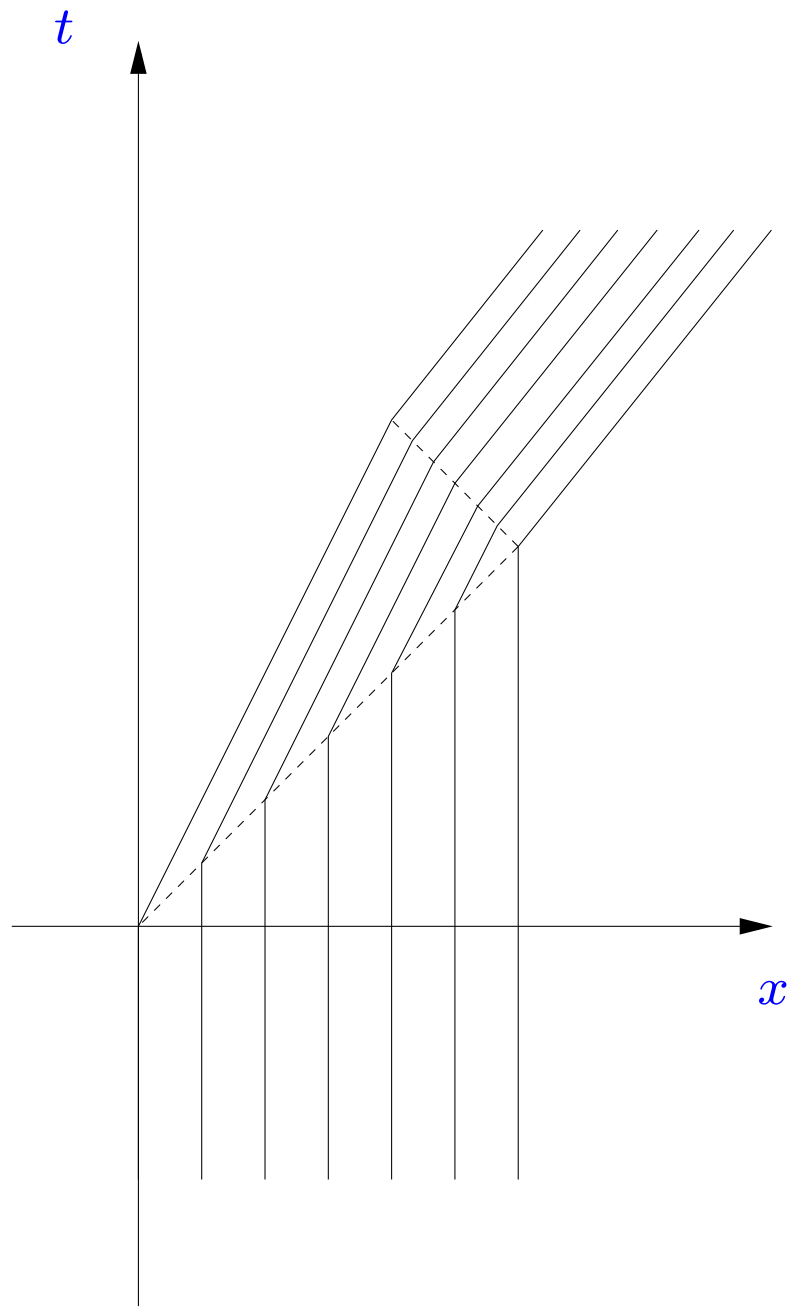


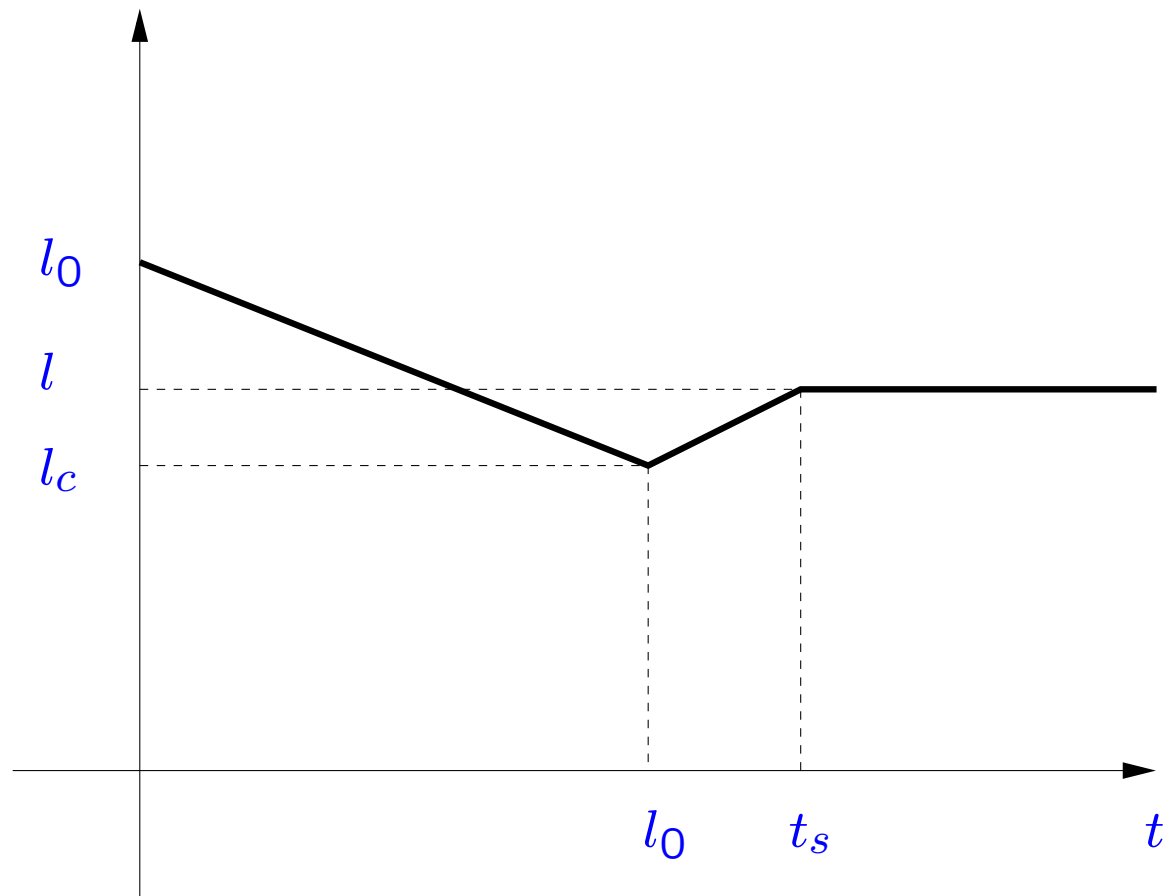




Pushing a rigid rod

- By changing frames we can reinterpret the solution above as describing a **finite rod** being pushed by a **constant force**.
- This offers a picture of how **Lorentz contraction** is attained in a physically realistic setting.





- If the force **keeps acting** then the rod will keep repeating these cycles of compression and rarefaction.
- If m_0 is the rest mass of the relaxed rod, a is the average proper acceleration and v is the velocity of the rear end as it starts being pushed then the **tension** is

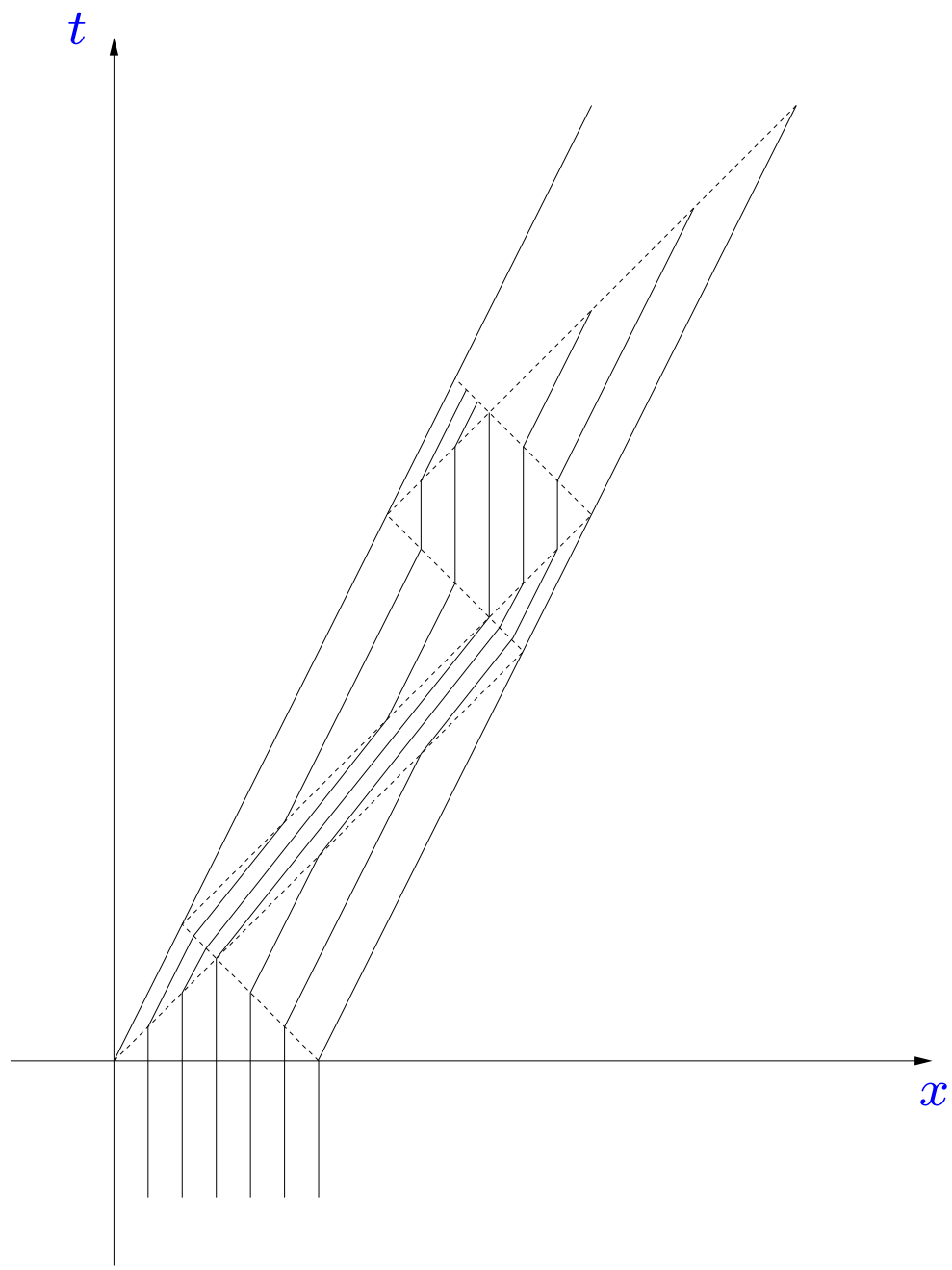
$$p = m_0 a \frac{\gamma v}{\operatorname{arctanh} v}.$$

Bell's spaceships paradox

- Two identical spaceships connected by a string start moving simultaneously with the same acceleration profile. Does the string stretch?
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- We can construct a solution modeling a rigid string initially at rest whose endpoints start moving with velocity v at time $t = 0$.

Bell's spaceships paradox

- Two identical spaceships connected by a string start moving simultaneously with the same acceleration profile. Does the string stretch?
- Yes, because it must compensate length contraction.
- We can construct a solution modeling a rigid string initially at rest whose endpoints start moving with velocity v at time $t = 0$.



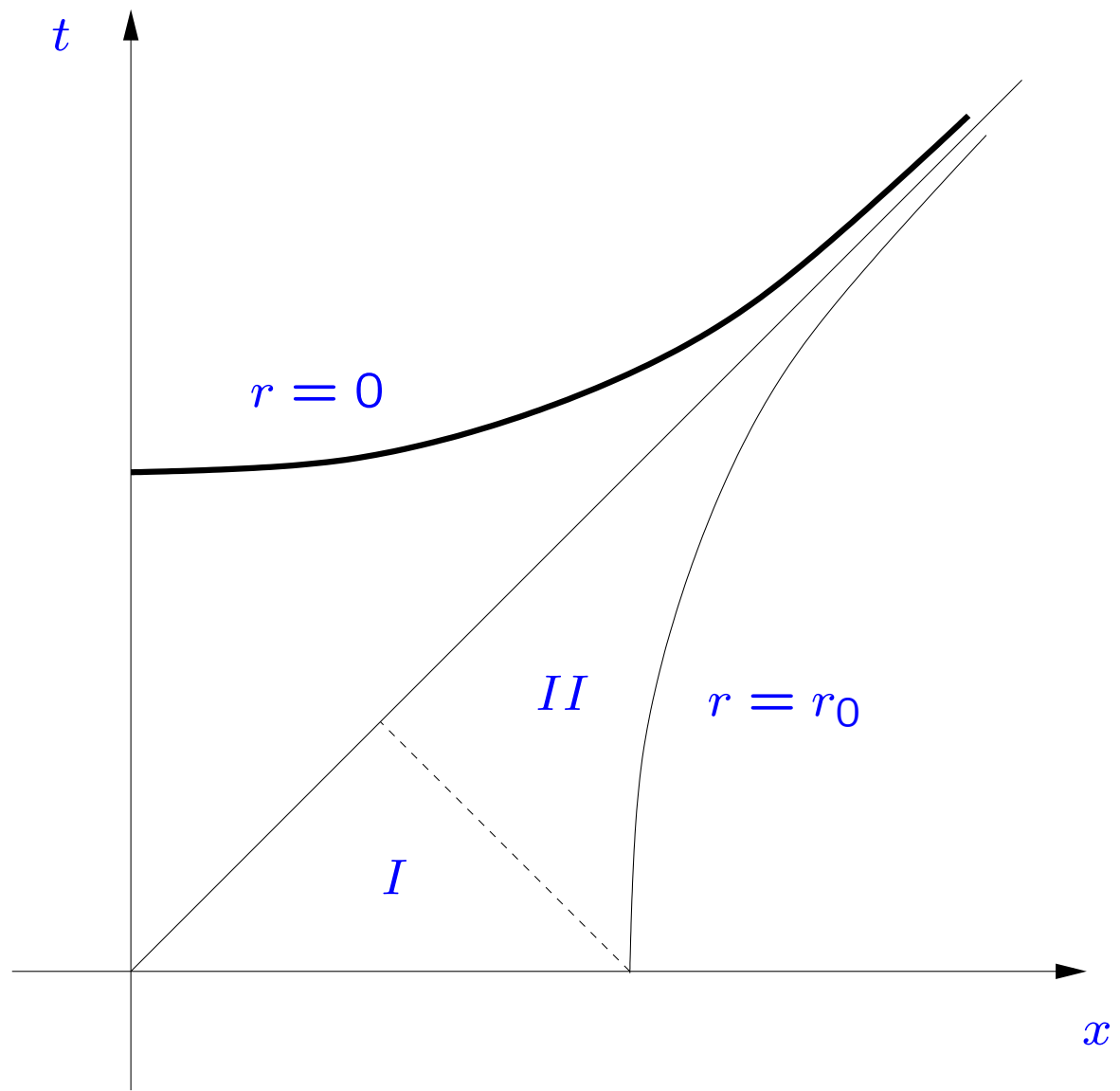
Fishing in black holes

- Kruskal-Szekeres coordinates ($2M = 1$):

$$g = 4r^{-1}e^{-r} (-dt^2 + dx^2), \quad x^2 - t^2 = (r - 1)e^r.$$

- If the string is being held at $r = r_0$ then we must solve the following **initial-boundary value problem**:

$$\left\{ \begin{array}{ll} \square \bar{x} = 0 & (t > 0, 1 < r < r_0) \\ \bar{x}(0, x) = \int_0^x 2r^{-\frac{1}{2}} e^{-\frac{r}{2}} dx & (0 < x < x_0) \\ \frac{\partial \bar{x}}{\partial t}(0, x) = 0 & (0 < x < x_0) \\ \bar{x}(x_0 \sinh u, x_0 \cosh u) = \int_0^{x_0} 2r^{-\frac{1}{2}} e^{-\frac{r}{2}} dx & (u > 0) \end{array} \right.$$



- It is possible to compute an **explicit exact solution**. We find that:
 - Eventually the **whole string** will cross the horizon.
 - The **force** necessary to hold the string **increases** indefinitely.
 - More generally, the **tension** of the string **increases** along any future-pointing causal direction, and indeed approaches $+\infty$.
- Although our **mathematical** model does not contemplate the string breaking, any **physical** string will certainly do so.

Conclusion and outlook

- **Elastic models** are useful tools to model extended bodies in general relativity.
- Many **questions** to explore:
 - Motion of strings and other extended bodies in black hole backgrounds, and relation with cosmic censorship.
 - Oscillations, stability and collapse of elastic (neutron) stars.
 - Modeling supernovas through collapse of two-phase models: fluid atmosphere surrounding an elastic core.