Asymptotic Quasinormal Frequencies for d–Dimensional Black Holes

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Outline

- What exactly do we want to compute?

- How to compute asymptotic quasinormal modes using monodromies: detailed calculation for the Schwarzschild solution.

- Main results (Reissner-Nordström; Schwarzschild de Sitter and Reissner-Nordström de Sitter; Schwarzschild anti-de Sitter and Reissner-Nordström anti-de Sitter).

- Further work

What is a black hole?

Spherically symmetric, static, black holes in d–dimensional space-time, with mass M, charge Q and background cosmological constant Λ. Defining equations are

$$
\begin{aligned}
& R_{\mu\nu} = \frac{2}{d-2}\Lambda g_{\mu\nu} + 8\pi G_d \left(T_{\mu\nu} - \frac{1}{d-2} T g_{\mu\nu} \right), \\
& \nabla^{\lambda} F_{\lambda\mu} = \nabla_{[\lambda} F_{\mu\nu]} = 0, \\
& T_{\mu\nu} = F_{\mu}{}^{\lambda} F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta},
\end{aligned}
$$

$$
g = -f(r)\ dt \otimes dt + f(r)^{-1}\ dr \otimes dr + r^2 d\Omega_{d-2}^2,
$$

$$
f(r) = 1 - \frac{2\boldsymbol{\mu}}{r^{d-3}} + \frac{\boldsymbol{\theta}^2}{r^{2d-6}} - \boldsymbol{\lambda} r^2.
$$

Parameters in the metric relate to black hole physical parameters:

$$\begin{aligned} M &= \frac{(d-2)\,\mathcal{A}_{d-2}}{8\pi G_d}\boldsymbol{\mu}, \\ Q^2 &= \frac{(d-2)\,(d-3)}{8\pi G_d}\boldsymbol{\theta}^2, \\ \Lambda &= \frac{1}{2}\,(d-1)\,(d-2)\,\boldsymbol{\lambda}. \end{aligned} \qquad \mathcal{A}_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)},$$

What is a quasinormal frequency?

- After the onset of a perturbation, the return to equilibrium of a black hole spacetime is dominated by damped, single frequency oscillations, describing the return to the initial configuration: **the quasinormal modes**.

- These modes are quite special: they depend *only* on the parameters of the given black hole spacetime, being *independent* of the details concerning the initial perturbation one started off with.

What is an *asymptotic* quasinormal frequency?

- Modes which damp infinitely fast do *not* radiate at all $\Rightarrow$ Can be interpreted as some sort of fundamental oscillations for the black hole spacetime: **asymptotic quasinormal modes**.

- Besides their natural role in the perturbation theory of general relativity, asymptotic quasinormal modes have recently been focus of attention following suggestions that they could have a role to play in a theory of quantum gravity [**Hod, Dreyer**].

Hod's remark:

- Remember Bohr's hydrogen atom:

$$L_n = n\hbar \quad (n \in \mathbb{N})$$

$$r_n = \frac{4\pi\varepsilon_0\hbar^2}{me^2} n^2$$

$$\omega_n = \frac{me^4}{(4\pi\varepsilon_0)^2\hbar^3}\frac{1}{n^3}$$

$$E_n = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2}$$

$$E_{n+1} - E_n = \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{2n+1}{(n+1)^2 n^2} = \frac{2n^2+n}{2n^2+4n+1}\hbar\omega_n$$

- Bohr's correspondence principle: transition frequencies at large quantum numbers should equal classical oscillation frequencies. But classical black holes do *not* radiate $\Rightarrow$ we should look at infinitely damped oscillations.

- Area spectrum of a black hole guess: $A_n = n\alpha$, where α is the quantum of area.

- Bekenstein-Hawking entropy: $S = \frac{A}{4} \quad (c = G_d = \hbar = k_B = 1)$.

- Boltzmann's formula $\Rightarrow \alpha = 4 \log \Omega$, where $\Omega \in \mathbb{N}$ is the number of microcopic states corresponding to the ground state (hence $S_n = \log \Omega^n$).

- First Law of black hole thermodynamics: $dM = \frac{1}{4} T_H dA$, where T_H is Hawking's temperature.

- Taking $dM = \hbar \mathbb{R}\text{e}(\omega_\infty)$, $dA = \alpha$ one gets $\mathbb{R}\text{e}(\omega_\infty) = T_H \log \Omega$.

- As we shall see $\mathbb{R}\text{e}(\omega_\infty) = T_H \log 3$ for the Schwarzshchild solution. *Is this a coincidence?*

Perturbations of Black Hole Spacetimes

- Black hole background spacetime metric: $\widehat{g}_{\mu\nu}$.
  (satisfies the Einstein–Maxwell equations)

- Linear perturbation: $g_{\mu\nu} = \widehat{g}_{\mu\nu} + h_{\mu\nu}$.
  (neglect terms at order $\mathcal{O}(\|h_{\mu\nu}\|^2)$)

- Linearized equations of motion describing $h_{\mu\nu}$:

$$
\begin{aligned}
\delta\Gamma^{\sigma}_{\mu\nu} &= \frac{1}{2}\widehat{g}^{\sigma\rho}\left(\widehat{\nabla}_{\mu}h_{\nu\rho} + \widehat{\nabla}_{\nu}h_{\mu\rho} - \widehat{\nabla}_{\rho}h_{\mu\nu}\right), \\
\delta R_{\mu\nu} &= \widehat{\nabla}_{\nu}\delta\Gamma^{\sigma}_{\mu\sigma} - \widehat{\nabla}_{\sigma}\delta\Gamma^{\sigma}_{\mu\nu}.
\end{aligned}
$$

- For vacuum solutions, Einstein's equations reduce to

$$\delta R_{\mu\nu} = 0$$

(in the Einstein–Maxwell case there will be other terms).

These are linear differential equations for the perturbation, $h_{\mu\nu}$, which can still be further simplified using the spherical symmetry of the background $\widehat{g}_{\mu\nu}$ [**Regge–Wheeler, Zerilli**].

Scalar Field in Black Hole Spacetime

- (M, Q, Λ) case: two types of fields can be excited, electromagnetic vector field A_μ and gravitational metric tensor field $g_{\mu\nu}$. However, start by studying the scalar wave equation.

- Massless, uncharged, scalar field, ϕ, in a background $g_{\mu\nu}$:

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi\right) = 0,$$

with

$$g = -f(r)\ dt \otimes dt + f(r)^{-1}\ dr \otimes dr + r^2 d\Omega_{d-2}^2,$$

- Harmonic decomposition of scalar field: $\phi = \sum_{\ell,m} r^{\frac{2-d}{2}} \psi_\ell(r,t)\, Y_{\ell m}(\theta_i)$.

- Time Fourier decompose: $\psi_\ell(r,t) = \Phi_\omega(r) e^{i\omega t}$.

- The wave equation can be recast in a Schrödinger–like form:

$$-\frac{d^2\Phi_\omega}{dx^2}(x) + V(x)\Phi_\omega(x) = \omega^2 \Phi_\omega(x).$$

(x is the tortoise coordinate and $V(x)$ is the potential, both determined from the function $f(r)$ in the background metric)

- The tortoise coordinate:

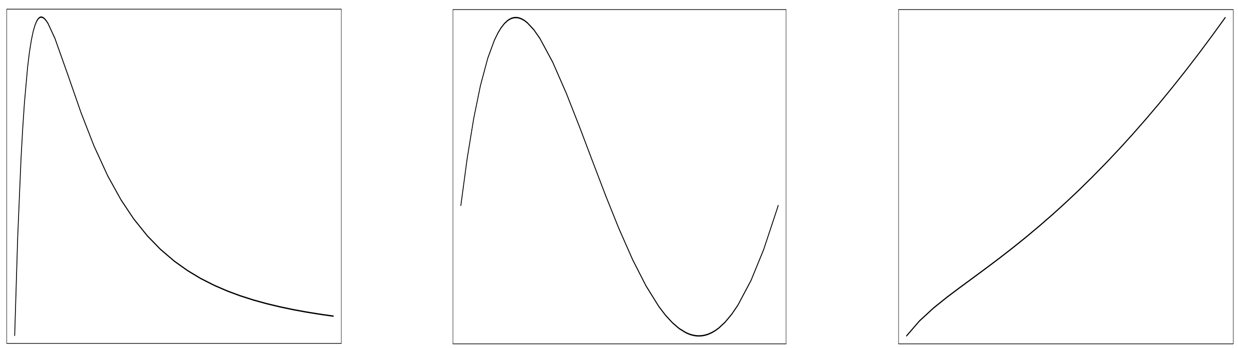
$$f(r)\frac{d}{dr}\left(f(r)\frac{d}{dr}\right)=\frac{d^2}{dx^2} \qquad \Rightarrow \qquad dx=\frac{dr}{f(r)}.$$

(x keeps infinity—or cosmological horizon, R_C, in dS case—at $x=+\infty$ and sends the black hole event horizon, R_H, to $x=-\infty$—in charged cases this refers to outer horizon)

- The potential in the Schrödinger–like equation:

$$V(r)=f(r)\left(\frac{\ell(\ell+d-3)}{r^2}+\frac{(d-2)(d-4)f(r)}{4r^2}+\frac{(d-2)f'(r)}{2r}\right).$$

• Typical potentials for asymptotically flat, asymptotically de Sitter and asymptotically Anti-de Sitter black holes:



Quasinormal Frequencies

- Quasinormal frequencies are found when imposing "outgoing" boundary conditions: nothing arrives neither from infinity nor from within the black hole horizon.

$$\begin{aligned} \Phi_\omega(x) &\sim e^{i\omega x} \text{ as } x \to -\infty, \\ \Phi_\omega(x) &\sim e^{-i\omega x} \text{ as } x \to +\infty. \end{aligned}$$

- The quasinormal frequencies, ω, are complex numbers: $\mathbb{R}\mathrm{e}(\omega)$ = actual frequency of oscillation, $\mathbb{I}\mathrm{m}(\omega)$ = damping.

- A spacetime solution is (linearly) stable if frequencies with negative imaginary part do not exist: time dependence for the perturbation is $e^{i\omega t}$, so that $\mathbb{I}\text{m}(\omega) > 0$ for stable solutions (implying that the perturbation vanishes exponentially in time).

  Our interest is $\mathbb{I}\text{m}(\omega) \to +\infty$.

- But $\mathbb{R}\text{e}(\pm i\omega x) \to +\infty$ as $x \to \mp\infty$ $\Rightarrow$ while solutions can oscillate, they must exponentially increase with $|x|$
  $\Rightarrow$ $\Phi_\omega(x)$ is not a normalizable wave function.

- General strategy of the monodromy method [Motl–Neitzke], begins with the question: how to properly define and impose quasinormal boundary conditions?

- With $x \in \mathbb{R}$ the quasinormal boundary conditions amount to distinguishing between an exponentially vanishing term and an exponentially growing term $\Rightarrow$ Hard!

- Instead do analytic continuation: $r \in \mathbb{C}$ and $x \in \mathbb{C}$.
If one picks the complex contour $\mathbb{I}\text{m}(\omega x) = 0$ in $\mathbb{C}$, asymptotic behavior of $e^{\pm i\omega x}$ is always oscillatory on this line and there will be no problems with exponentially growing versus exponentially vanishing terms.

- One should restrict to studying the boundary conditions on the **Stokes line**, $\mathbb{I}\text{m}(\omega x) = 0$.

- Schwarzschild, RN, Schwarzschild dS and RN dS, asymptotic quasinormal frequencies are such that $\mathbb{I}\text{m}(\omega) \gg \mathbb{R}\text{e}(\omega)$:

$$\begin{aligned} &\omega = \omega_R + in\omega_I, && n \to +\infty, \\ &\omega_R, \omega_I \in \mathbb{R}, && \omega_I > 0. \end{aligned}$$

In $n \to +\infty$ limit, contour $\mathbb{I}\text{m}(\omega x) = 0$ approximated by the curve $\mathbb{R}\text{e}(x) = 0$ which is immediate to plot in $\mathbb{C}$.

- Schwarzschild AdS and RN AdS is different, as asymptotic quasinormal frequencies behave as $\mathbb{I}\mathrm{m}(\omega) \sim \mathbb{R}\mathrm{e}(\omega)$:

$$\begin{aligned} &\omega = n(\omega_R + i\omega_I) + \omega_0, && n \to +\infty, \\ &\omega_R, \omega_I \in \mathbb{R}, && \omega_I > 0. \end{aligned}$$

Monodromies and Asymptotic Quasinormal Frequencies

- First address regions of the complex r–plane around a horizon (be it black hole horizon or a cosmological horizon).

- An horizon is defined by the zeroes of $f(r)$, *i.e.*, $f(R_H) = 0$, and besides physical real horizons $R_H \in \mathbb{R}$ there can be other, non–physical complex horizons $R_H \in \mathbb{C}$.

- Expansion near any horizon yields $f(r) \simeq (r - R_H)\, f'(R_H) + \cdots$, and it follows for the tortoise

$$
\begin{aligned}
x &= \int \frac{dr}{f(r)} \sim \int \frac{dr}{(r - R_H) f'(R_H)} = \frac{1}{f'(R_H)} \log(r - R_H) = \\
&= \frac{1}{2k_H} \log(r - R_H) = \frac{1}{4\pi T_H} \log(r - R_H).
\end{aligned}
$$

- Around any nondegenerate horizon tortoise coordinate will be multivalued and can ask for monodromy of tortoise plane waves in clockwise contours around *any* horizon $f(R_H) = 0$.

- Consider horizon $R_H \in \mathbb{C}$, and clockwise contour $\gamma \subset \mathbb{C}$ centered at R_H and not including any other horizon. As tortoise coordinate goes around γ it increases by $-\frac{i\pi}{k_H}$. This implies the monodromy of plane waves along the selected contour:

$$\mathfrak{M}_{\gamma,R_H}\left[e^{\pm i\omega x}\right] = e^{\pm\frac{\pi\omega}{k_H}}.$$

- Can recast quasinormal boundary conditions as **monodromy conditions** $\Rightarrow$ At the black hole event horizon the quasinormal boundary condition is

$$\Phi_\omega(x) \sim e^{i\omega x} \text{ as } x \to -\infty,$$

$x \to -\infty$ as $r \to R_H$ $\Rightarrow$ Write this boundary condition as monodromy condition for the solution at the black hole horizon:

$$\mathfrak{M}_{\gamma,R_H}\left[\Phi_\omega(x)\right] = e^{\frac{\pi\omega}{k_H}}.$$

(γ is a clockwise contour)

- The other quasinormal boundary condition is $\Phi_\omega(x) \sim e^{-i\omega x}$ as $x \to +\infty$, living at $r \sim +\infty$, as $x \to +\infty$ when $r \to +\infty$ (except asymptotically dS spacetimes). This boundary condition *cannot* be recast as a quasinormal monodromy condition.

- For asymptotically dS spacetimes, above boundary condition is located at cosmological event horizon, as $x \to +\infty$ when $r \to R_C$ $\Rightarrow$ Recast this boundary condition as a monodromy condition for the solution at the cosmological horizon:

$$\mathfrak{M}_{\gamma,R_C}\left[\Phi_\omega(x)\right] = e^{-\frac{\pi\omega}{k_C}}.$$

The Ishibashi–Kodama Master Equations

- Recently the black hole stability problem was addressed within a d–dimensional setting [Ishibashi-Kodama].

- Perturbation theory of static, spherically symmetric black holes in any spacetime dimension $d > 3$, allowing for both electromagnetic charge and background cosmological constant $\Rightarrow$ Generalize the Regge–Wheeler equation.

- (M, Q, Λ) case: two types of fields can be excited, electromagnetic vector field A_μ and gravitational metric tensor field $g_{\mu\nu}$. Want to study perturbations to these fields.

- Perturbations come in three types: tensor type, vector type and scalar type. Nomenclature refers to tensorial behavior on the sphere, $\mathbb{S}^{d-2}$, of each Einstein–Maxwell gauge invariant type of perturbation, and is *not* related to perturbations associated to external particles.

- Perturbations of $g_{\mu\nu}$ include all three types of perturbations, while perturbations of A_μ only include vector type and scalar type perturbations. As Einstein–Maxwell system couples $g_{\mu\nu}$ and A_μ, master equations are coupled $\Rightarrow$ Can always decouple the master equations and these are the relevant gauge invariant equations when studying quasinormal modes. Quasinormal modes are studied as perturbations to the electrovacuum $\Rightarrow$ IK master equations will all be homogeneous.

Tensor Type Perturbations

Only $g_{\mu\nu}$ displays this type of perturbations. The IK master equation is of Schrödinger–like form, with potential

$$V_{\mathsf{T}}(r) = f(r) \left(\frac{\ell\,(\ell + d - 3)}{r^2} + \frac{(d-2)\,(d-4)\,f(r)}{4r^2} + \frac{(d-2)\,f'(r)}{2r} \right).$$

This is the same potential as in massless, uncharged, scalar wave equation ⇒ Quasinormal modes of tensor type perturbations of $g_{\mu\nu}$ will coincide with quasinormal modes of scalar wave equation.

Vector Type Perturbations

Both $g_{\mu\nu}$ and A_μ display this type of perturbations. The IK master equations decouple and are of Schrödinger–like form. If $Q = 0$ there is only one IK master equation

$$V_{\mathsf{V}}(r) = f(r)\left(\frac{\ell\,(\ell + d - 3)}{r^2} + \frac{(d-2)\,(d-4)\,f(r)}{4r^2} - \frac{(d-1)\,(d-2)\,\boldsymbol{\mu}}{r^{d-1}}\right).$$

If $Q \neq 0$ there are two decoupled IK master equations

$$V_{\mathsf{V}^{\pm}}(r) = f(r)\left(\frac{\ell\,(\ell + d - 3)}{r^2} + \frac{(d-2)\,(d-4)\,f(r)}{4r^2} - \frac{(d-1)\,\boldsymbol{\mu}}{r^{d-1}} + \frac{(d-2)^2\,\boldsymbol{\theta}^2}{r^{2d-4}} \pm \frac{\boldsymbol{\Delta}}{r^{d-1}}\right),$$

- Φ^{+} equation represents the electromagnetic mode.

- Φ^{-} equation represents the gravitational mode.

- When $\boldsymbol{\theta} = 0 \Rightarrow V_{\mathsf{V}^-}(r) = V_{\mathsf{V}}(r)$.

- $\Delta \equiv \sqrt{(d-1)^2 (d-3)^2 \boldsymbol{\mu}^2 + 2\,(d-2)\,(d-3) \Big(\ell\,(\ell + d - 3) - (d-2)\Big) \boldsymbol{\theta}^2}$.

- Potentials clearly different from previous ones characterizing tensor type perturbations of $g_{\mu\nu}$ and scalar wave equation.

Scalar Type Perturbations

Both $g_{\mu\nu}$ and A_μ display this type of perturbations. The IK master equations decouple and are of Schrödinger–like form. If $Q = 0$ there is only one IK master equation

$$V_{\mathsf{S}}(r) = \frac{f(r)U(r)}{16r^2H^2(r)},$$

where

$$H(r) = \ell\,(\ell + d - 3) - (d-2) + \frac{(d-1)\,(d-2)\,\boldsymbol{\mu}}{r^{d-3}},$$

and $U(r)$ too long to display.

If $Q \neq 0$ there are two decoupled IK master equations

$$V_{\mathsf{S}^{\pm}}(r) = \frac{f(r)U_{\pm}(r)}{64r^2 H_{\pm}^2(r)},$$

where

$$\begin{aligned}
H_{+}(r) &= 1 + \frac{(d-1)(d-2)(1-\Omega)}{2\Big\{\ell(\ell+d-3)-(d-2)\Big\}}\,\frac{\boldsymbol{\mu}}{r^{d-3}}, \\
H_{-}(r) &= \ell(\ell+d-3)-(d-2)+\frac{1}{2}(d-1)(d-2)(1+\Omega)\,\frac{\boldsymbol{\mu}}{r^{d-3}}, \\
\Omega &= \sqrt{1+\frac{4\Big\{\ell(\ell+d-3)-(d-2)\Big\}}{(d-1)^2}\,\frac{\boldsymbol{\theta}^2}{\boldsymbol{\mu}^2}},
\end{aligned}$$

and $U_{\pm}(r)$ way too long to display.

- Φ^{+} equation represents the electromagnetic mode.

- Φ^{-} equation represents the gravitational mode.

- When $\boldsymbol{\theta} = 0$, one observes

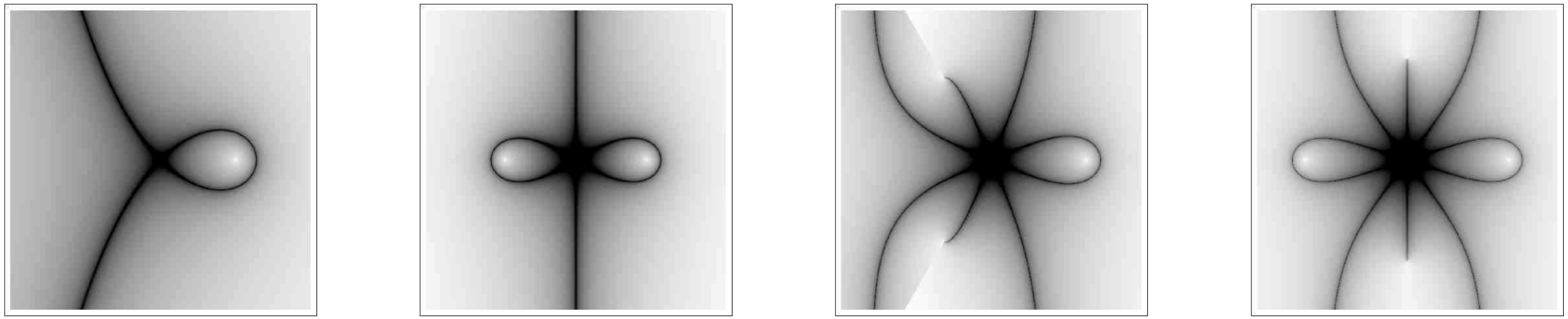
$$
\begin{aligned}
H_{-}(r)\Big|_{\theta=0} &= H(r), \\
U_{-}(r)\Big|_{\theta=0} &= 4U(r),
\end{aligned}
$$

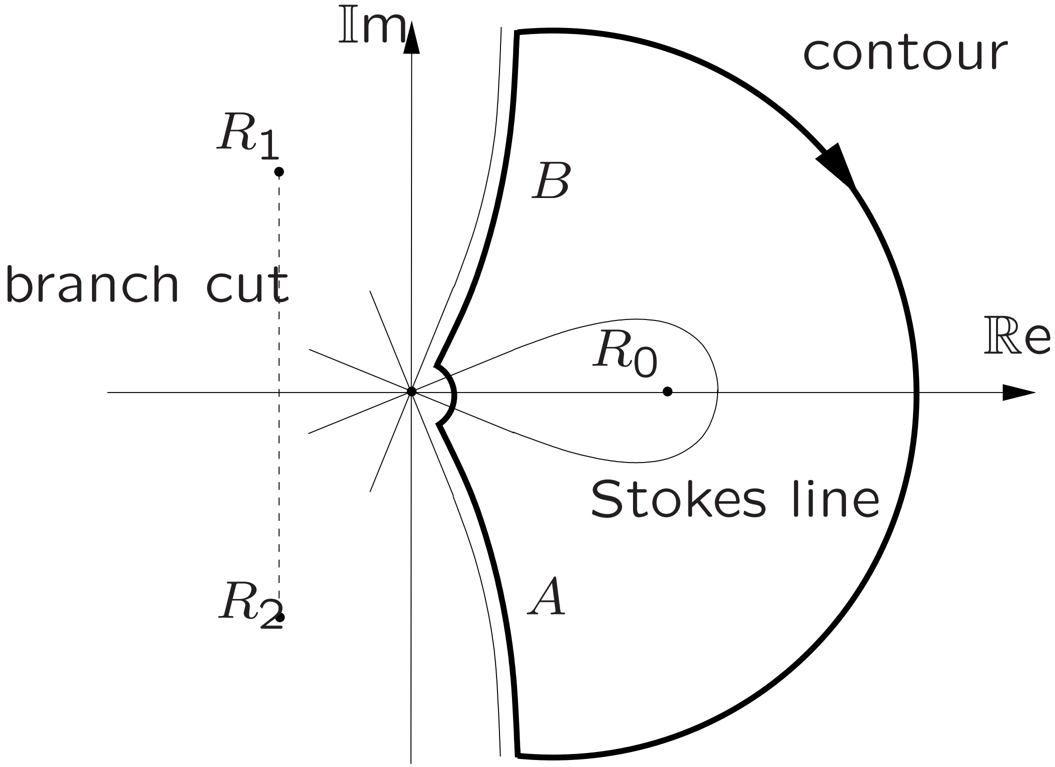
  so $V_{\mathsf{S}-}(r) = V_{\mathsf{S}}(r)$ at $\boldsymbol{\theta} = 0$ as expected.

- Potentials clearly different from previous ones characterizing tensor, vector type perturbations and scalar wave equation.

The Schwarzschild Solution

Stokes lines for Schwarzschild black hole in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$:





Stokes line for Schwarzschild black hole, along with the chosen contour for monodromy matching, in $d = 6$.

The Schwarzschild Solution: First studied by [Motl, Motl–Neitzke]. For all types of perturbations, tensor, vector and scalar type perturbations, equation for asymptotic quasinormal frequencies is the same and is

$$e^{\frac{\omega}{T_H}} + 3 = 0.$$

This case is particularly simple and can solve for the asymptotic quasinormal frequency

$$\omega = T_H \log 3 + 2\pi i T_H \left(n + \frac{1}{2} \right) \qquad (n \in \mathbb{N}, \quad n \gg 1)\,.$$

Result is independent of spacetime dimension.

The Schwarzschild Solution: Analytical Details

If only interested in *asymptotic* quasinormal modes ⇒ No need to solve IK master equation exactly (only need approximate solutions near infinity, origin, and black hole horizon) ⇒ Obtain asymptotic quasinormal frequencies via monodromy matching.

Consider Schrödinger–like master equation with $r, x \in \mathbb{C}$

$$-\frac{d^2\Phi}{dx^2}(x) + V\big[r(x)\big]\Phi(x) = \omega^2\Phi(x).$$

At infinity $r \sim \infty$, $V(r)$ vanishes ⇒ Solution of IK equation is

$$\Phi(x) \sim A_+ e^{i\omega x} + A_- e^{-i\omega x}$$

Boundary condition for quasinormal modes at infinity is: $A_+ = 0$.

Next, study $\Phi(x)$ near singularity $r = 0$. Potential for tensor and scalar type perturbations is

$$V\left[r(x)\right] \sim -\frac{1}{4x^2} = \frac{j^2 - 1}{4x^2},$$

with $j = 0$. Schrödinger–like master equation approximates to

$$-\frac{d^2\Phi}{dx^2}(x) + \frac{j^2 - 1}{4x^2}\Phi(x) = \omega^2\Phi(x)$$

with solution

$$\Phi(x) \sim B_+\sqrt{2\pi\omega x}\ J_{\frac{j}{2}}(\omega x) + B_-\sqrt{2\pi\omega x}\ J_{-\frac{j}{2}}(\omega x)\,.$$

Next link solution at origin with solution at infinity.

For Schwarzschild asymptotic quasinormal modes, may consider asymptotic expansion of Bessel functions

$$
\begin{aligned}
J_\nu(z) &\sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \qquad z \gg 1, \\
\Phi(x) &\sim 2B_+ \cos(\omega x - \alpha_+) + 2B_- \cos(\omega x - \alpha_-) \\
&= \left(B_+ e^{-i\alpha_+} + B_- e^{-i\alpha_-}\right) e^{i\omega x} + \left(B_+ e^{i\alpha_+} + B_- e^{i\alpha_-}\right) e^{-i\omega x}, \\
&\left(\alpha_\pm = \frac{\pi}{4}(1 \pm j)\right).
\end{aligned}
$$

This asymptotic expression near origin ideal to match with asymptotic expression at infinity. Matching must be along Stokes line, so that neither of exponentials $e^{\pm i\omega x}$ dominates the other.

Consider contour obtained by closing unlimited portions of Stokes line near $r \sim \infty$. At point A, $\omega x \gg 0$, and "Bessel" expansion holds. Imposing asymptotic quasinormal condition follows:

$$B_+e^{-i\alpha_+} + B_-e^{-i\alpha_-} = 0.$$

For $z \sim 0$ one has $J_\nu(z) = z^\nu w(z)$ with w even holomorphic function. Rotating from branch containing point A to branch containing point B, x rotates through angle of 3π, and

$$\begin{aligned}\Phi(x) &\sim 2B_+e^{6i\alpha_+}\cos(-\omega x - \alpha_+) + 2B_-e^{6i\alpha_-}\cos(-\omega x - \alpha_-) \\ &= \left(B_+e^{7i\alpha_+} + B_-e^{7i\alpha_-}\right)e^{i\omega x} + \left(B_+e^{5i\alpha_+} + B_-e^{5i\alpha_-}\right)e^{-i\omega x}\end{aligned}$$

holds at point B. Closing contour near $r \sim \infty$, have $x \sim r$ and hence $\mathbb{R}\mathrm{e}(x) > 0 \Rightarrow e^{i\omega x}$ exponentially small in this part of contour $\Rightarrow$ Only coefficient of $e^{-i\omega x}$ should be trusted. Completing contour, this coefficient gets multiplied by

$$\frac{B_+ e^{5i\alpha_+} + B_- e^{5i\alpha_-}}{B_+ e^{i\alpha_+} + B_- e^{i\alpha_-}}.$$

As monodromy of $e^{-i\omega x}$ going clockwise around contour is $e^{-\frac{\pi\omega}{k}}$, clockwise monodromy of Φ around contour is

$$\frac{B_+ e^{5i\alpha_+} + B_- e^{5i\alpha_-}}{B_+ e^{i\alpha_+} + B_- e^{i\alpha_-}} e^{-\frac{\pi\omega}{k}}.$$

Can deform chosen contour—without crossing singularities—so that it becomes small clockwise circle around black hole event horizon. Near horizon, again $V(r) \sim 0$, hence

$$\Phi(x) \sim C_+ e^{i\omega x} + C_- e^{-i\omega x}.$$

Condition for quasinormal modes at horizon is $C_- = 0$. Can restate boundary condition as condition that monodromy of Φ going clockwise around contour should be

$$e^{\frac{\pi\omega}{k}}.$$

Because monodromy is invariant under this deformation of contour, condition for quasinormal modes at horizon follows

$$\frac{B_+ e^{5i\alpha_+} + B_- e^{5i\alpha_-}}{B_+ e^{i\alpha_+} + B_- e^{i\alpha_-}} e^{-\frac{\pi\omega}{k}} = e^{\frac{\pi\omega}{k}}.$$

Final condition for this equation, together with

$$B_+ e^{-i\alpha_+} + B_- e^{-i\alpha_-} = 0,$$

(equations which reflect quasinormal mode boundary conditions at infinity and horizon), to have nontrivial solutions (B_+, B_-) is

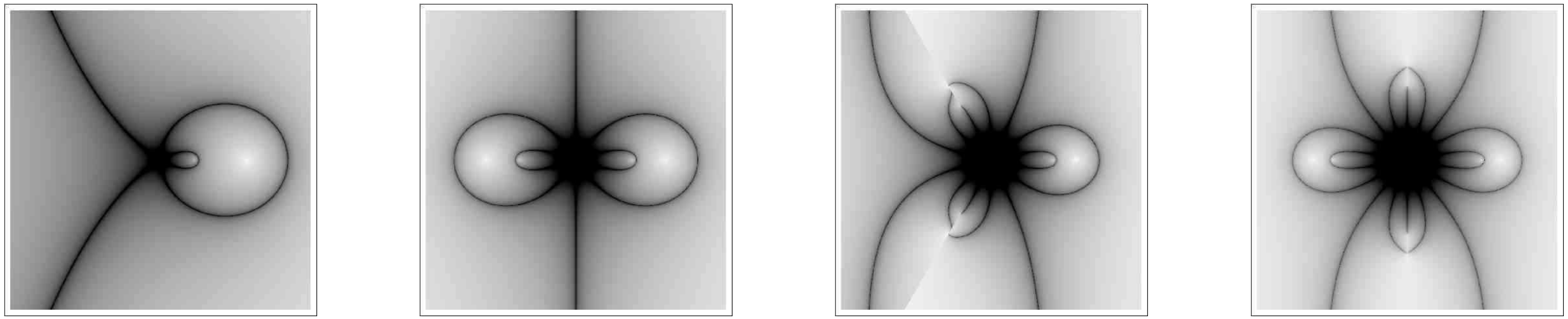
$$\begin{vmatrix} e^{-i\alpha_+} & e^{-i\alpha_-} \\ e^{5i\alpha_+} e^{-\frac{\pi\omega}{k}} - e^{i\alpha_+} e^{\frac{\pi\omega}{k}} & e^{5i\alpha_-} e^{-\frac{\pi\omega}{k}} - e^{i\alpha_-} e^{\frac{\pi\omega}{k}} \end{vmatrix} = 0.$$

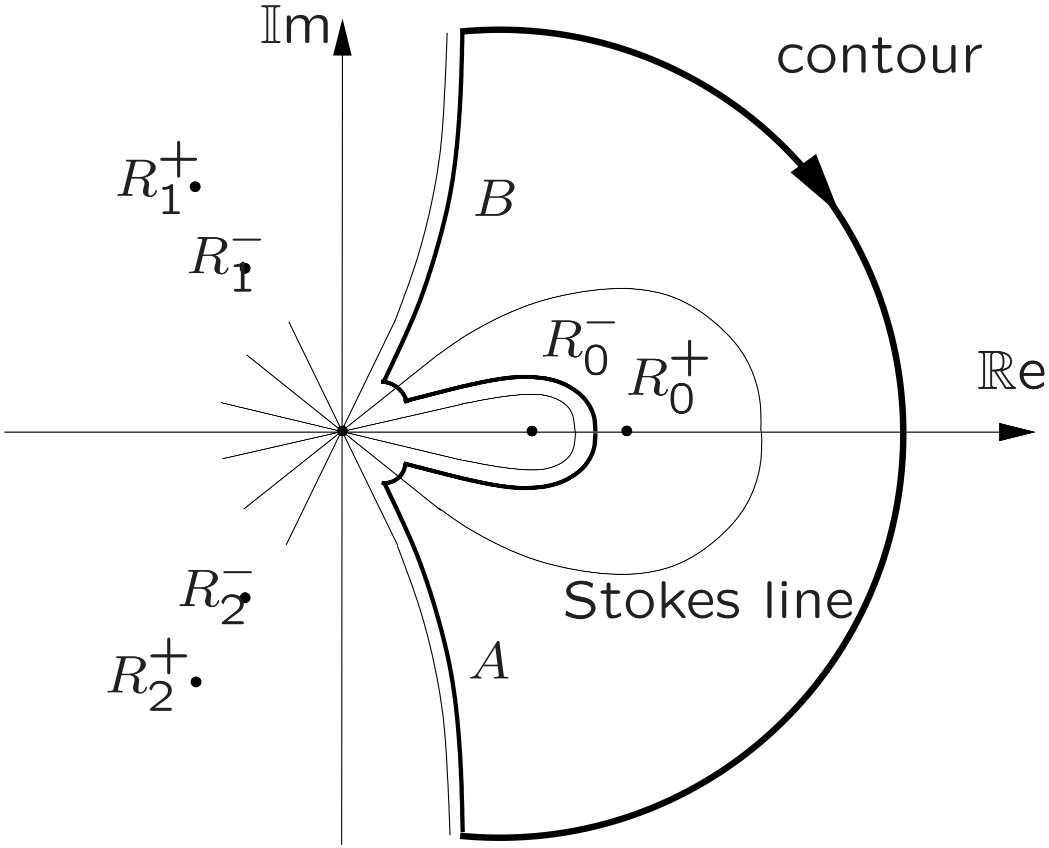
Equation automatically satisfied for $j = 0$ $\Rightarrow$ Expected, as for $j = 0$ Bessel functions $J_{\pm\frac{j}{2}}$ coincide and do not form basis for space of solutions of IK master equation near origin $\Rightarrow$ Consider this equation for j nonzero and take limit as $j \to 0$ $\Rightarrow$ Require that derivative of determinant with respect to j be zero for $j = 0$

$$\omega = T_H \log 3 + ik_H \left(n + \frac{1}{2}\right).$$

The Reissner–Nordström Solution

Stokes lines for Reissner–Nordström black hole in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$:





Stokes line for Reissner–Nordström black hole, along with the chosen contour for monodromy matching, in $d = 6$.

The RN Solution: For all types of perturbations, tensor, vector and scalar perturbations, equation for asymptotic quasinormal frequencies is the same and is

$$e^{\frac{\omega}{T_H^+}} + \Big(1 + 2\cos(\pi j)\Big) + \Big(2 + 2\cos(\pi j)\Big) e^{-\frac{\omega}{T_H^-}} = 0,$$

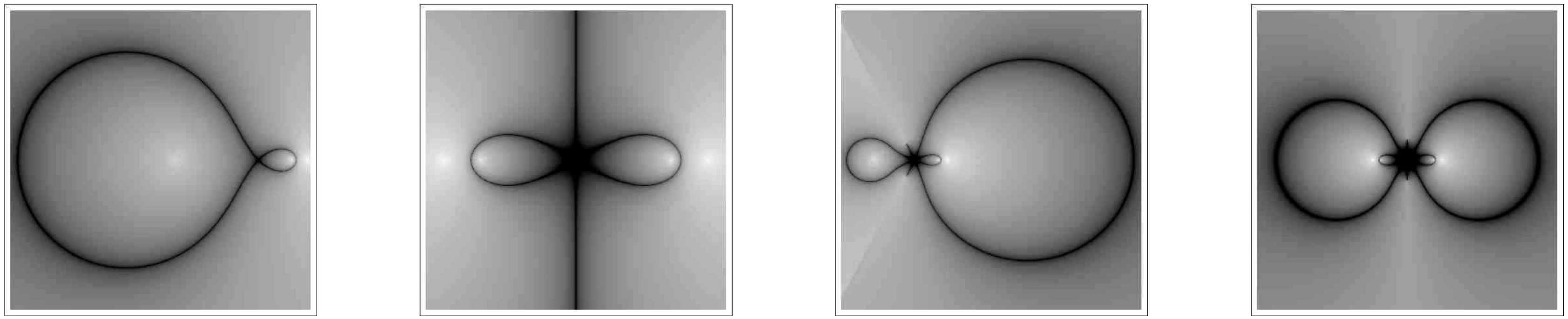
where $T_H^{\pm}$ are the Hawking temperatures at outer and inner horizons (notice that $T_H^- < 0$), and where

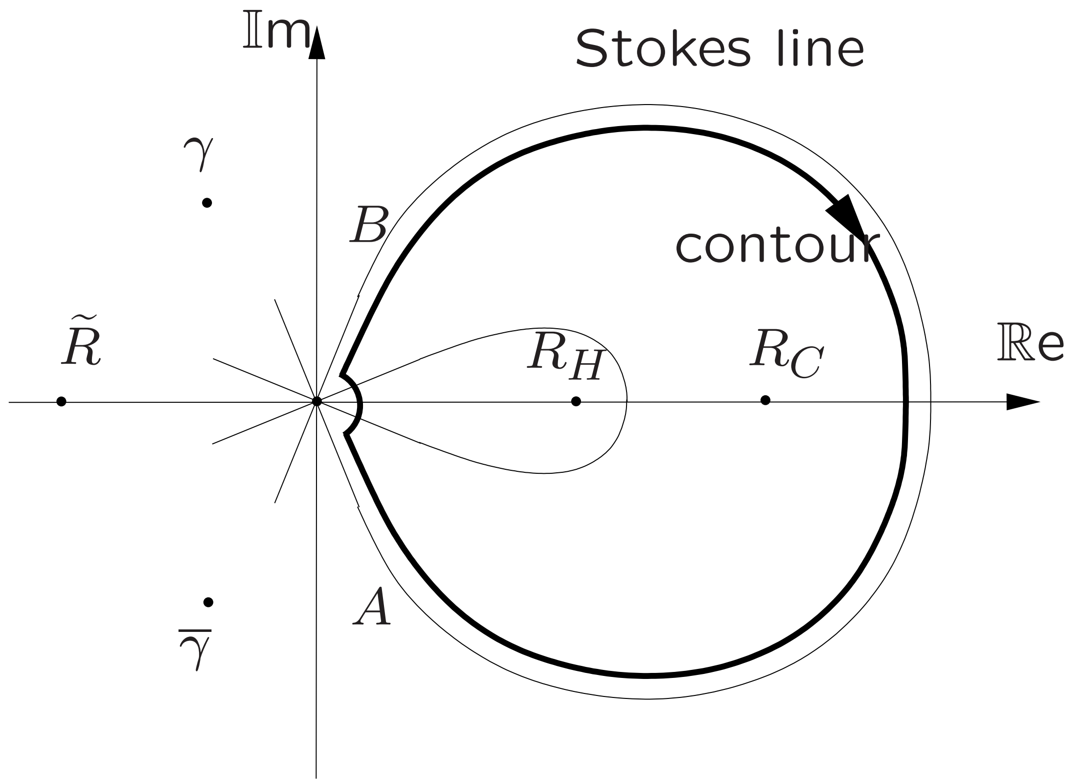
$$j = \frac{d-3}{2d-5}.$$

There is no known closed form solution in ω for the above equation.

The Schwarzschild de Sitter Solution

Stokes lines for Schwarzschild de Sitter black hole in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$:





Stokes line for Schwarzschild de Sitter black hole, along with the chosen contour for monodromy matching, in $d = 6$.

The Schwarzschild dS Solution: First studied in $d = 4$ by [Cardoso–JN–Schiappa]. For all types of perturbations, tensor, vector and scalar perturbations, equation for asymptotic quasinormal frequencies is the same and is

$$\cosh\left(\frac{\omega}{2T_H} - \frac{\omega}{2T_C}\right) + 3\cosh\left(\frac{\omega}{2T_H} + \frac{\omega}{2T_C}\right) = 0,$$

where T_H is Hawking temperature at black hole event horizon and T_C is [negative] Hawking temperature at cosmological horizon. There is no known closed form solution in ω for the above equation.

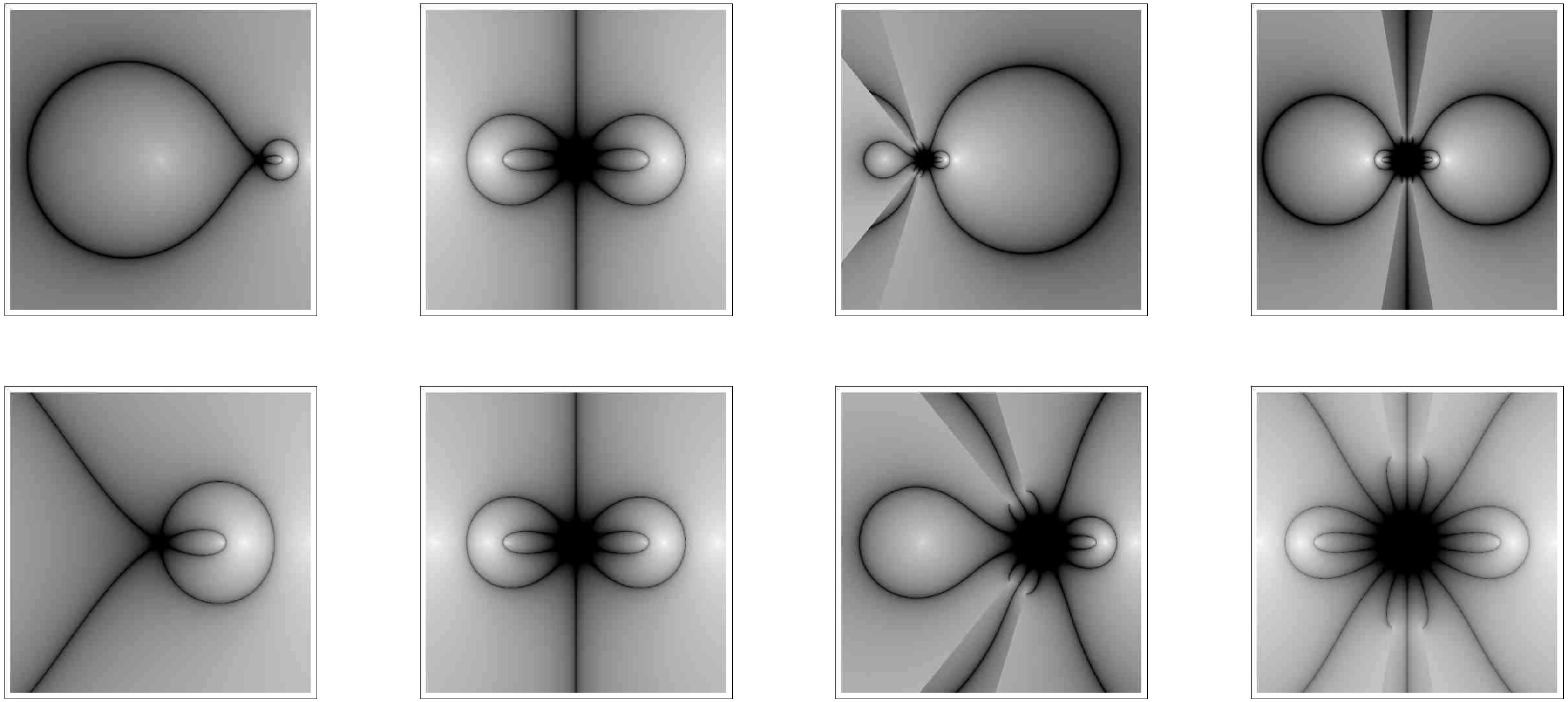
Result is independent of spacetime dimension, except in dimension $d = 5$ where the formula above must be replaced by:

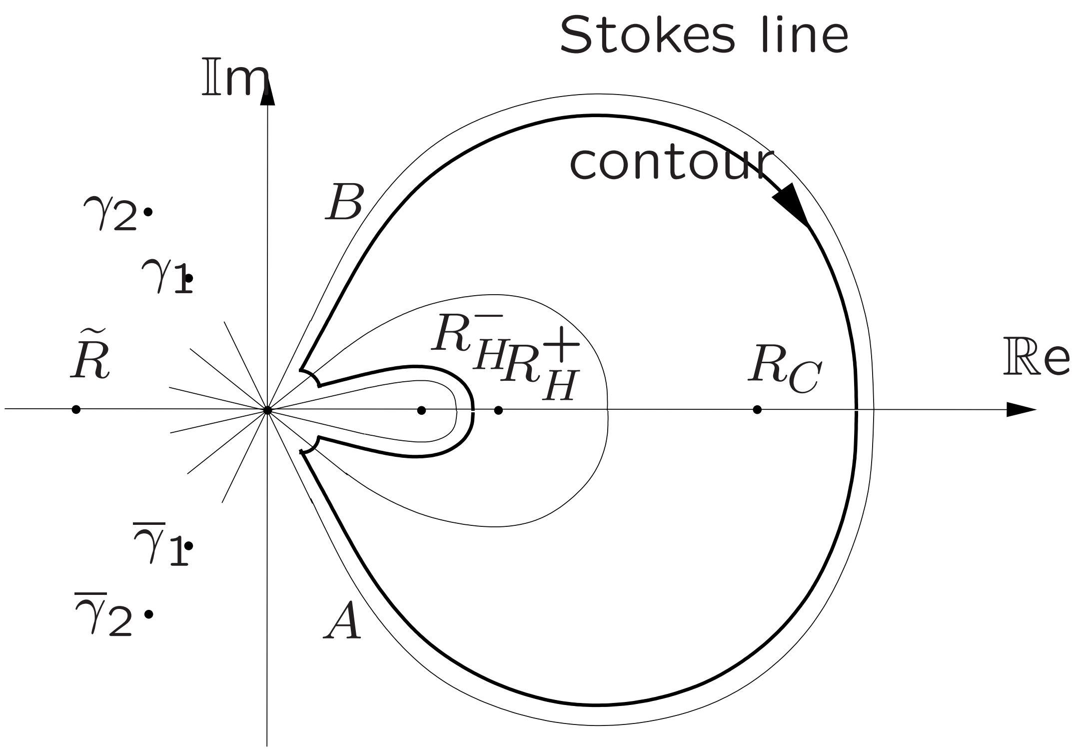
$$\sinh\left(\frac{\omega}{2T_H} - \frac{\omega}{2T_C}\right) - 3\sinh\left(\frac{\omega}{2T_H} + \frac{\omega}{2T_C}\right) = 0.$$

For this solution one can actually take Schwarzschild limit $\boldsymbol{\lambda} \to 0$ without provoking any topology change in complex plane where the monodromy analysis is performed.

The Reissner–Nordström de Sitter Solution

Stokes lines for Reissner–Nordström de Sitter black hole in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$ (with close–up):





Stokes line for Reissner–Nordström de Sitter black hole, along with the chosen contour for monodromy matching, in $d = 6$.

The RN dS Solution: For all types of perturbations, tensor, vector and scalar perturbations, equation for asymptotic quasinormal frequencies is the same and is

$$\cosh\left(\frac{\omega}{2T_H^+} - \frac{\omega}{2T_C}\right) + (1 + 2\cos(\pi j))\cosh\left(\frac{\omega}{2T_H^+} + \frac{\omega}{2T_C}\right) +$$
$$+ (2 + 2\cos(\pi j))\cosh\left(\frac{\omega}{T_H^-} + \frac{\omega}{2T_H^+} + \frac{\omega}{2T_C}\right) = 0,$$

where $T_H^{\pm}$ are Hawking temperatures at outer and inner black hole event horizons and T_C is temperature at cosmological horizon. There is no known closed form solution in ω for the above equation.

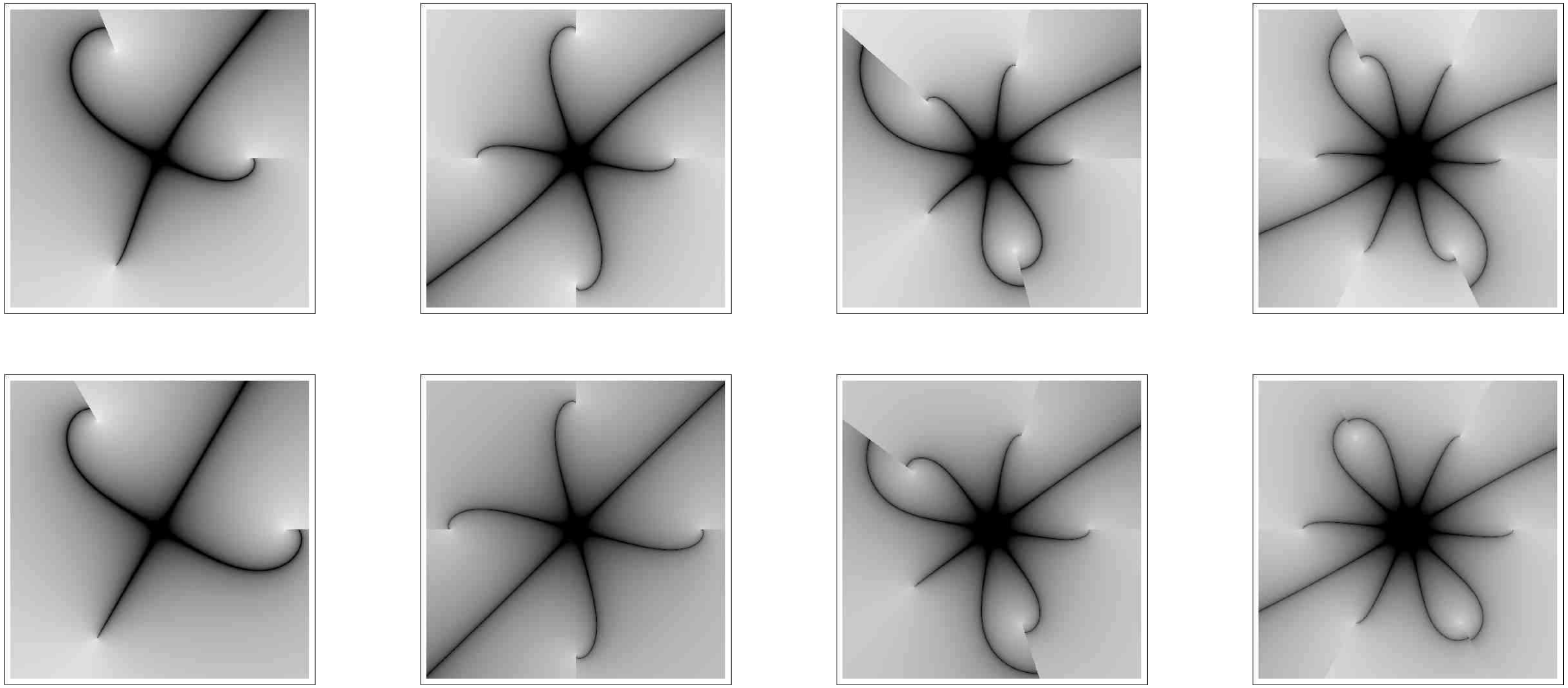
Result explicitly depends on spacetime dimension (j depends on d), but formula not valid in $d = 5$, where it must be replaced:

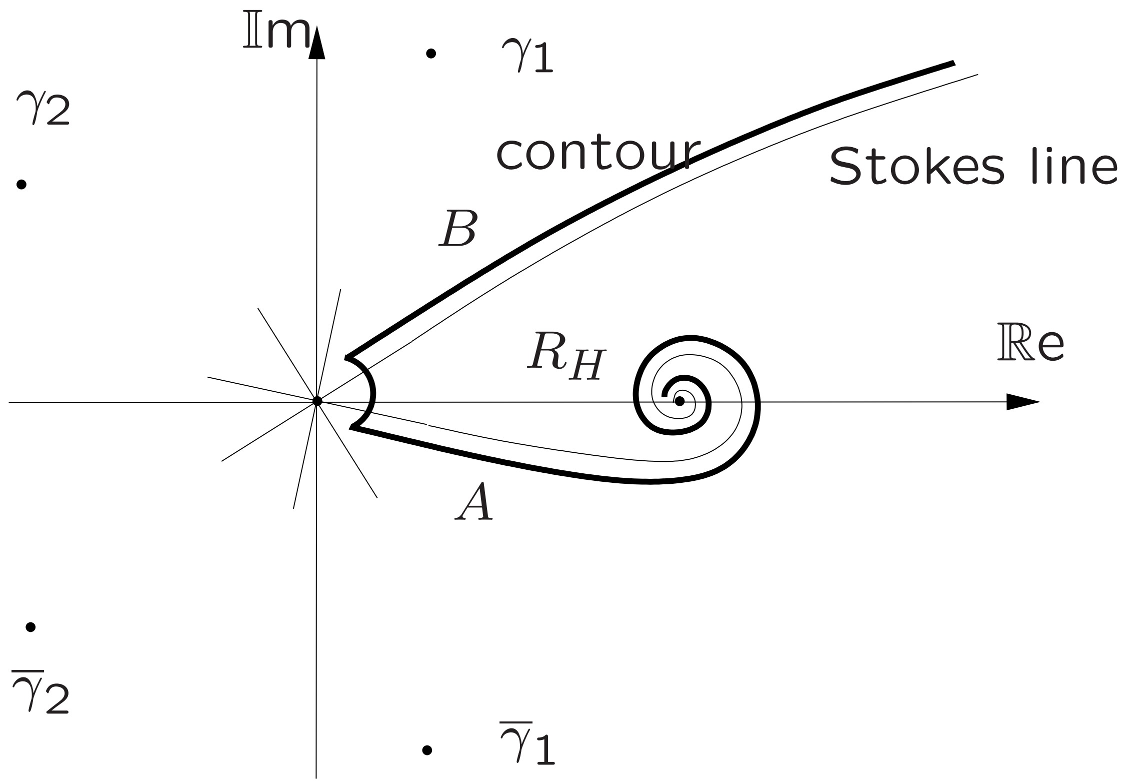
$$\sinh\left(\frac{\omega}{2T_C} - \frac{\omega}{2T_H^+}\right) + \frac{1+\sqrt{5}}{2}\sinh\left(\frac{\omega}{2T_H^+} + \frac{\omega}{2T_C}\right) +$$
$$+\frac{3+\sqrt{5}}{2}\sinh\left(\frac{\omega}{T_H^-} + \frac{\omega}{2T_H^+} + \frac{\omega}{2T_C}\right) = 0.$$

As in Schwarzschild dS case, can take pure RN limit $\boldsymbol{\lambda} \to 0$ without provoking any topology change in complex plane where the monodromy analysis is performed.

The Schwarzschild Anti–de Sitter Solution

Stokes lines for Schwarzschild Anti–de Sitter black hole in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$ (plus *large* black holes where horizon singularities now located at $(d-1)$ roots of unity):





Stokes line for Schwarzschild Anti–de Sitter black hole, along with the chosen contour for monodromy matching, in $d = 6$.

The Schwarzschild AdS Solution: First studied in $d = 4$ large black hole by [Cardoso–JN–Schiappa], and in $d = 5$ large black hole by [Starinets, Nuñez–Starinets, Fidkowski–Hubeny–Kleban–Shenker, Musiri–Siopsis] using different methods. For all types of perturbations, tensor, vector and scalar perturbations, equation for asymptotic quasi-normal frequencies is the same and is

$$\tan\left(\frac{\pi}{4}(d+1) - \omega x_0\right) = \frac{i}{3},$$

where x_0 is parameter related to tortoise coordinate at spatial infinity and given by

$$x_0 = \sum_{n=1}^{d-1} \frac{1}{2k_n} \log\left(-\frac{1}{R_n}\right).$$

Above R_n are $(d-1)$ complex horizons and k_n surface gravities at each complex horizon. There is no general analytic solution for x_0. However, for *large* black holes can compute exactly

$$\frac{1}{x_0} = 4T_H \sin\left(\frac{\pi}{d-1}\right) \exp\left(\frac{i\pi}{d-1}\right)$$

with T_H Schwarzschild AdS Hawking temperature. Can solve for asymptotic quasinormal frequency as

$$\omega x_0 = \frac{\pi}{4}(d+1) - \arctan\left(\frac{i}{3}\right) + n\pi \qquad (n \in \mathbb{N}, \quad n \gg 1)\,.$$

For *large* Schwarzschild AdS black holes (of particular relevance to describe thermal gauge theories within AdS/CFT), formulae above lead to following analytical result for the leading term

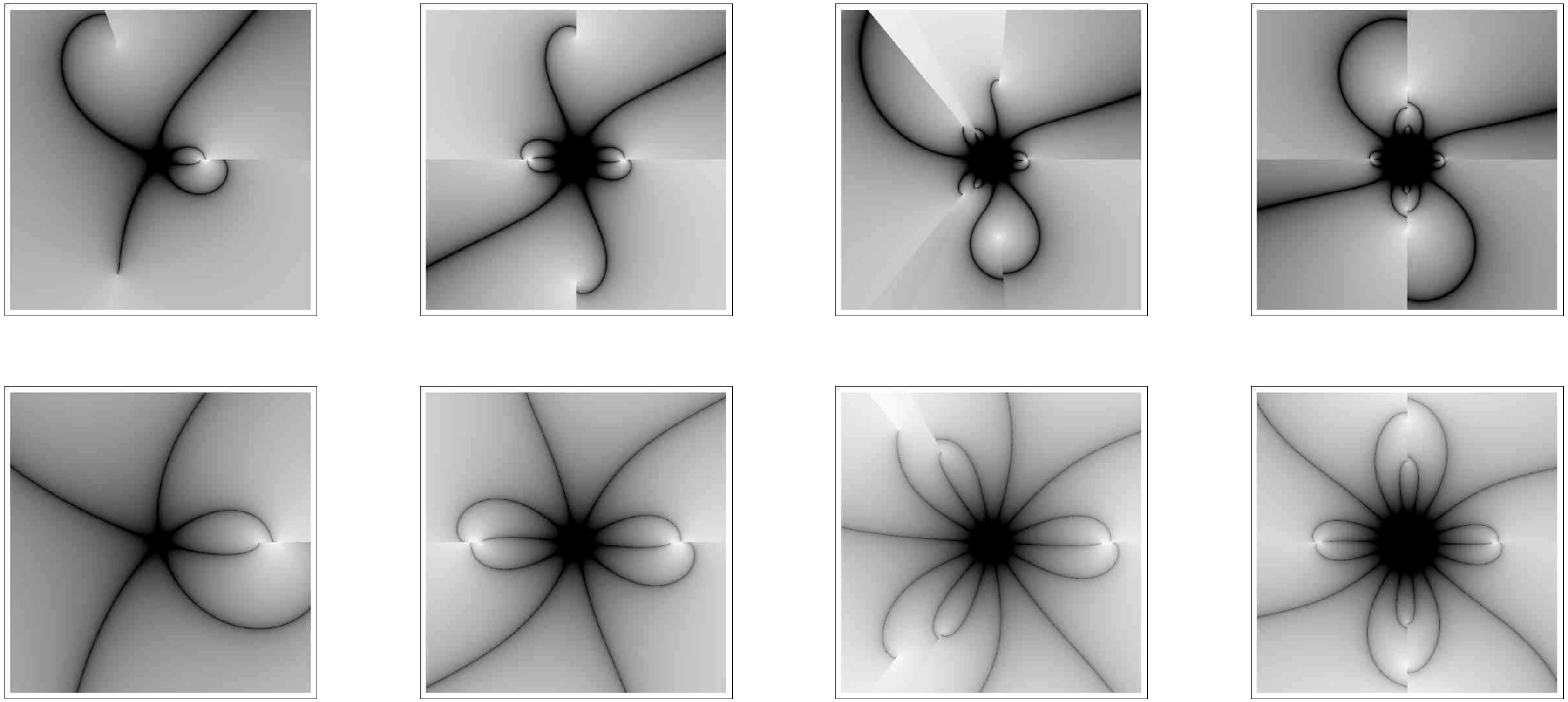
$$\omega = 4\pi n T_H \sin\left(\frac{\pi}{d-1}\right) \exp\left(\frac{i\pi}{d-1}\right) + \cdots .$$

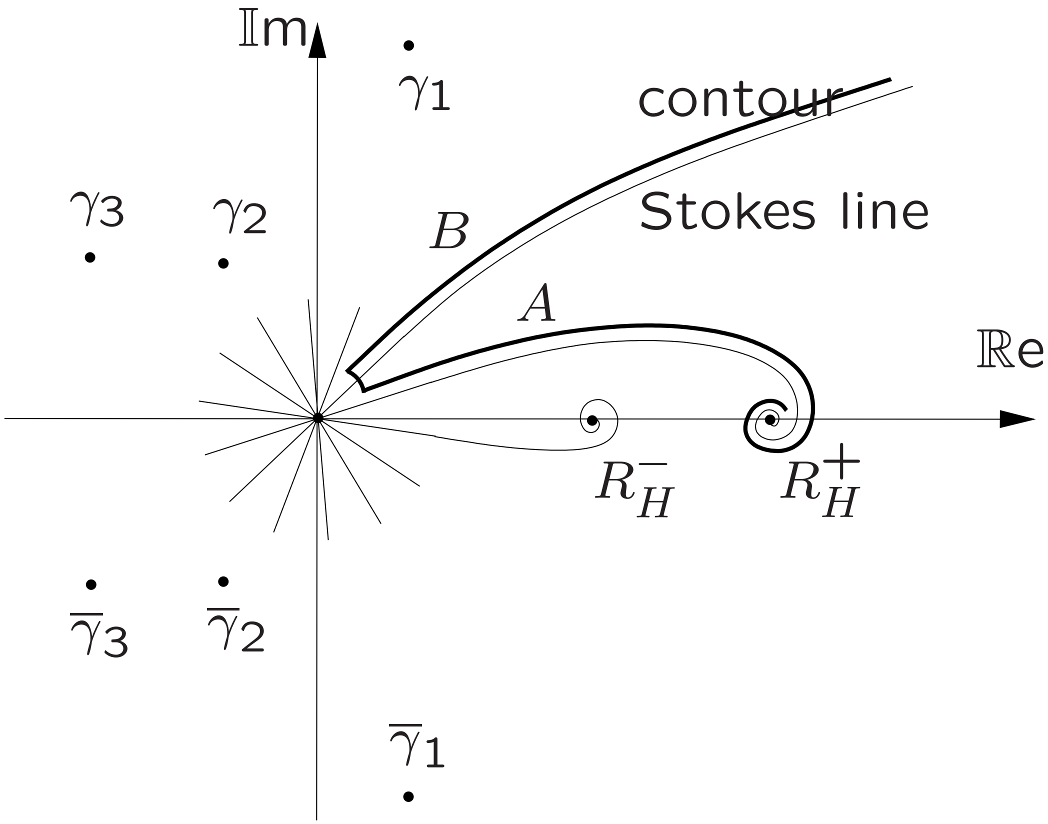
For most popular AdS/CFT dimensions, $d = 4, 5, 7$, obtain

$$\omega_{d=4} \sim 2\sqrt{3}\pi n T_H e^{\frac{i\pi}{3}}, \qquad \omega_{d=5} \sim 2\sqrt{2}\pi n T_H e^{\frac{i\pi}{4}} \qquad \omega_{d=7} \sim 2\pi n T_H e^{\frac{i\pi}{6}}.$$

The Reissner–Nordström Anti–de Sitter Solution

Stokes lines for Reissner–Nordström Anti–de Sitter black hole in dimensions $d = 4$, $d = 5$, $d = 6$ and $d = 7$ (with close–up):





Stokes line for Reissner–Nordström Anti–de Sitter black hole, along with chosen contour for monodromy matching, in $d = 6$.

The RN AdS Solution: For all types of perturbations, tensor, vector and scalar perturbations, equation for asymptotic quasinormal frequencies is the same and is

$$\sin(\pi j)\, e^{i\left(\frac{\pi}{4}(d-1)-\omega x_0\right)} + \sin\left(\frac{\pi j}{2}\right) e^{-i\left(\frac{\pi}{4}(d-1)-\omega x_0\right)} = 0,$$

with x_0 as before (only recall this time around have $(2d-4)$ complex horizons as any charged situation) and j as in previous RN cases. Again, no general analytic solution for x_0. In spite of this, can still solve for asymptotic quasinormal frequency as

$$\omega x_0 = \frac{\pi}{4}(d+1) - \frac{i}{2}\log\left(2\cos\left(\frac{\pi j}{2}\right)\right) + n\pi \qquad (n \in \mathbb{N}, \quad n \gg 1)\,.$$

Directions of Further Work

- Stationary spacetimes: the Kerr solution.

- Quasinormal frequencies are poles in black hole greybody factors, which play pivotal role in study of Hawking radiation. Results obtained for asymptotic greybody factors could be of help in identifying dual conformal field theory which microscopically describes the black hole.

- String theory black holes.