# Gravito-electromagnetic analogies

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## Outline

- Analogy between tidal tensors
- Analogy between inertial and electromagnetic fields
- Formal analogy between tidal tensors and electromagnetic fields

## Analogy between tidal tensors

- Idea: compare electromagnetic and gravitational interactions using only the physical (covariant) forces of the two theories.

- No gravitational force analogous to the Lorentz force, but tidal forces exist in both theories. Two basic effects:
  - the relative acceleration of two nearby test particles;
  - the net force exerted on particles with dipole moments.

- Worldline deviation:

$$\frac{D^2\delta x^\alpha}{d\tau^2} = \frac{q}{m} E^\alpha{}_\beta \delta x^\beta \,, \qquad\qquad E^\alpha{}_\beta \equiv F^\alpha{}_{\mu;\beta} U^\mu \,.$$

- Geodesic deviation:

$$\frac{D^2\delta x^\alpha}{d\tau^2} = -\mathbb{E}^\alpha{}_\beta \delta x^\beta \,, \qquad\qquad \mathbb{E}^\alpha{}_\beta \equiv R^\alpha{}_{\mu\beta\nu} U^\mu U^\nu \,.$$

- Force on a magnetic dipole:

$$F_{EM}^{\beta} = B_{\alpha}{}^{\beta}\mu^{\alpha}\,, \qquad\qquad B^{\alpha}{}_{\beta} \equiv \star F^{\alpha}{}_{\mu;\beta}U^{\mu}\,.$$

- Force on a gyroscope:

$$F_{G}^{\beta} = -\mathbb{H}_{\alpha}{}^{\beta}S^{\alpha}\,, \qquad\qquad \mathbb{H}^{\alpha}{}_{\beta} \equiv \star R^{\alpha}{}_{\mu\beta\nu}U^{\mu}U^{\nu}\,.$$

- A third effect: differential precession.
  - Of a magnetic dipole at rest with respect to a system of nearby magnetic dipoles:

$$\delta\Omega^i_{\mathsf{EM}} = -\sigma B^i{}_\beta \delta x^\beta$$

    ($\sigma$ gyromagnetic ratio).
  - Of a gyroscope at rest in the Fermi frame determined by system of nearby gyroscopes:

$$\delta\Omega^i_{\mathsf{G}} = \mathbb{H}^i{}_\beta \delta x^\beta \, .$$

- Time projection of the Maxwell source equations ($\star\, d \star F \;=\; 4\pi\; J$):

$$E^{\alpha}{}_{\alpha} = 4\pi\rho_c\,.$$

- Time-time projection of the Einstein equations:

$$\mathbb{E}^{\alpha}{}_{\alpha} = 4\pi\,(2\rho + T^{\alpha}_{\alpha})\,.$$

- Space projection of the Maxwell source equations:

$$B_{[\alpha\beta]} = \frac{1}{2} \star F_{\alpha\beta;\gamma} U^{\gamma} - 2\pi\epsilon_{\alpha\beta\sigma\gamma} j^{\sigma} U^{\gamma}\,.$$

- Time-space projection of the Einstein equations:

$$\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma} J^{\sigma} U^{\gamma}\,.$$

- Time projection of the Bianchi identity ($dF = 0$):

$$B^{\alpha}{}_{\alpha} = 0\,.$$

- Time-time projection of the algebraic Bianchi identity:

$$\mathbb{H}^{\alpha}{}_{\alpha} = 0\,.$$

- Space projection of the Bianchi identity:

$$E_{[\alpha\beta]} = \frac{1}{2} F_{\alpha\beta;\gamma} U^{\gamma}\,.$$

- Space-time projection of the algebraic Bianchi identity:

$$\mathbb{E}_{[\alpha\beta]} = 0\,.$$

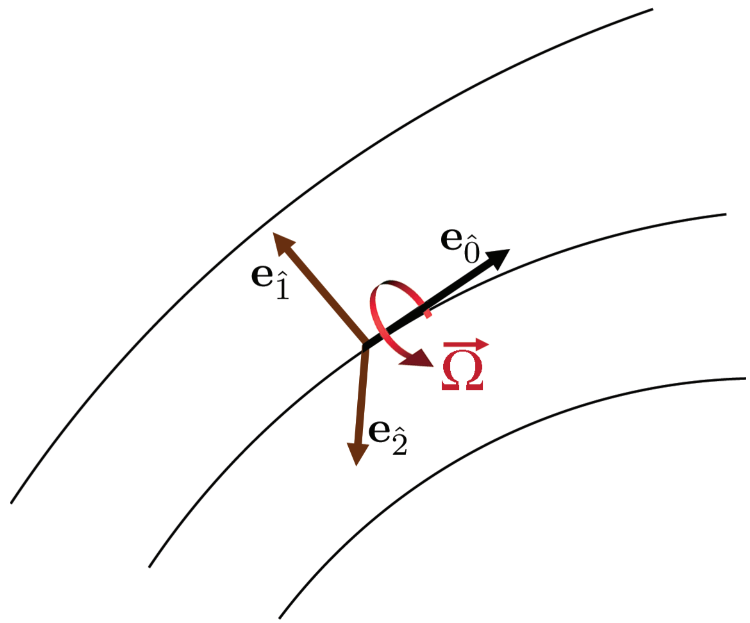
- Antisymmetric parts of electromagnetic tidal tensors encode the laws of electromagnetic induction.
  - $E_{[\alpha\beta]}$ encodes Faraday's law of induction;
  - $B_{[\alpha\beta]}$ encodes Maxwell's displacement current.
- No analogous effects in the gravitational physical forces.

## Analogy between inertial and electromagnetic fields

- Different spirit from the approach based on tidal tensors, where we compared physical forces of both theories.

- Here the parallelism is between physical electromagnetic forces and inertial “forces” (i.e. reference frame effects).

- Threading: congruence of observers with 4-velocity $\mathbf{e}_{\hat{0}}$ carrying a spatial triad $\{\mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}}\}$.



- Kinematical quantities:
  - acceleration $a^{\hat{i}} = \Gamma^{\hat{i}}_{\hat{0}\hat{0}} = \Gamma^{\hat{0}}_{\hat{0}\hat{i}}$ ;
  - rotation relative to Fermi-Walker transport $\Omega^{\hat{k}} = \frac{1}{2}\epsilon_{\hat{i}\hat{j}\hat{k}}\Gamma^{\hat{j}}_{\hat{0}\hat{i}}$ ;
  - kinematical tensor $K_{\hat{i}\hat{j}} \equiv \nabla_{\hat{j}} u_{\hat{i}} = \Gamma^{\hat{i}}_{\hat{j}\hat{0}} = \Gamma^{\hat{0}}_{\hat{j}\hat{i}}$ ;
  - expansion $\theta = K^{\hat{i}}{}_{\hat{i}}$ ;
  - vorticity $\omega_{\hat{i}\hat{j}} = K_{[\hat{i}\hat{j}]}$ ;
  - shear $\sigma_{\hat{i}\hat{j}} = K_{(\hat{i}\hat{j})} - \frac{1}{3}\theta\delta_{\hat{i}\hat{j}}$ .

- If $X$ is a connecting vector and $Y$ is its spatial part then

$$\dot{Y}_{\hat{i}} = \left(\sigma_{\widehat{i}\widehat{j}} + \frac{1}{3}\theta\delta_{\widehat{i}\widehat{j}} + \omega_{\widehat{i}\widehat{j}} - \Omega_{\widehat{i}\widehat{j}}\right) Y^{\hat{j}} .$$

- Geodesic equation:

$$\frac{\tilde{D}\vec{U}}{d\tau} = U^{\hat{0}}\left[U^{\hat{0}}\vec{G} + \vec{U} \times \vec{H} - \sigma^{\hat{i}}{}_{\hat{j}}U^{\hat{j}}\mathbf{e}_{\hat{i}} - \frac{1}{3}\theta\vec{U}\right].$$

- $\vec{G} = -\vec{a}$ is the gravitoelectric field and $\vec{H} = \vec{\omega} + \vec{\Omega}$ is the gravitomagnetic field.

- Lorentz force law:

$$\frac{D\vec{U}}{d\tau} = \frac{q}{m}\left(U^0\vec{E} + \vec{U} \times \vec{B}\right).$$

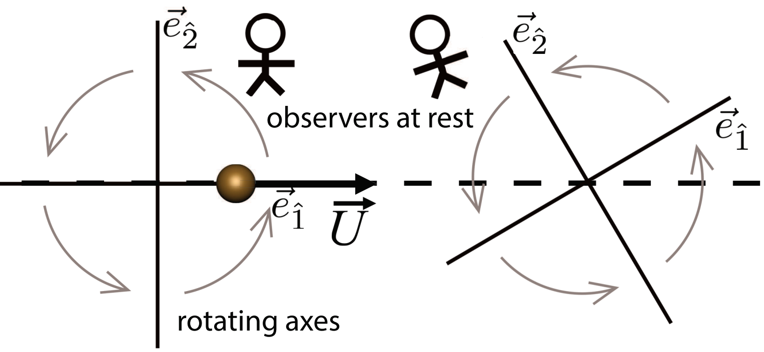
- In the equation of motion

$$\frac{\tilde{D}X^{\hat{i}}}{d\tau} = \frac{dX^{\hat{i}}}{d\tau} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{j}} X^{\hat{k}} \,.$$

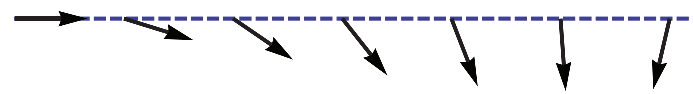
- Mathematically, this is the covariant derivative associated to a connection $\tilde{\nabla}$ on the bundle of spatial vectors (ortogonal to the congruence).

- It corrects the Fermi-Walker connection $\nabla^{\perp}$ (projected Levi-Civita connection) so that the triad $\{\mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}}\}$ is constant along the congruence:

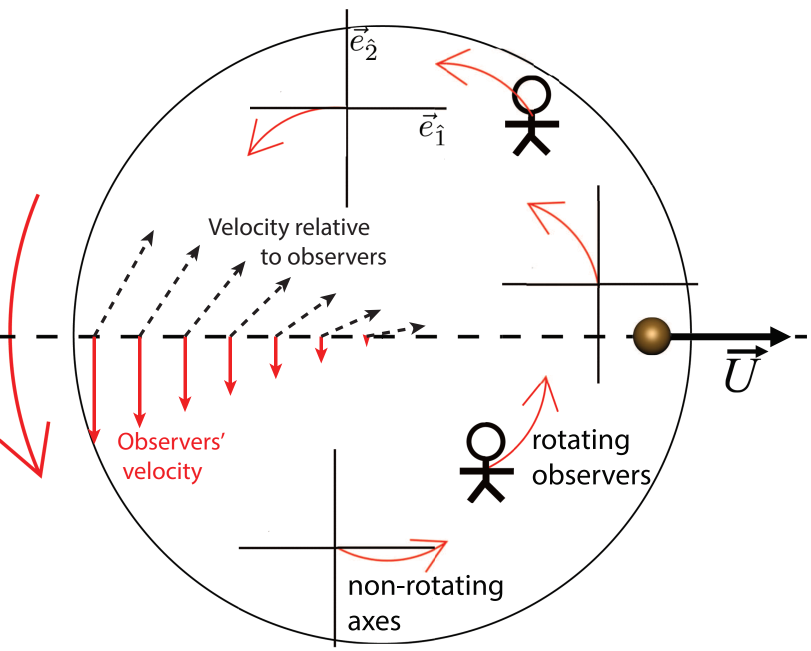
$$\frac{D^{\perp}X^{\hat{i}}}{d\tau} = \frac{dX^{\hat{i}}}{d\tau} + \Omega^{\hat{i}}{}_{\hat{j}} U^{\hat{0}} X^{\hat{j}} + \Gamma^{\hat{i}}_{\hat{j}\hat{k}} U^{\hat{j}} X^{\hat{k}} \,.$$




In the tetrad:



$$\vec{G} = 0; \quad \vec{H} = \vec{\Omega}$$

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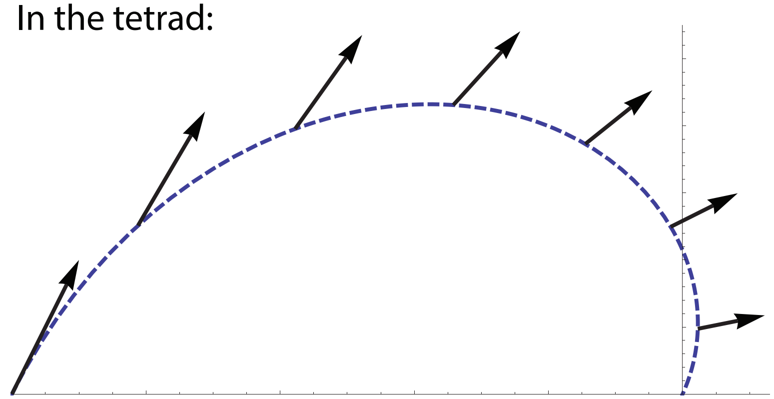
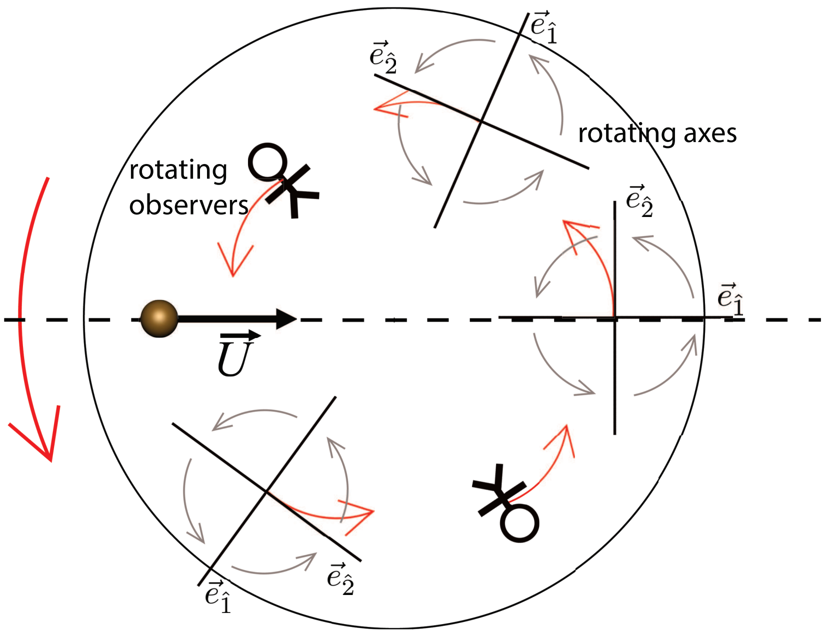
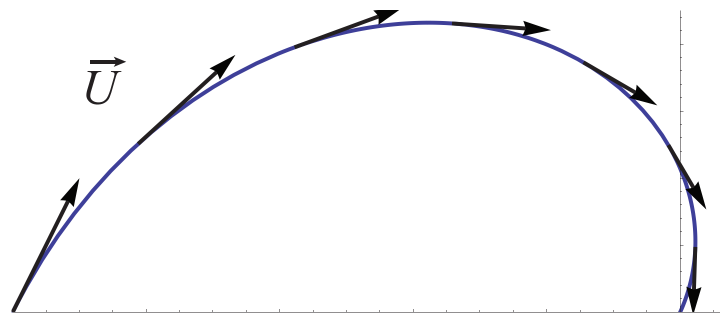


$$\vec{G} = \vec{\omega} \times (\vec{r} \times \vec{\omega}); \quad \vec{H} = \vec{\omega}$$




In the tetrad:




$$\vec{G} = \vec{\omega} \times (\vec{r} \times \vec{\omega}); \quad \vec{H} = \vec{\Omega} + \vec{\omega} = 2\vec{\omega}$$

- Important particular cases:
  - Quasi-Maxwell: rigid, congruence adapted ($\vec{\Omega} = \vec{\omega}$) frame in a stationary spacetime. Yields the gravitomagnetic field $\vec{H}$ measured by LARES/LAGEOS. Gives the rotation of local frames relative to the distant stars, as measured by Gravity Probe B.
  - Locally non-rotating observers: vorticity-free congruence with axes fixed to some coordinate system. Useful when studying black holes.

- If the Mathisson-Pirani condition $S^{\alpha\beta}U_\beta = 0$ holds, the spin vector of a pole-dipole particle in a gravitational field is Fermi-Walker transported:

$$\frac{D_F S^\alpha}{d\tau} = 0 \,.$$

- In the congruence adapted orthonormal frame:

$$\frac{d\vec{S}}{d\tau} = \frac{1}{2}\vec{S} \times \vec{H} \,.$$

- Resembles the precession of a magnetic dipole:

$$\frac{D\vec{S}}{d\tau} = \vec{\mu} \times \vec{B} \,.$$

- The force on a gyroscope at rest in a quasi-Maxwell frame is

$$\vec{F}_G = \frac{1}{2}\left[\tilde{\nabla}(\vec{H}\cdot\vec{S}) - \vec{S}(\tilde{\nabla}\cdot\vec{H}) - 2(\vec{S}\cdot\vec{H})\vec{G}\right].$$

- Similar to the electromagnetic force on a magnetic dipole:

$$\vec{F}_{EM} = \tilde{\nabla}(\vec{B}\cdot\vec{\mu}) - \frac{1}{2}\left[\vec{\mu}(\tilde{\nabla}\cdot\vec{B}) + (\vec{\mu}\cdot\vec{H})\vec{E}\right].$$

- Splitting of the electromagnetic and gravitational field equations into time and space projections for a quasi-Maxwell frame is similar.

- In this case $\tilde{\nabla}$ is the Levi-Civita connection of the Riemannian quotient manifold (space manifold).

- Maxwell equations:

$$\tilde{\nabla} \cdot \vec{E} = 4\pi\rho_c + \vec{H} \cdot \vec{B}\,;$$
$$\tilde{\nabla} \cdot \vec{B} = -\vec{H} \cdot \vec{E}\,;$$
$$\tilde{\nabla} \times \vec{E} = \vec{G} \times \vec{E}\,;$$
$$\tilde{\nabla} \times \vec{B} = \vec{G} \times \vec{B} + 4\pi\vec{j}\,.$$

- Gravitational field equations (time-time and time-space Einstein equations plus algebraic Bianchi identities):

$$\tilde{\nabla} \cdot \vec{G} = -4\pi(2\rho + T^{\alpha}{}_{\alpha}) + \vec{G}^2 + \frac{1}{2}\vec{H}^2\,;$$
$$\tilde{\nabla} \cdot \vec{H} = -\vec{H} \cdot \vec{G}\,;$$
$$\tilde{\nabla} \times \vec{G} = \vec{0}\,;$$
$$\tilde{\nabla} \times \vec{H} = 2\vec{G} \times \vec{H} - 16\pi\vec{J}\,.$$

- Gravitational tidal tensors:

$$\mathbb{E}_{ij} = -\tilde{\nabla}_j G_i + G_i G_j + \frac{1}{4}\left(\vec{H}^2 \gamma_{ij} - H_j H_i\right);$$

$$\mathbb{H}_{ij} = -\frac{1}{2}\left[\tilde{\nabla}_j H_i + (\vec{G}\cdot\vec{H})\gamma_{ij} - 2G_j H_i\right].$$

- Electromagnetic tidal tensors:

$$E_{ij} = \tilde{\nabla}_j E_i - \frac{1}{2}\left[\vec{B}\cdot\vec{H}\gamma_{ij} - B_j H_i\right]; \qquad E_{i0} = \frac{1}{2}(\vec{H}\times\vec{E})_i + (\vec{G}\times\vec{B})_i;$$

$$B_{ij} = \tilde{\nabla}_j B_i + \frac{1}{2}\left[\vec{E}\cdot\vec{H}\gamma_{ij} - E_j H_i\right]; \qquad B_{i0} = \frac{1}{2}(\vec{H}\times\vec{B})_i - (\vec{G}\times\vec{E})_i.$$

- Example: the Gödel universe is given by

$$ds^2 = \left(dt + e^{\sqrt{2}\omega x} dy\right)^2 + dx^2 + \frac{1}{2} e^{2\sqrt{2}\omega x} dy^2 + dz^2.$$

- We have $\vec{G} = \vec{0}$ and $\vec{H} = 2\omega \vec{e}_z$, so the gravitomagnetic tidal tensor vanishes: $\mathbb{H}_{\alpha\beta} = 0$.

- On the other hand,

$$\mathbb{E}_{ij} = \omega^2 \left(\gamma_{ij} - \delta_i^z \delta_j^z\right)$$

is similar to the Newtonian tidal tensor $\nabla_i \nabla_j V$ of a potential $V = \omega^2(x^2 + y^2)/2$, corresponding to the Newtonian analogue of the Gödel Universe: an infinite cylinder of dust rotating rigidly with angular velocity $\omega$.

- Linear GEM:

$$ds^2 = -\left(1 + 2\Phi\right) dt^2 + 2\mathcal{A}_j dt dx^j + \left(1 - 2\Phi\right) \delta_{ij} dx^i dx^j \,.$$

- Gravitational fields:

$$\vec{G} = -\nabla\Phi - \frac{\partial\vec{\mathcal{A}}}{\partial t}\,; \qquad\qquad \vec{H} = \nabla \times \vec{\mathcal{A}}\,.$$

- Geodesic equation:

$$\frac{d\vec{U}}{dt} = \vec{G} + \vec{v} \times \vec{H} + 2\frac{\partial\Phi}{\partial t}\vec{v}\,.$$

- Linearized Einstein equations:

$$\nabla \cdot \vec{G} = -4\pi\rho + 3\frac{\partial^2 \Phi}{\partial t^2} \ ;$$

$$\nabla \times \vec{G} = -\frac{\partial \vec{H}}{\partial t} \ ;$$

$$\nabla \cdot \vec{H} = 0 \ ;$$

$$\nabla \times \vec{H} = -16\pi \vec{J} + 4\frac{\partial \vec{G}}{\partial t} - 4\frac{\partial^2 \vec{\mathcal{A}}}{\partial t^2} \ ;$$

$$\frac{\partial}{\partial t}\mathcal{A}_{(i,j)} - \left(\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi\right)\delta_{ij} = -4\pi\rho\delta_{ij} \ .$$

- Linearized tidal tensors:

$$\mathbb{E}_{ij} \approx -G_{i,j} + \frac{1}{2}\epsilon_{ijk}\frac{\partial H^k}{\partial t} - \frac{\partial \Phi}{\partial t}\delta_{ij} = -G_{(i,j)} - \frac{\partial \Phi}{\partial t}\delta_{ij} \; ;$$

$$\mathbb{H}_{ij} \approx -\frac{1}{2}\left[H_{i,j} - 2\epsilon_{ijl}\left(\frac{\partial G^l}{\partial t} - \frac{\partial^2 \mathcal{A}^l}{\partial t^2}\right)\right] .$$

## Formal analogy between tidal tensors and electromagnetic fields

- Faraday tensor splits, with respect to a unit timelike vector $U^\alpha$, into its electric and magnetic parts:

$$E^\alpha = F^\alpha{}_\beta U^\beta\,, \qquad B^\alpha = \star F^\alpha{}_\beta U^\beta\,.$$

- Two algebraically independent invariants:

$$E^\alpha E_\alpha - B^\alpha B_\alpha = -\frac{F_{\alpha\beta}F^{\alpha\beta}}{2}\,, \qquad E^\alpha B_\alpha = -\frac{F_{\alpha\beta}\star F^{\alpha\beta}}{4}\,.$$

- The Weyl tensor splits, with respect to a unit timelike vector $U^\alpha$, into its electric and magnetic parts:

$$\mathcal{E}_{\alpha\beta} = C_{\alpha\gamma\beta\sigma}U^\gamma U^\sigma \,, \qquad \mathcal{H}_{\alpha\beta} = \star C_{\alpha\gamma\beta\sigma}U^\gamma U^\sigma \,.$$

- Two algebraically independent invariants:

$$\mathcal{E}^{\alpha\beta}\mathcal{E}_{\alpha\beta} - \mathcal{H}^{\alpha\beta}\mathcal{H}_{\alpha\beta} = \frac{C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}}{8}\,, \qquad \mathcal{E}^{\alpha\beta}\mathcal{H}_{\alpha\beta} = \frac{C_{\alpha\beta\mu\nu} \star C^{\alpha\beta\mu\nu}}{16}\,.$$

- There are also two cubic invariants.

- Maxwell's equations:

$$E^i{}_{,i} = 0\,; \qquad \epsilon^{ikl} E_{l,k} = -\frac{\partial B^i}{\partial t}\,;$$

$$B^i{}_{,i} = 0\,; \qquad \epsilon^{ikl} B_{l,k} = \frac{\partial E^i}{\partial t}\,.$$

- Matte's equations (linearized Einstein plus second Bianchi):

$$\mathbb{E}^{ij}{}_{,i} = 0\,; \qquad \epsilon^{ikl} \mathbb{E}^j{}_{l,k} = -\frac{\partial \mathbb{H}^{ij}}{\partial t}\,;$$

$$\mathbb{H}^{ij}{}_{,i} = 0\,; \qquad \epsilon^{ilk} \mathbb{H}^j{}_{l,k} = \frac{\partial \mathbb{E}^{ij}}{\partial t}\,.$$

- Wave equations:

$$\left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) E^i = 0\,;$$
$$\left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) B^i = 0\,;$$

$$\left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) \mathbb{E}_{ij} = 0\,;$$
$$\left(\frac{\partial^2}{\partial t^2} - \partial^k \partial_k\right) \mathbb{H}_{ij} = 0\,.$$

- Gravitational waves as propagating tidal tensors (do not couple to monopole particles). Superenergy?

- However, this analogy is purely formal, as it compares objects which are not physically analogous.

- In particular, the invariants give conditions for the vanishing of $\mathbb{H}$, not $\vec{H}$, and cannot be used to discuss effects such as gravitomagnetic forces on test particles or precession of gyroscopes.

- Future work with Filipe Costa and Lode Wylleman.