Formation of Higher-dimensional Topological Black Holes

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(based on `arXiv:0906.3216` *with Filipe Mena and Paul Tod)*

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Outline

1. Introduction
   - Topological black holes
   - Einstein manifolds
   - Einstein equations
2. Collapse to higher dimensional topological black holes
   - Generalized Kottler solutions
   - Generalized Friedman-Lemaître-Robertson-Walker solutions
   - Matching
   - Global properties
3. Collapse with gravitational wave emission
   - Exterior: Bizoń-Chmaj-Schmidt metric
   - Interiors
   - Matching

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Topological black holes

- 4-dimensional topological black holes have been considered in the literature.
- In these one replaces the spherically symmetric Kottler solution (i.e. Schwarzschild–de Sitter or Schwarzschild–anti–de Sitter) by its planar or hyperbolic counterparts.
- Modding out by discrete subgroup of $\mathbb{R}^2$ or $SL(2,\mathbb{R})$ yields black holes with toroidal or higher genus horizons.

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Topological black holes

- Caveat 1: needs negative cosmological constant to work.
- Caveat 2: $\mathscr{I}$ has the same topology (times $\mathbb{R}$).
- Formation of these black holes was studied in [Mena, Natário and Tod]. Many spherical collapses (e.g. Oppenheimer-Snyder) generalize to this setting.

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Einstein manifolds

- $(N, d\sigma^2)$ is a n-dimensional Einstein manifold with $Ricci = \lambda \, d\sigma^2$.
- Examples: S^n, T^n, H^n, $\mathbb{C}P^{\frac{n}{2}}$, Taub-NUT, Eguchi-Hanson, Calabi-Yau.
- Can be used to construct $(n+1)$-dimensional Einstein metrics

$$d\Sigma^2 = d\rho^2 + (f(\rho))^2 d\sigma^2$$

with $Ricci = k \, d\Sigma^2$.

- If $k > 0$ then $k = \nu^2 n$, $\lambda = \nu^2(n-1)$, $f = \sin(\nu\rho)$ for some $\nu > 0$.
  If $k = 0$ then $\lambda = n - 1$, $f = \rho$ or $\lambda = 0$, $f = 1$.
  If $k < 0$ then $k = -\nu^2 n$, $\lambda = \nu^2(n-1)$, $f = \sinh(\nu\rho)$ or $k = -\nu^2 n$, $\lambda = 0$, $f = e^{\pm\nu\rho}$ or $k = -\nu^2 n$, $\lambda = -\nu^2(n-1)$, $f = \cosh(\nu\rho)$ for some $\nu > 0$.

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Einstein manifolds

- $d\Sigma^2$ typically has a singularity at $\rho = 0$: the Kretschmann scalar K is related to the square C^2 of the Weyl tensor of $d\sigma^2$ by

$$K = \frac{C^2}{f^4} + \mathrm{const}.$$

- This can be avoided for $k < 0$ with $f = e^{\pm\nu\rho}$, when the metric has an internal infinity (cusp), or $f = \cosh(\nu\rho)$, when the metric has a minimal surface and a second asymptotic region (wormhole).

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Einstein equations

- We consider the $(n+2)$-dimensional Einstein equations

$$R_{ab} = \Lambda g_{ab} + \kappa\left(T_{ab} - \frac{1}{n} T g_{ab}\right),$$

or

$$R_{ab} - \frac{1}{2} R g_{ab} + \frac{n\Lambda}{2} g_{ab} = \kappa T_{ab}.$$

Outline

Generalized Kottler solutions

- The generalized Kottler solutions are vacuum ($T_{ab} = 0$) solutions given by
$$ds^2 = -V(r)dt^2 + (V(r))^{-1}dr^2 + r^2 d\sigma^2,$$
where
$$V(r) = \frac{\lambda}{n-1} - \frac{2m}{r^{n-1}} - \frac{\Lambda r^2}{n+1}.$$

Generalized Friedman-Lemaître-Robertson-Walker solutions

- The Generalized Friedman-Lemaître-Robertson-Walker solutions are dust ($T_{ab} = \mu\, u_a u_b$) solutions given by

$$ds^2 = -d\tau^2 + R^2(\tau)\, d\Sigma^2,$$

where $d\Sigma^2$ is any $(n+1)$-dimensional Riemannian Einstein metric with $Ricci = k\, d\Sigma^2$ and $R(\tau)$, $\mu(\tau)$ satisfy the conservation equation

$$\mu R^{n+1} = \mu_0$$

and the generalized Friedman equation

$$\frac{\dot{R}^2}{R^2} + \frac{k}{nR^2} = \frac{2\kappa\mu}{n(n+1)} + \frac{\Lambda}{n+1}.$$

Matching

- Can match generalized Kottler to generalized FLRW at $\rho = \rho_0$ provided that $f'(\rho_0) > 0$ and

$$m = \frac{\kappa \mu_0 (f(\rho_0))^{n+1}}{n(n+1)}.$$

- Similar results for generalized Lemaître-Tolman-Bondi.

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Global properties

- If $\Lambda = 0$ (hence $\lambda > 0$) and $(N, d\sigma^2)$ is not an n-sphere then the locally naked singularity is always visible from $\mathscr{I}^+$ for $k \leq 0$, but can be hidden if $k > 0$ and $n \geq 4$ (cf. [Ghosh and Beesham]).

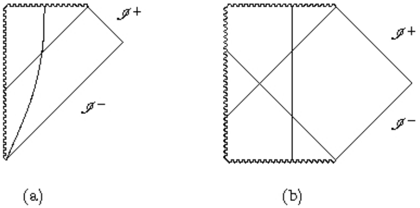

Figure 1: Penrose diagram for $\Lambda = 0$ and (a) $k \leq 0$; (b) $k > 0$, showing the matching surfaces and the horizons.

Global properties

- If $\Lambda > 0$ (hence $\lambda > 0$) and $(N, d\sigma^2)$ is not an n-sphere then the locally naked singularity can be always be hidden except if the FLRW universe is recollapsing (hence $k > 0$) and $n < 4$.

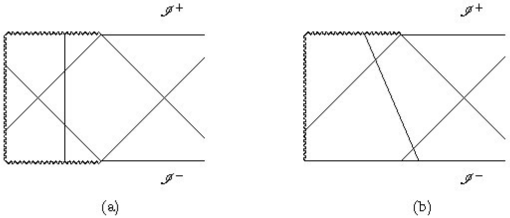

Figure 2: Penrose diagram for $\Lambda > 0$ with the FLRW universe (a) recollapsing; (b) non-recollapsing, showing the matching surfaces and the horizons.

Global properties

- For $\Lambda < 0$ we have:
  If $\lambda > 0$ and $(N, d\sigma^2)$ is not an n-sphere then the locally naked singularity can always be hidden.
  If $\lambda = 0$ then the cusp singularity is not locally naked.
  If $\lambda < 0$ then no causal curve can cross the wormhole from one $\mathscr{I}$ to the other (cf. [Galloway]).

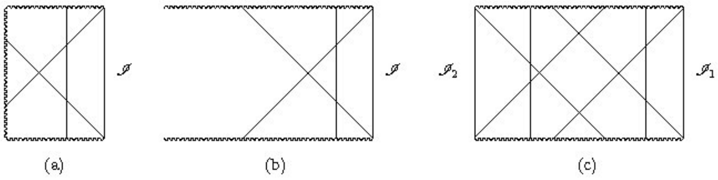

Figure 3: Penrose diagram for $\Lambda < 0$ and (a) $\lambda > 0$; (b) $\lambda = 0$; (c) $\lambda < 0$, showing the matching surfaces and the horizons.

Global properties

- Global properties are much more diverse in the generalized Lemaître-Tolman-Bondi case. For instance, one can easily find examples of black hole formation with wormholes inside the matter with positive λ and $\Lambda = 0$ (previously seen in 4 dimensions [Hellaby]).

Outline

Exterior: Bizoń-Chmaj-Schmidt metric

- The Bizoń-Chmaj-Schmidt metric is a vacuum ($T_{ab} = 0$) solution of the Einstein equations with $\Lambda = 0$ given by

$$ds^{2+} = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + \frac{r^2}{4}e^{2B}(\sigma_1^2 + \sigma_2^2) + \frac{r^2}{4}e^{-4B}\sigma_3^2,$$

where σ_i are the standard left-invariant 1-forms on S^3 and A, δ and B are functions of t and r satisfying

$$\partial_r A = -\frac{2A}{r} + \frac{1}{3r}(8e^{-2B} - 2e^{-8B}) - 2r(e^{2\delta}A^{-1}(\partial_t B)^2 + A(\partial_r B)^2);$$
$$\partial_t A = -4rA(\partial_t B)(\partial_r B);$$
$$\partial_r \delta = -2r(e^{2\delta}A^{-2}(\partial_t B)^2 + (\partial_r B)^2);$$
$$\partial_t(e^{\delta}A^{-1}r^3(\partial_t B)) - \partial_r(e^{-\delta}Ar^3(\partial_r B)) + \frac{4}{3}e^{-\delta}r(e^{-2B} - e^{-8B}) = 0.$$

Interiors

- Eguchi-Hanson ($S^3/\mathbb{Z}_2$):

$$d\Sigma^2 = \left(1 - \frac{a^4}{\rho^4}\right)^{-1} d\rho^2 + \frac{\rho^2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{\rho^2}{4}\left(1 - \frac{a^4}{\rho^4}\right)\sigma_3^2.$$

- k-Eguchi-Hanson ($S^3/\mathbb{Z}_p$):

$$d\Sigma^2 = \Delta^{-1} d\rho^2 + \frac{\rho^2}{4}(\sigma_1^2 + \sigma_2^2) + \frac{\rho^2}{4}\Delta\sigma_3^2,$$

where $k < 0$, $p \geq 3$, $\Delta = 1 - \frac{a^4}{\rho^4} - \frac{k}{6}\rho^2$ and $a^4 = \frac{4}{3k^2}(p-2)^2(p+1)$.

- k-Taub-NUT:

$$d\Sigma^2 = \frac{1}{4}\Sigma^{-1} d\rho^2 + \frac{1}{4}(\rho^2 - L^2)(\sigma_1^2 + \sigma_2^2) + L^2\Sigma\sigma_3^2,$$

where $\Sigma = \frac{(\rho - L)(1 - \frac{k}{12}(\rho - L)(\rho + 3L))}{\rho + L}$.

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where $\Sigma = \frac{(\rho-L)(1-\frac{k}{12}(\rho-L)(\rho+3L))}{\rho+L}$.

Matching

- In each case, the interior metric gives consistent data for the Bizoń-Chmaj-Schmidt metric at a comoving timelike hypersurface. Local existence of the radiating exterior in the neighbourhood of the matching surface is then guaranteed. In the case of Eguchi-Hanson (asymptotically locally Euclidean) and k-Taub-NUT with $k < 0$ (includes hyperbolic space), the data can be chosen to be close to the data for the Schwarzschild solution. Since this solution is known to be stable [Dafermos and Holzegel], it is reasonable to expect that the exterior will settle down to the Schwarzschild solution.

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