Strong cosmic censorship in spherical symmetry

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Outline

- Penrose diagrams
- Strong cosmic censorship
- What do you mean by “inextendible”?
- (Weak cosmic censorship)
- Einstein-Maxwell-scalar field equations in spherical symmetry
- Christodoulou’s results
- Dafermos’ results
- Our results

Penrose diagrams

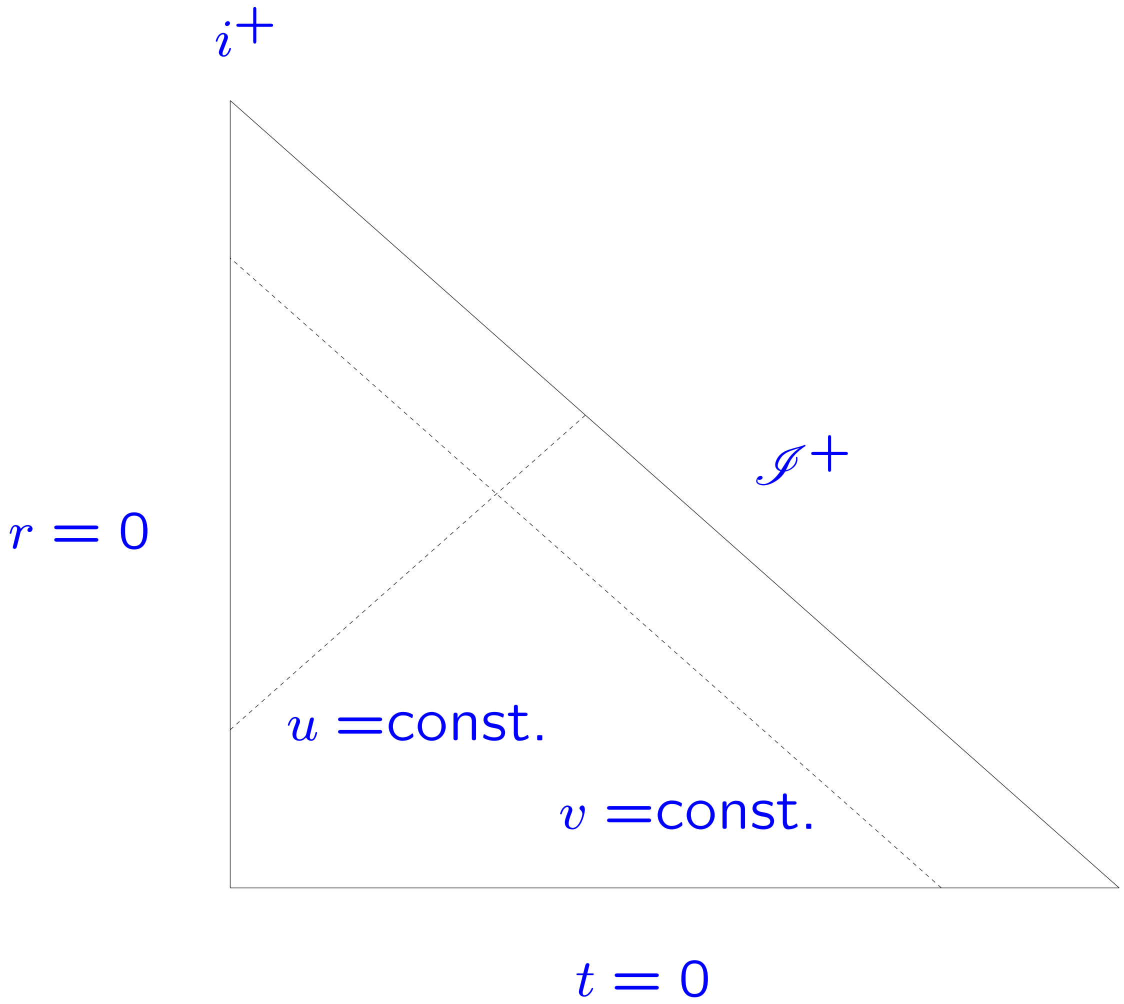
- The Minkowski metric is spherical coordinates reads

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

- Defining the retarded time $u = t - r$ and the advanced time $v = t + r$ yields

$$ds^2 = -du\, dv + r^2(u,v) \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

- These null coordinates can easily be rescaled to make their range bounded, yielding the Penrose diagram.

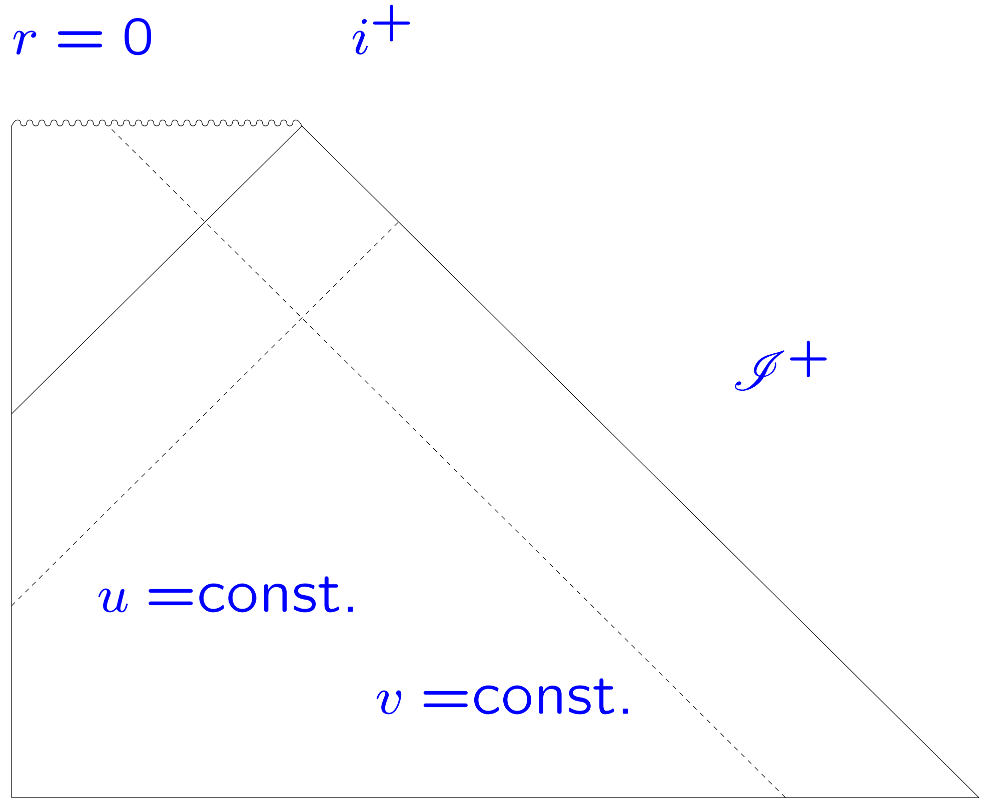


i^+
$\mathscr{I}^+$
$r = 0$
u =const.
v =const.
$t = 0$

- The metric of any spherically symmetric spacetime can be written as

$$ds^2 = -\Omega^2(u,v)du\,dv + r^2(u,v)\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right).$$

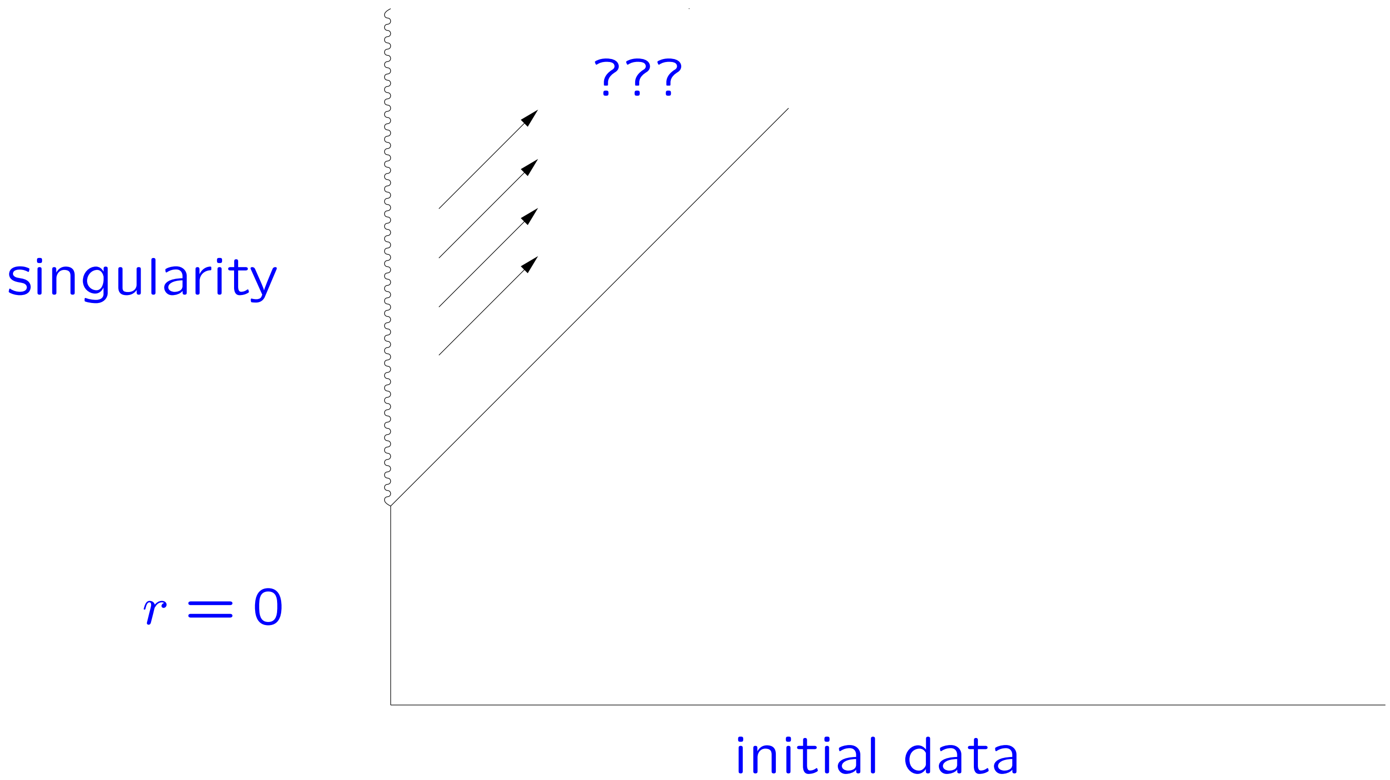
- Example: Penrose diagram for the formation of a black hole.



$r = 0$
i^+
$\mathscr{I}^+$
u =const.
v =const.

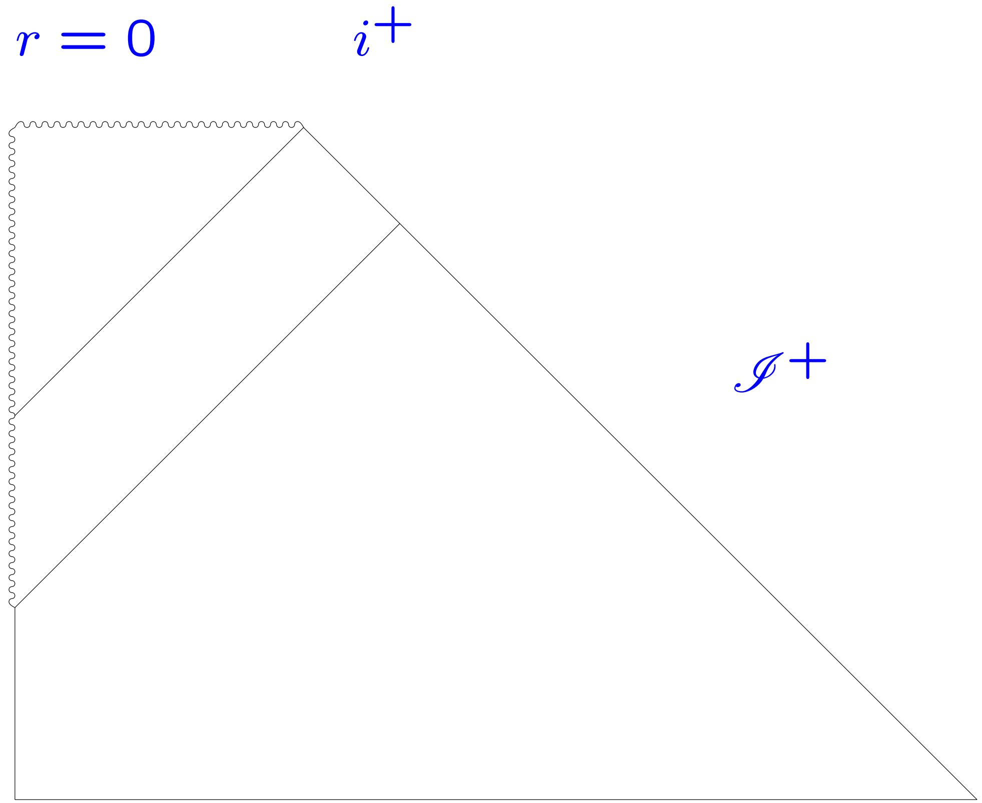
Strong cosmic censorship

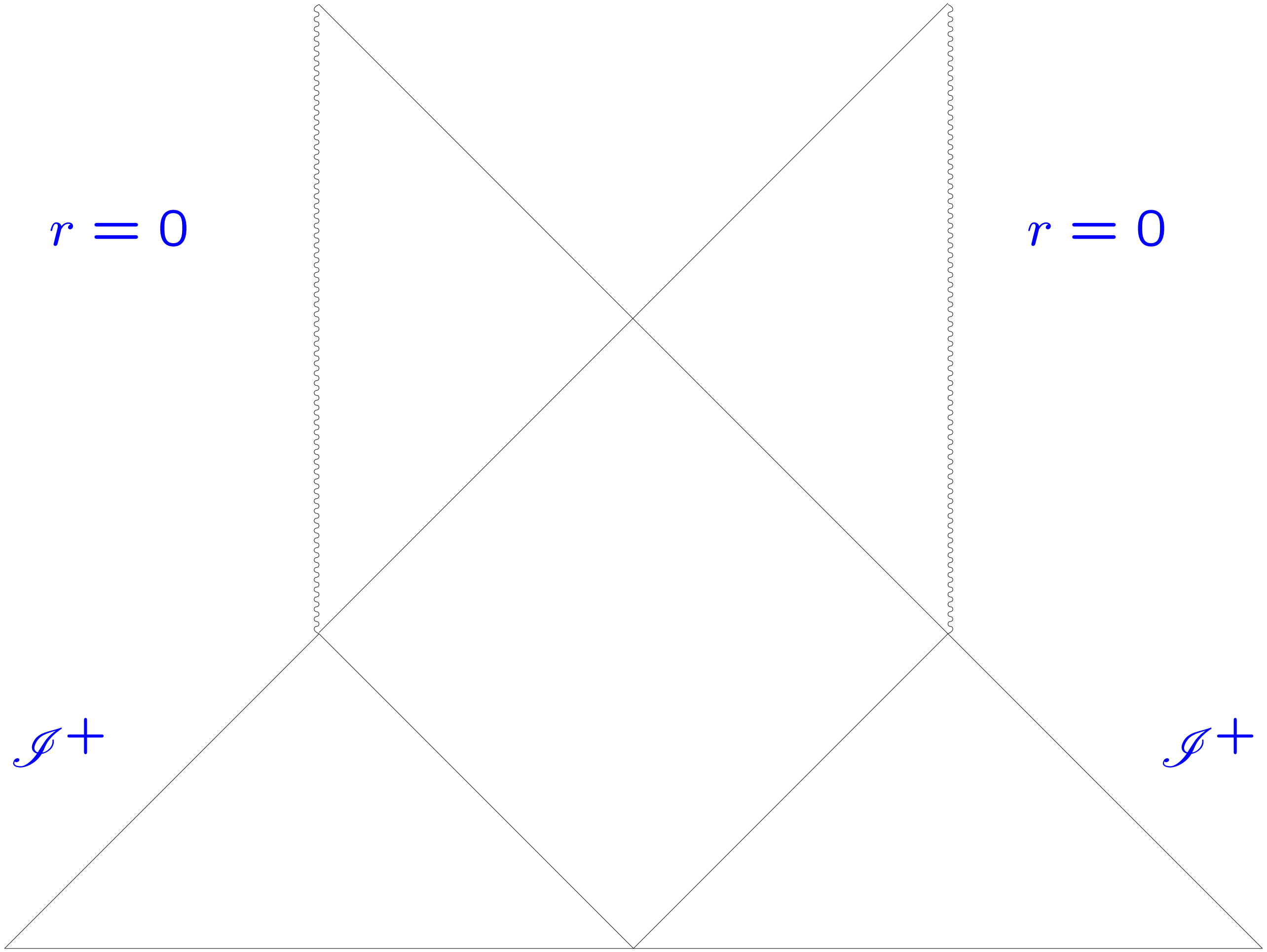
- The appearance of a visible singularity destroys determinism: the singularity can radiate gravitationally or otherwise (mathematically it is a singular boundary).


???
singularity
$r = 0$
initial data

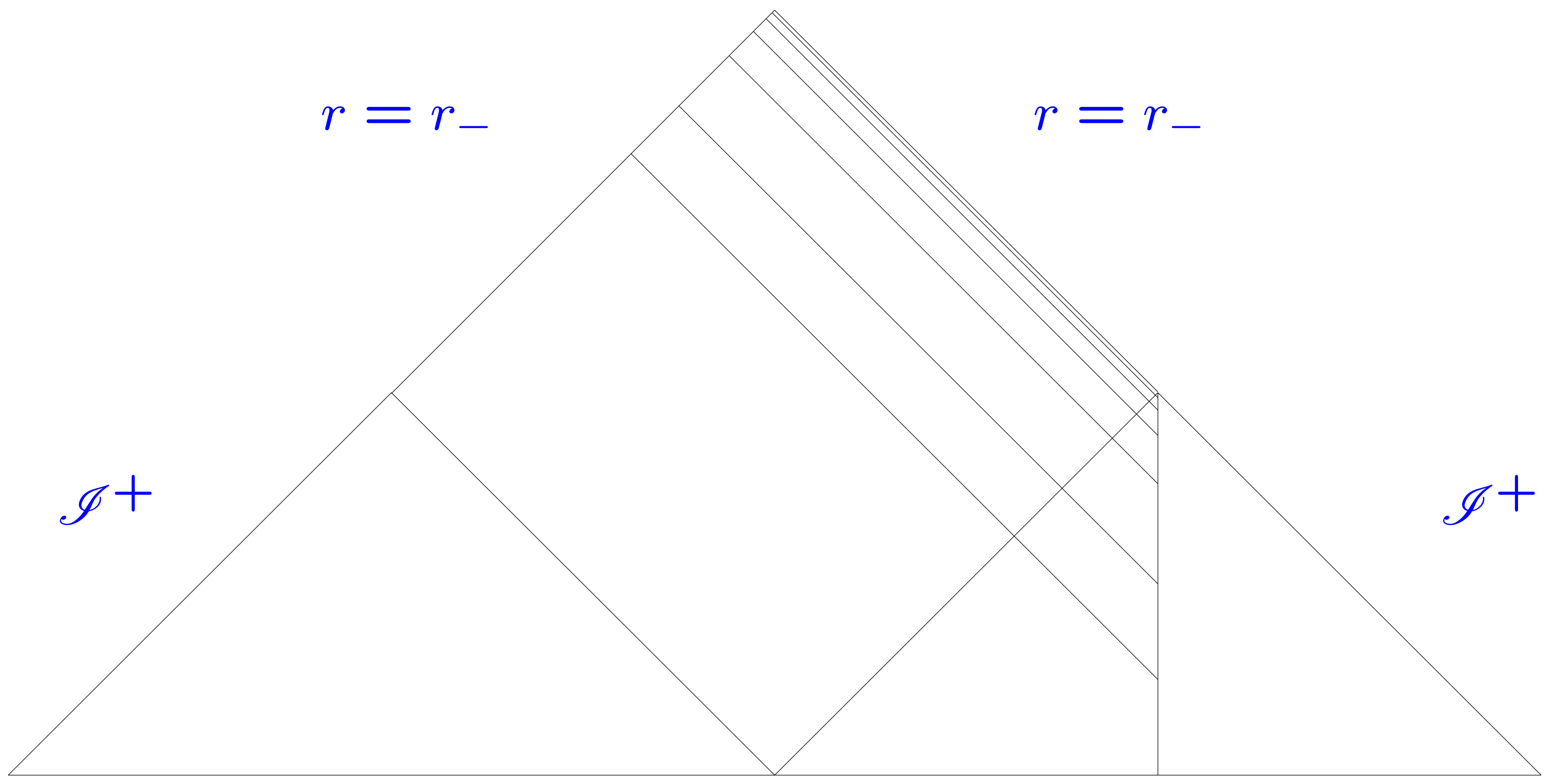
- Obvious conjecture: the singularities that form are never locally naked, that is, visible by some observer.

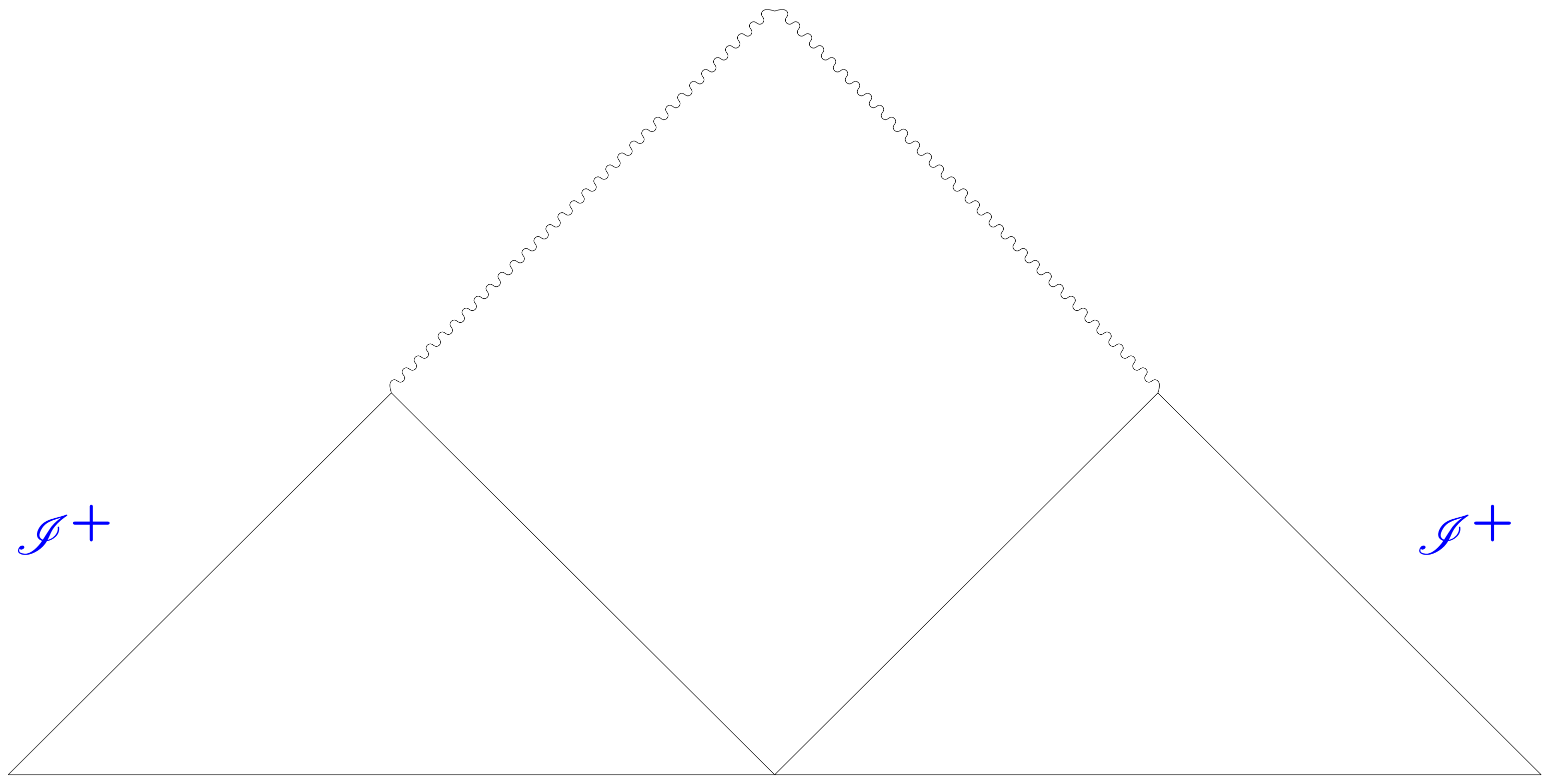
- Well known to be false: counter-examples are the dust cloud solutions of Christodoulou, or the Reissner-Nordström solution.


$r = 0$
i^+
$\mathscr{I}^+$


$r = 0$
$r = 0$
$\mathscr{I}^+$
$\mathscr{I}^+$

- However, the blueshift effect should make the Cauchy horizon unstable.


$r = r_-$
$r = r_-$
$\mathscr{I}^+$
$\mathscr{I}^+$


$\mathscr{I}^+$
$\mathscr{I}^+$

- Corrected conjecture: For generic solutions of the Einstein field equations with reasonable matter models, the singularities that form are never locally naked.

- PDE version: For generic asymptotically flat or compact initial data the future maximal globally hyperbolic development is inextendible.

What do you mean by “inextendible”?

- Inextendible as a Lorentzian manifold or inextendible as a solution?

- Strongest possible form: no C^0 extensions (that is, extensions with continuous metric).

- What you get if curvature blows up: no C^2 extensions.

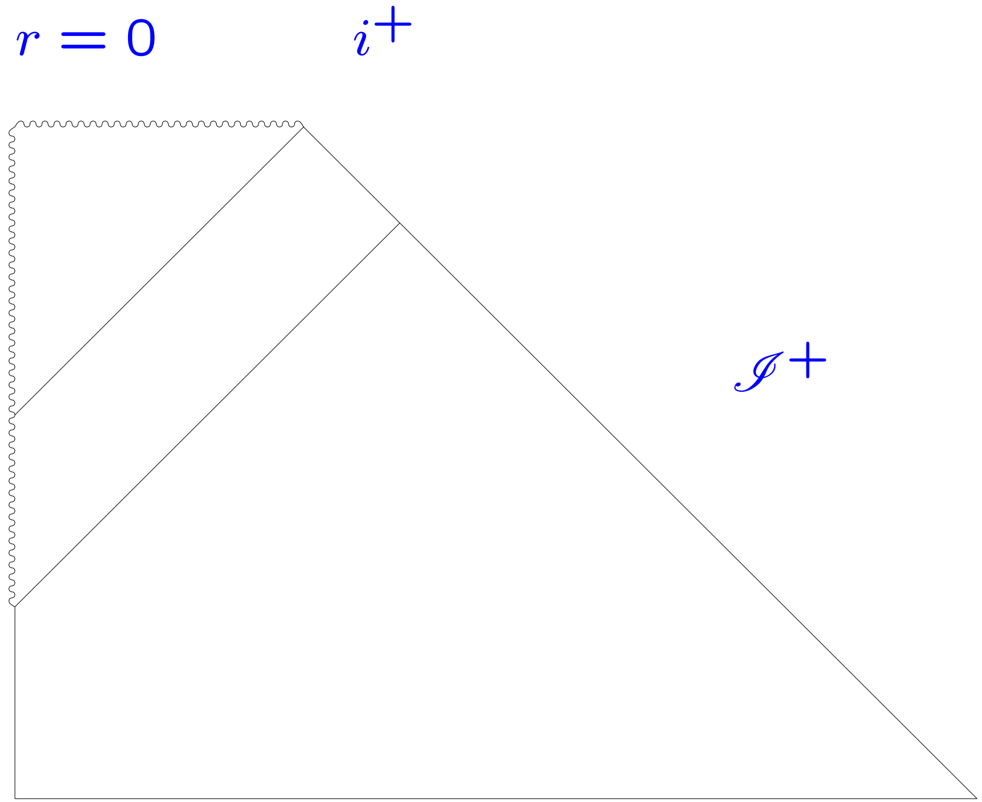
- What you really want: inextendible as a solution, in the class of functions you are considering (not necessarily C^2: shocks, impulsive gravitational waves, ...).

- If you want to prevent any conceivable extension as solution: no C^0 extensions with Christoffel symbols in $L^2_{\text{loc}} \subset L^1_{\text{loc}}$ (Christodoulou).

$$0 = \int_M Ric \cdot \varphi = \int_M (\partial\Gamma + \Gamma\Gamma) \cdot \varphi = \int_M (-\Gamma \cdot \partial\varphi + \Gamma\Gamma \cdot \varphi)$$

(Weak cosmic censorship)

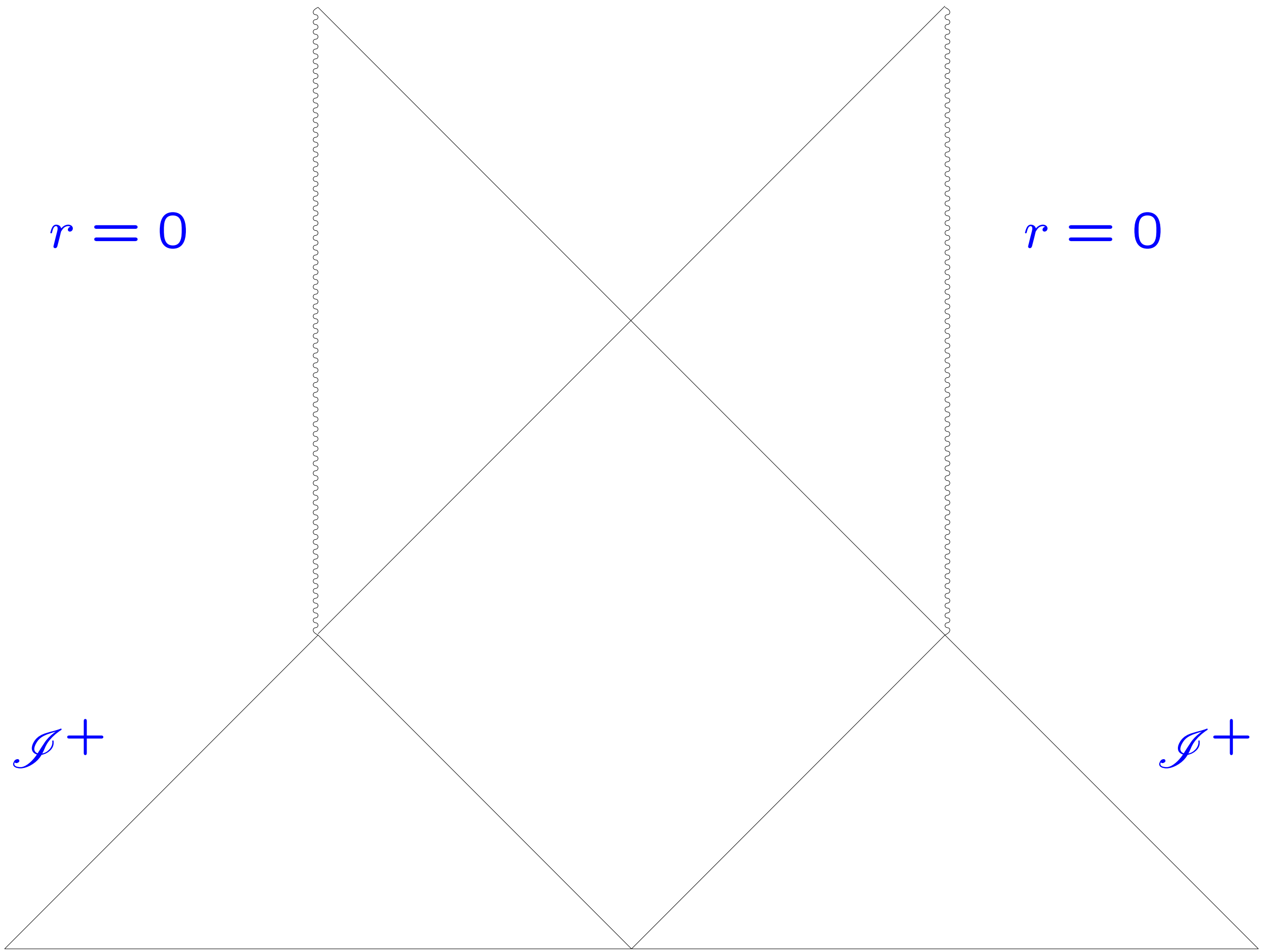
- For generic asymptotically flat solutions of the Einstein field equations with reasonable matter models, the singularities that form are never naked, that is, visible by observers at infinity.



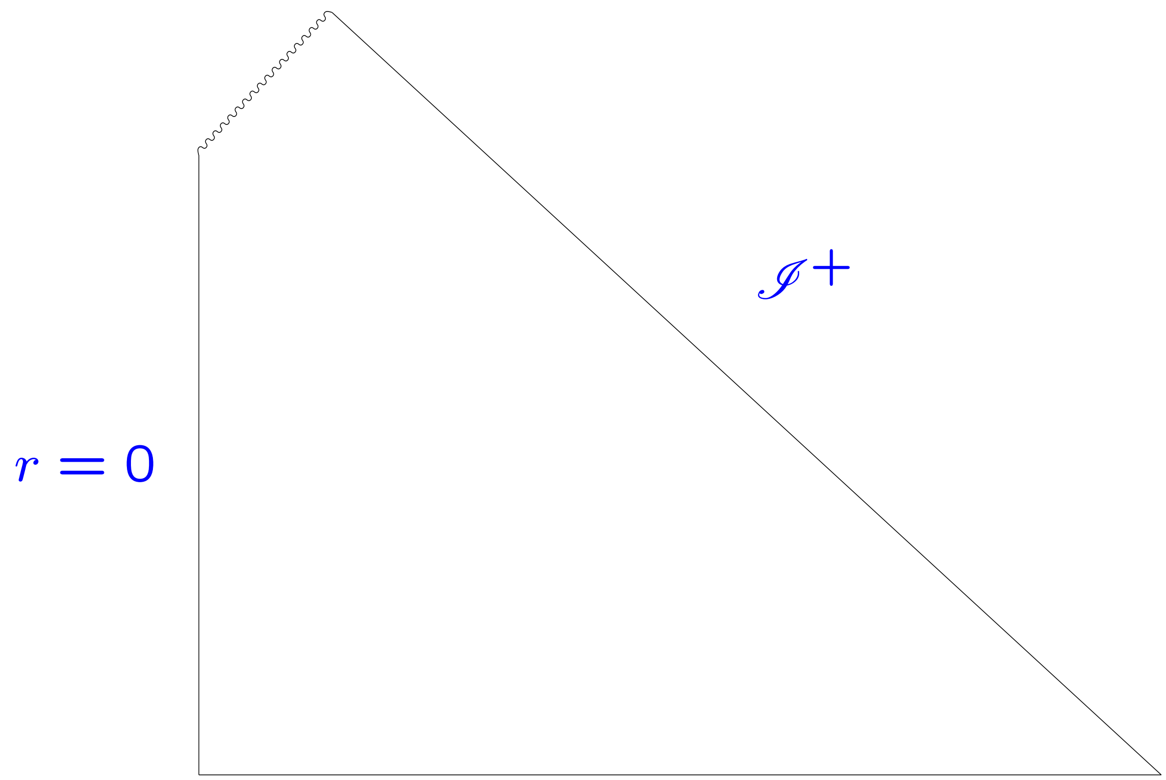
$r = 0$
i^+
$\mathscr{I}^+$

- PDE version: For generic asymptotically flat initial data the future maximal globally hyperbolic development possesses a complete future null infinity $\mathscr{I}^+$.

- Strong and weak CCCs are logically independent (examples are Reissner-Nordström and thunderbolts).



$r = 0$
$r = 0$
$\mathscr{I}^+$
$\mathscr{I}^+$



$\mathscr{I}^+$
$r = 0$

Einstein-Maxwell-scalar field equations in spherical symmetry

- Birkhoff's theorem: there are no gravitational degrees of freedom in spherical symmetry.

- Simplest hyperbolic matter model: massless scalar field.

- Simplest spherically symmetric solution containing a Cauchy horizon: Reissner-Nordström (electromagnetic field).

- The equations for a gravitating massless scalar field ϕ in a sourceless electromagnetic field F with a cosmological constant Λ are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 2T_{\mu\nu}$$

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi\,g_{\mu\nu} + F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}$$

$$\Box\phi = 0$$

$$dF = d \star F = 0$$

(using units for which $c = 4\pi G = \varepsilon_0 = 1$)

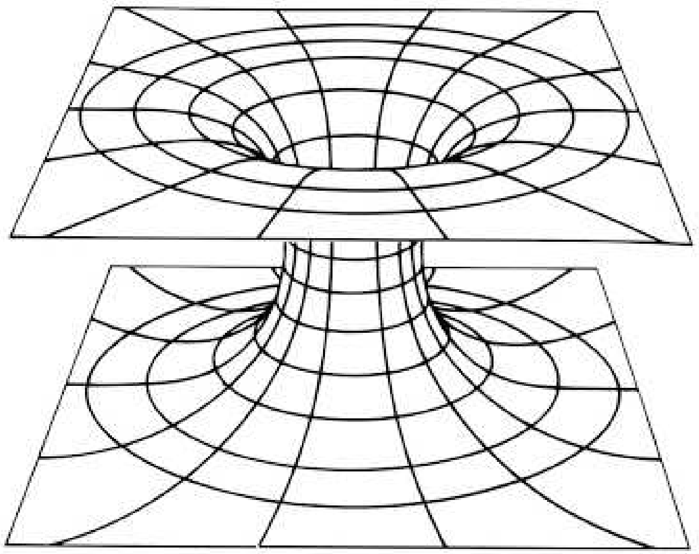
- If we impose spherical symmetry then the metric and the fields become

$$ds^2 = -\Omega^2(u,v)du\,dv + r^2(u,v)\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)$$
$$\phi = \phi(u,v)$$
$$F = -E(u,v)\frac{\Omega^2(u,v)}{2}du \wedge dv$$

- In particular the electromagnetic field completely decouples:

$$\star F = E(u,v)r^2(u,v)\sin\theta\,d\theta \wedge d\varphi \;\Rightarrow\; E(u,v) = \frac{e}{r^2(u,v)}$$

- The total charge $4\pi e$ is topological: initial surface $t = 0$ in Reissner-Nordström is



- Introducing the renormalized Hawking mass ϖ through

$$1-\frac{2\varpi}{r}+\frac{e^2}{r^2}-\frac{\Lambda}{3}r^2=-\frac{4\partial_u r\partial_v r}{\Omega^2}=(\mathrm{grad}\, r)^2$$

the Einstein-Maxwell-scalar field equations become

$$\partial_u\partial_v\phi=-\frac{\partial_u r\,\partial_v\phi}{r}-\frac{\partial_v r\,\partial_u\phi}{r}$$

$$\partial_u\partial_v r=\partial_u r\,\partial_v r\,\frac{\frac{2\varpi}{r^2}-\frac{2e^2}{r^3}-\frac{2\Lambda}{3}r}{1-\frac{2\varpi}{r}+\frac{e^2}{r^2}-\frac{\Lambda}{3}r^2}$$

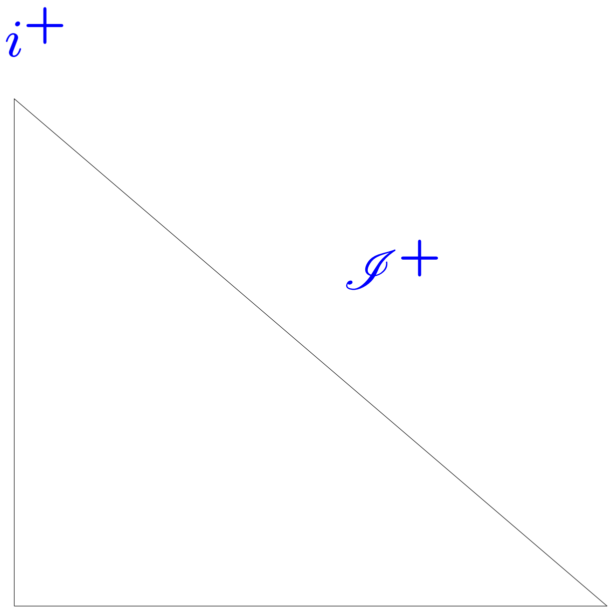
$$\partial_u\varpi=\left(1-\frac{2\varpi}{r}+\frac{e^2}{r^2}-\frac{\Lambda}{3}r^2\right)\frac{(r\partial_u\phi)^2}{2\partial_u r}$$

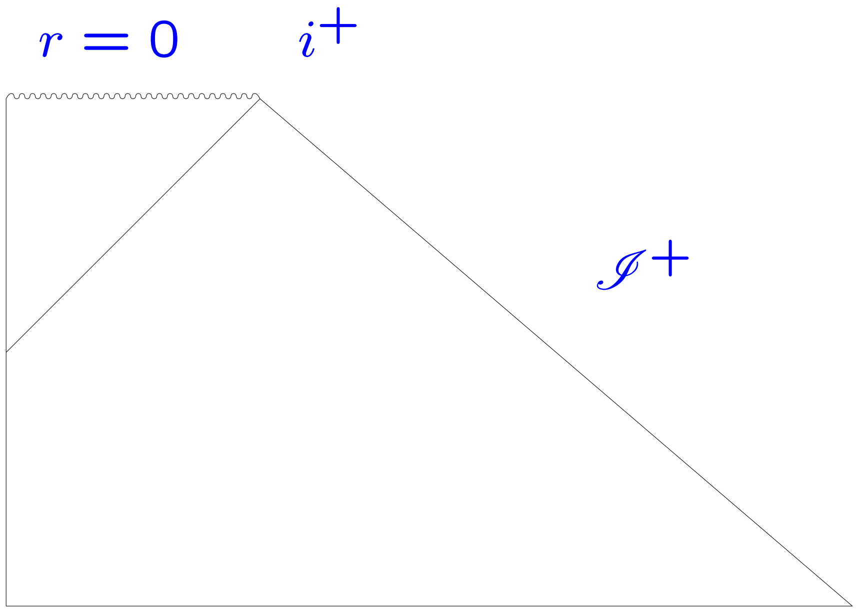
$$\partial_v\varpi=\left(1-\frac{2\varpi}{r}+\frac{e^2}{r^2}-\frac{\Lambda}{3}r^2\right)\frac{(r\partial_v\phi)^2}{2\partial_v r}$$

Christodoulou's results

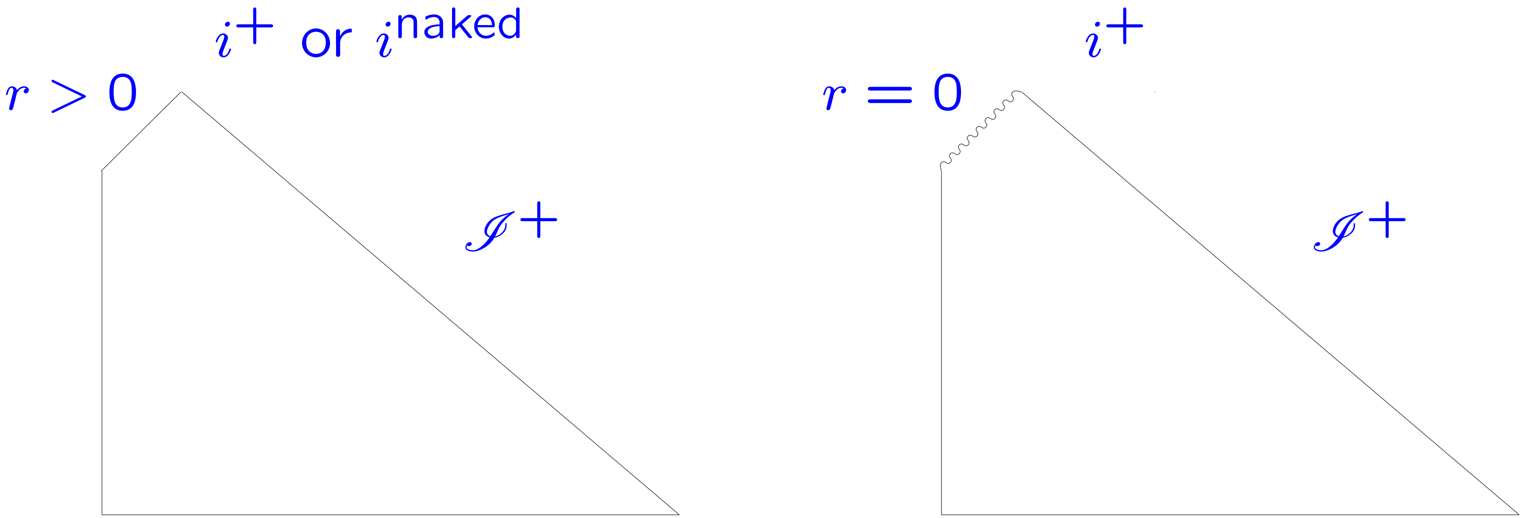
- Christodoulou (1999): Strong cosmic censorship is true for the Einstein-scalar field system ($e = \Lambda = 0$) in the C^0 formulation.

- Generic: dispersive or black hole.

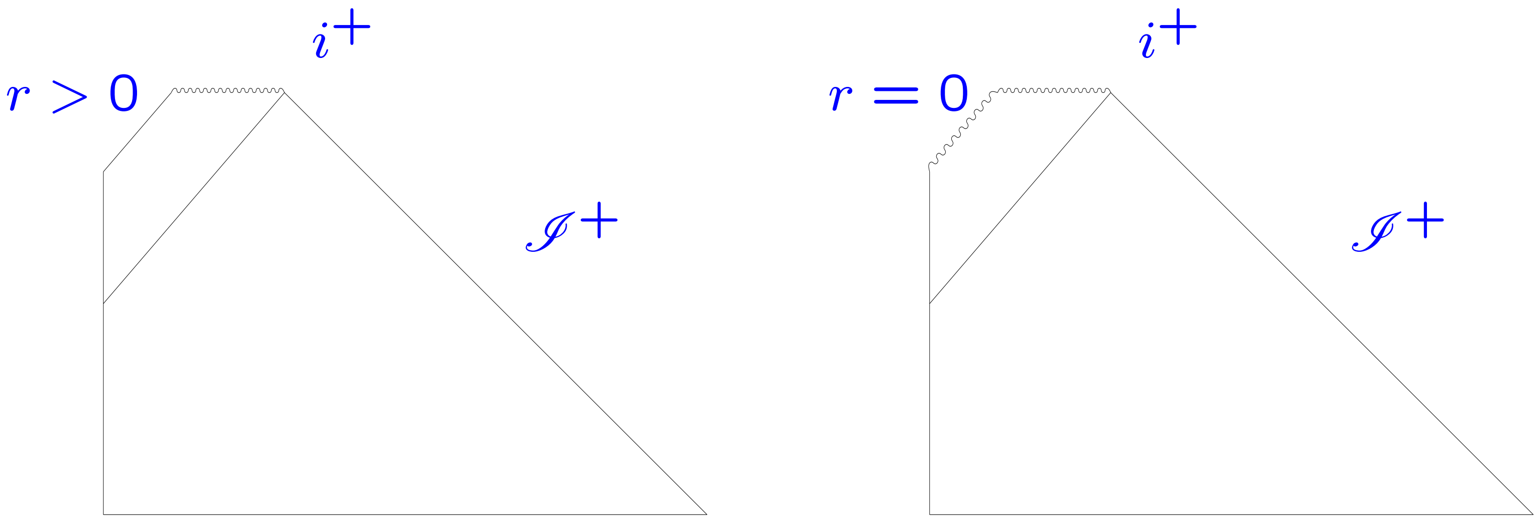

i^+
$\mathscr{I}^+$

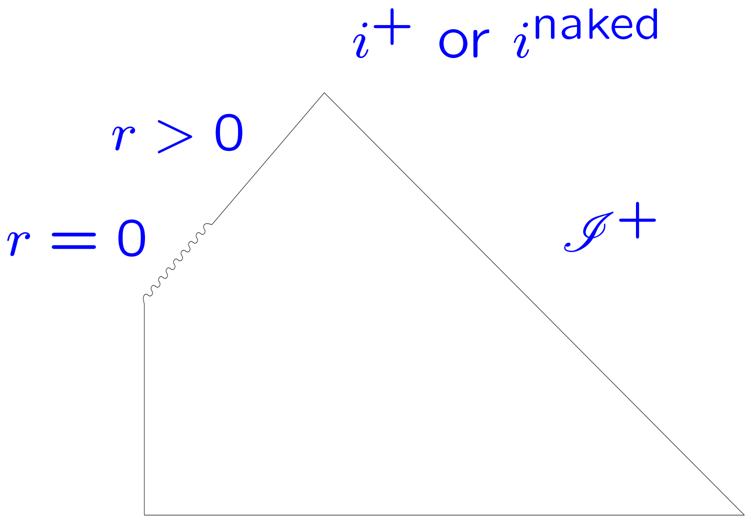

$r = 0$
i^+
$\mathscr{I}^+$

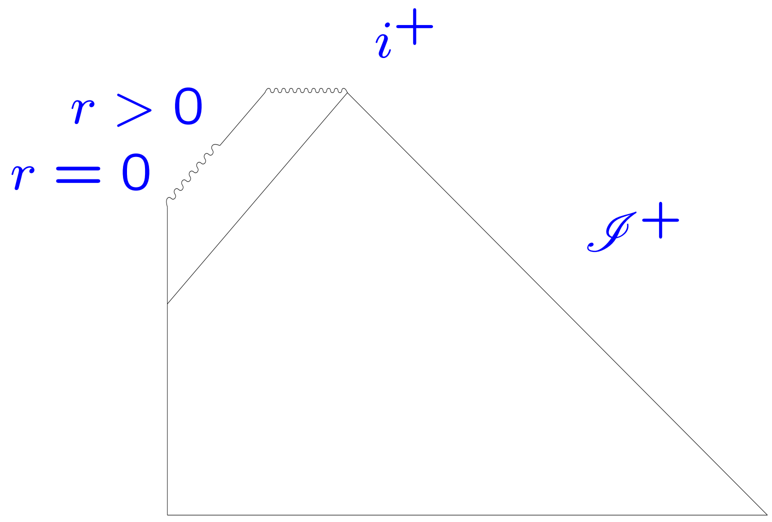
- Non-generic: light cone singularities (possibly naked) and black holes with light cone singularities.



i^+ or i^{naked}
$r > 0$
$\mathscr{I}^+$
i^+
$r = 0$
$\mathscr{I}^+$

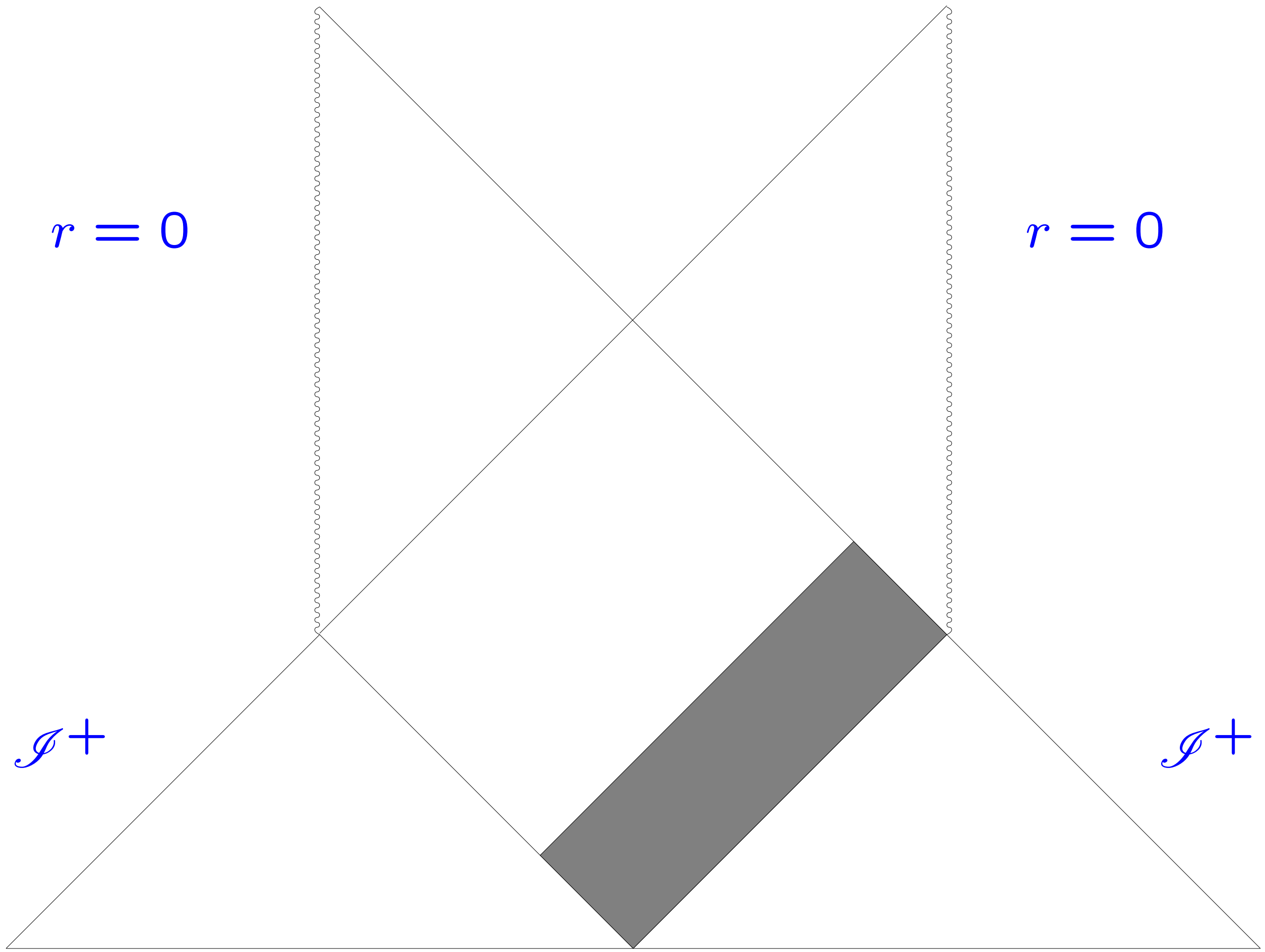

$r > 0$
i^+
$\mathscr{I}^+$
$r = 0$
i^+
$\mathscr{I}^+$


i^+ or i^{naked}
$r > 0$
$r = 0$
$\mathscr{I}^+$


i^+
$r > 0$
$r = 0$
$\mathscr{I}^+$

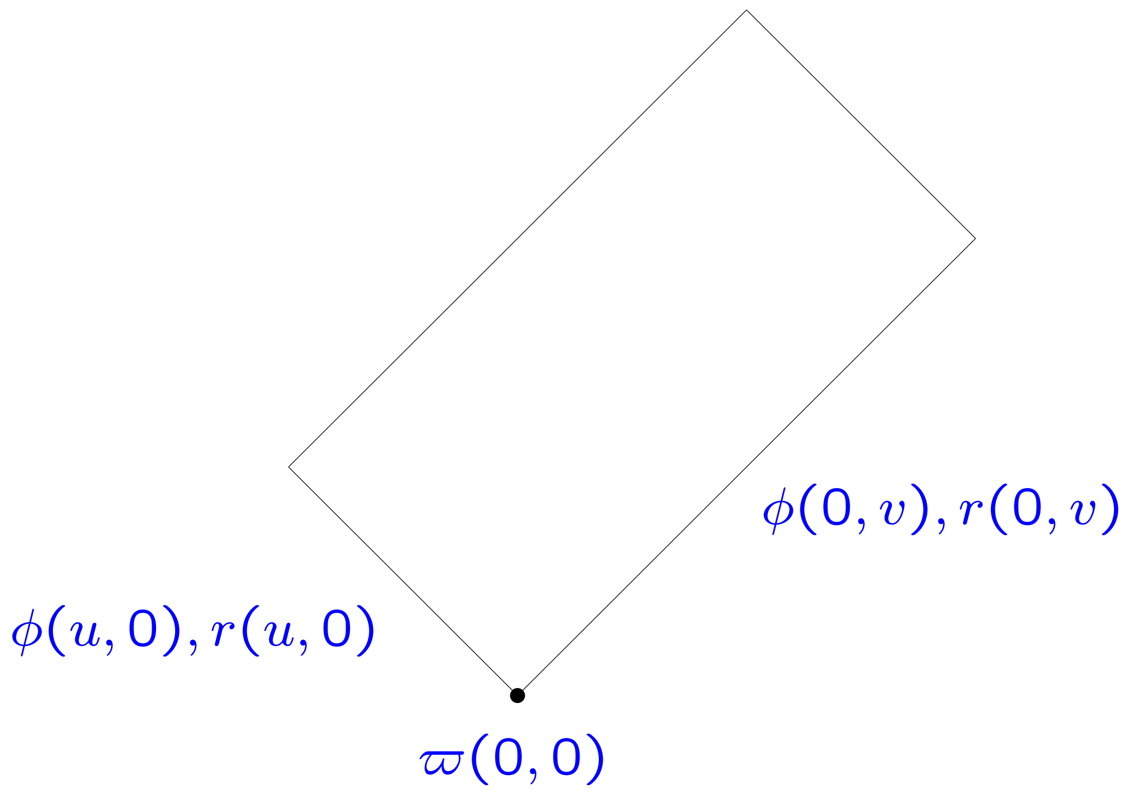
Dafermos' results

- Idea: perturb the Reissner-Nordström black hole interior.



$r = 0$
$r = 0$
$\mathscr{I}^+$
$\mathscr{I}^+$

• Characteristic initial data:



- Poisson and Israel (1989) gave a nonlinear heuristic analysis suggesting that for $\Lambda = 0$ the Cauchy horizon of generic solutions ($\phi \neq 0$) still has $r \sim r_-$, but $\varpi \to +\infty$ (mass inflation).

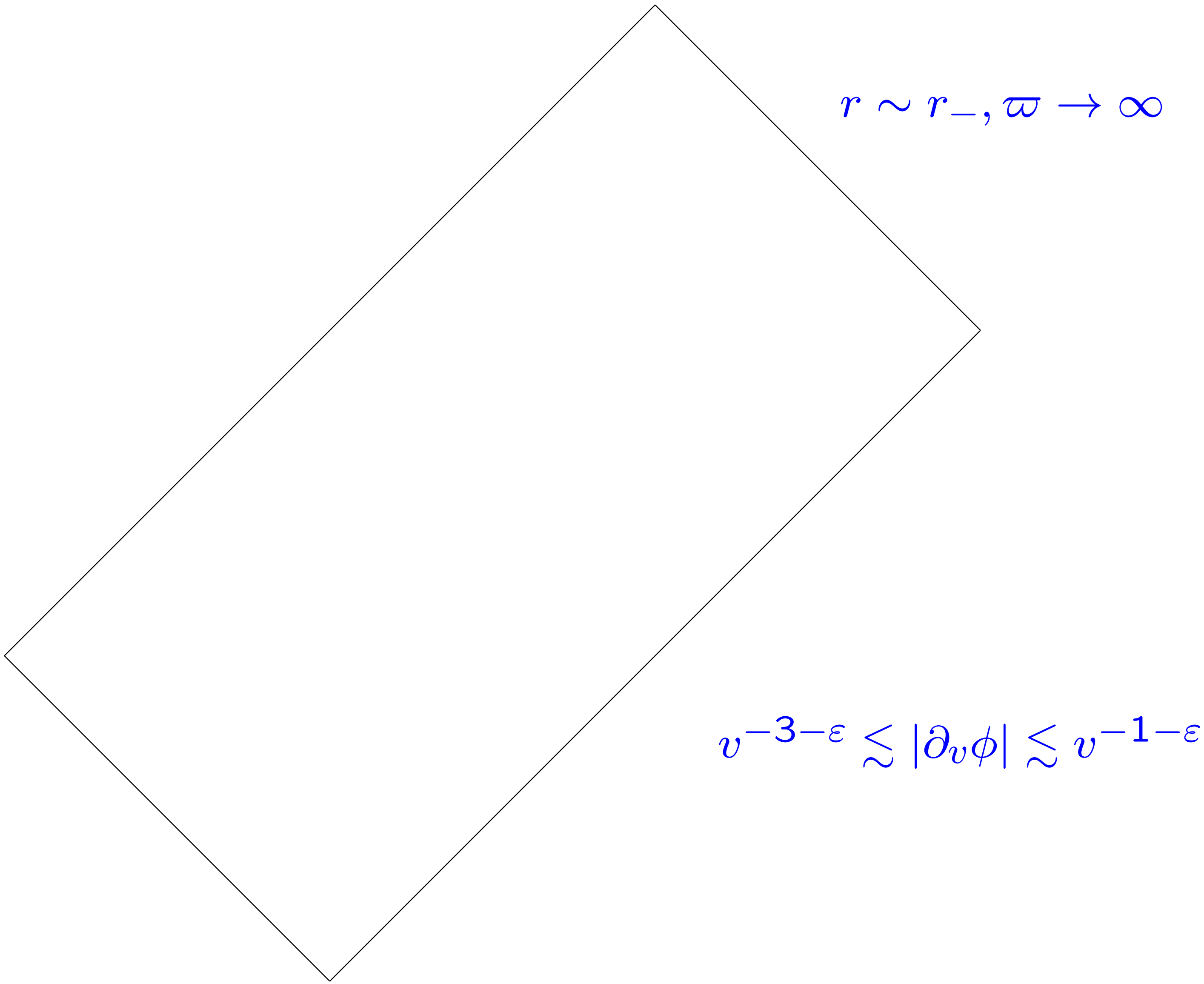
- Brady, Moss and Myers (1998) performed a linear analysis suggesting that mass inflation might not occur for $\Lambda > 0$ near extremality (but the curvature still blows up at the Cauchy horizon).

- Dafermos (2005) proved the following two results for the spherically symmetric Einstein-Maxwell-scalar field system.

1. If $|\partial_v \phi| \lesssim v^{-1-\varepsilon}$ along the event horizon then r can be extended to a nonvanishing continuous function on the Cauchy horizon, and so the metric can be extended as a C^0 metric.

2. If $v^{-3-\varepsilon} \lesssim |\partial_v \phi| \lesssim v^{-1-\varepsilon}$ along the event horizon then the Hawking mass blows up identically along the Cauchy horizon, and so the metric is inextendible as a C^1 metric.

- The stonger form $|\partial_v \phi| \lesssim v^{-3+\varepsilon}$ of the first hypothesis (Price's law) was subsequently proved to occur by Dafermos and Rodnianski (2005).

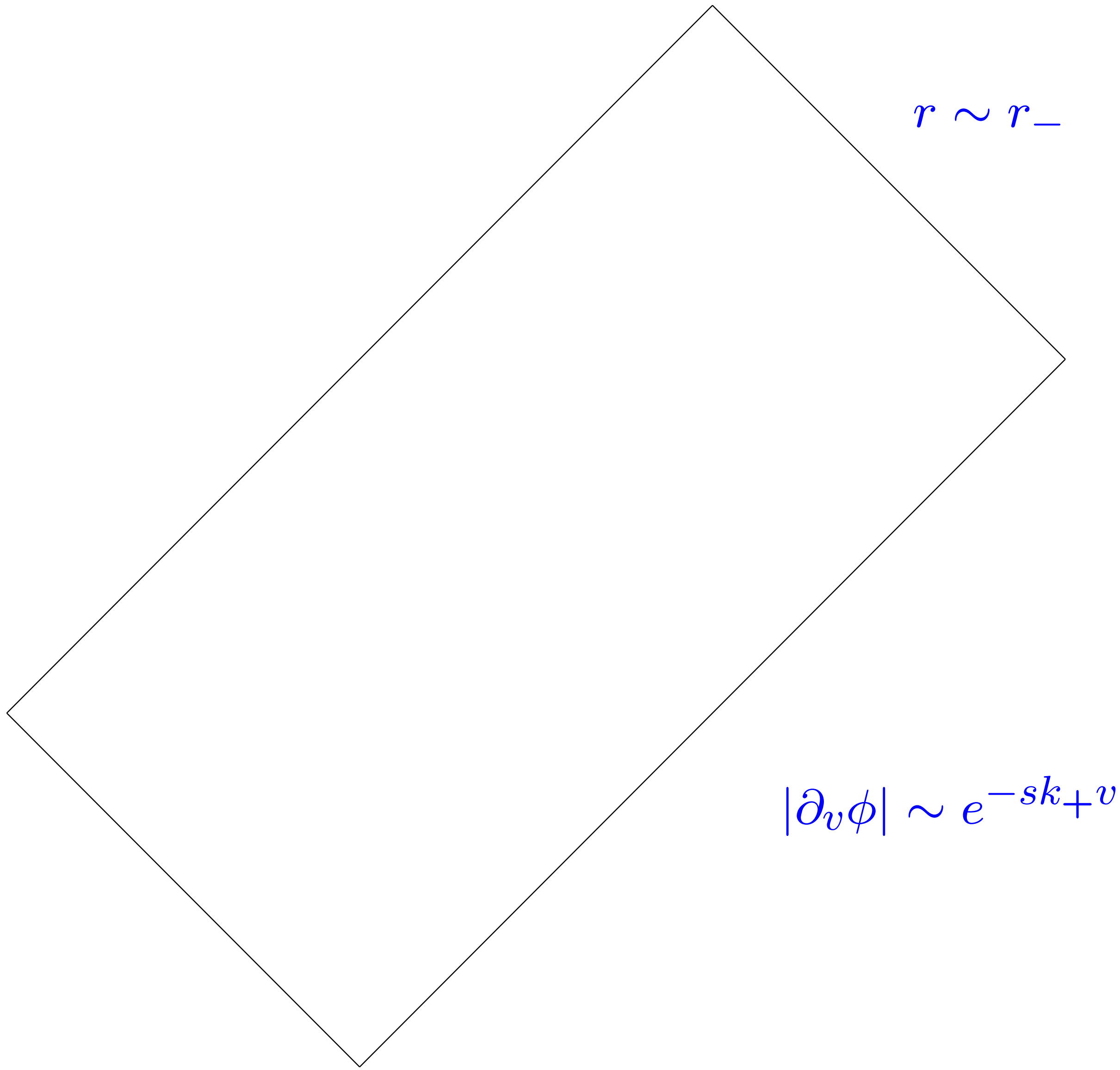

$r \sim r_-, \varpi \to \infty$
$v^{-3-\varepsilon} \lesssim |\partial_v \phi| \lesssim v^{-1-\varepsilon}$

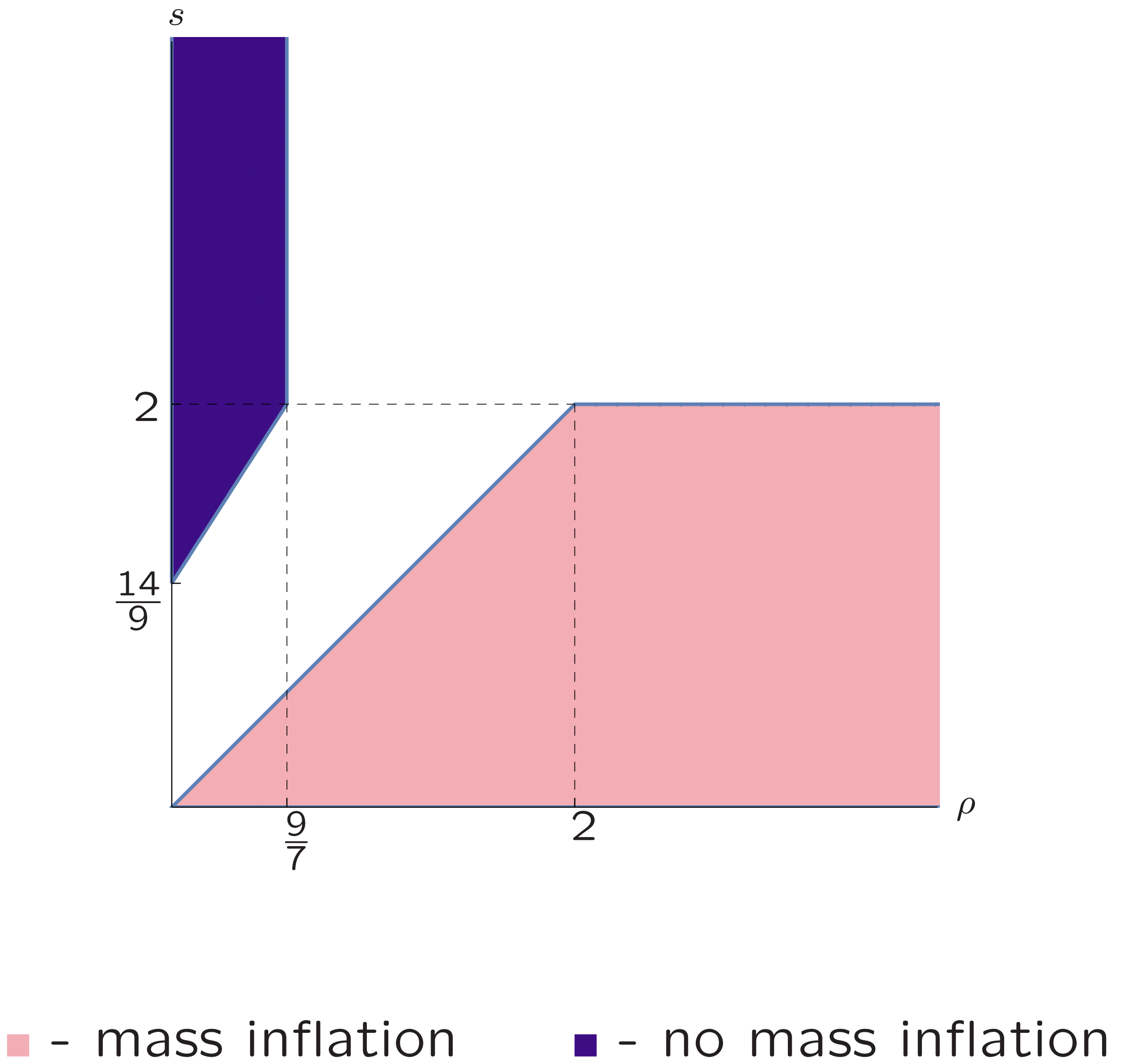
Our results

- We (João Costa, Pedro Girão, J. N., Jorge Drumond Silva) consider the case $|\partial_v \phi| \sim e^{-sk_+ v}, s > 0$, for any Λ.

- r can always be extended to a nonvanishing continuous function on the Cauchy horizon, and so the metric can be extended as a C^0 metric.

- Mass inflation depends on s and $\rho = \frac{k_-}{k_+} > 1$ (we exclude the extremal case $\rho = 1$).


$r \sim r_-$
$|\partial_v \phi| \sim e^{-sk_+ v}$



s
2
14/9
ρ
9/7
2
- mass inflation
- no mass inflation

- For $\Lambda > 0$ one expects an exponential decay in Price's law, which can be as fast as $|\partial_v \phi| \sim e^{-(2-\varepsilon)k+v}$, that is, $s = 2 - \varepsilon$.

- So it is likely that there is no mass inflation near extremality.

- Moreover, it is possible to construct extensions with Christoffel symbols in L^2_{loc} !!! What about cosmic censorship?

$$
\begin{aligned}
\partial_u r &= \nu, \\
\partial_v r &= \lambda, \\
\partial_u \lambda &= -2\nu\kappa\frac{1}{r^2}\left(\frac{e^2}{r}+\frac{\Lambda}{3}r^3-\varpi\right), \\
\partial_v \nu &= -2\nu\kappa\frac{1}{r^2}\left(\frac{e^2}{r}+\frac{\Lambda}{3}r^3-\varpi\right), \\
\partial_u \varpi &= \frac{1}{2}\left(1-\frac{2\varpi}{r}+\frac{e^2}{r^2}-\frac{\Lambda}{3}r^2\right)\left(\frac{\zeta}{\nu}\right)^2\nu, \\
\partial_v \varpi &= \frac{1}{2}\frac{\theta^2}{\kappa}, \\
\partial_u \theta &= -\frac{\zeta\lambda}{r}, \\
\partial_v \zeta &= -\frac{\theta\nu}{r}, \\
\partial_u \kappa &= \kappa\nu\frac{1}{r}\left(\frac{\zeta}{\nu}\right)^2, \\
\lambda &= \kappa(1-\mu).
\end{aligned}
$$

$$
\begin{aligned}
\kappa(u,v) &= \kappa_0(v)e^{\int_0^u \left(\frac{\zeta^2}{r\nu}\right)(u',v)\,du'}, \\
\nu(u,v) &= \nu_0(u)e^{-\int_0^v \left(2\kappa\frac{1}{r^2}\left(\frac{e^2}{r}+\frac{\Lambda}{3}r^3-\varpi\right)\right)(u,v')\,dv'}, \\
\lambda(u,v) &= \lambda_0(v) - \int_0^u \left(2\nu\kappa\frac{1}{r^2}\left(\frac{e^2}{r}+\frac{\Lambda}{3}r^3-\varpi\right)\right)(u',v)\,du', \\
\theta(u,v) &= \theta_0(v) - \int_0^u \left(\frac{\zeta\lambda}{r}\right)(u',v)\,du', \\
\zeta(u,v) &= \zeta_0(u) - \int_0^v \left(\frac{\theta\nu}{r}\right)(u,v')\,dv', \\
\varpi(u,v) &= \varpi_0(v)e^{-\int_0^u \left(\frac{\zeta^2}{r\nu}\right)(u',v)\,du'} \\
&\quad + \int_0^u e^{-\int_s^u \frac{\zeta^2}{r\nu}(u',v)\,du'}\left(\frac{1}{2}\left(1+\frac{e^2}{r^2}-\frac{\Lambda}{3}r^2\right)\frac{\zeta^2}{\nu}\right)(s,v)\,ds, \\
r(u,v) &= r_0(u) + \int_0^v \lambda(u,v')\,dv'.
\end{aligned}
$$

- Prescribe a continuous integrable function $\widehat{f} :]0, r_+[\to \mathbb{R}_0^+$ so that

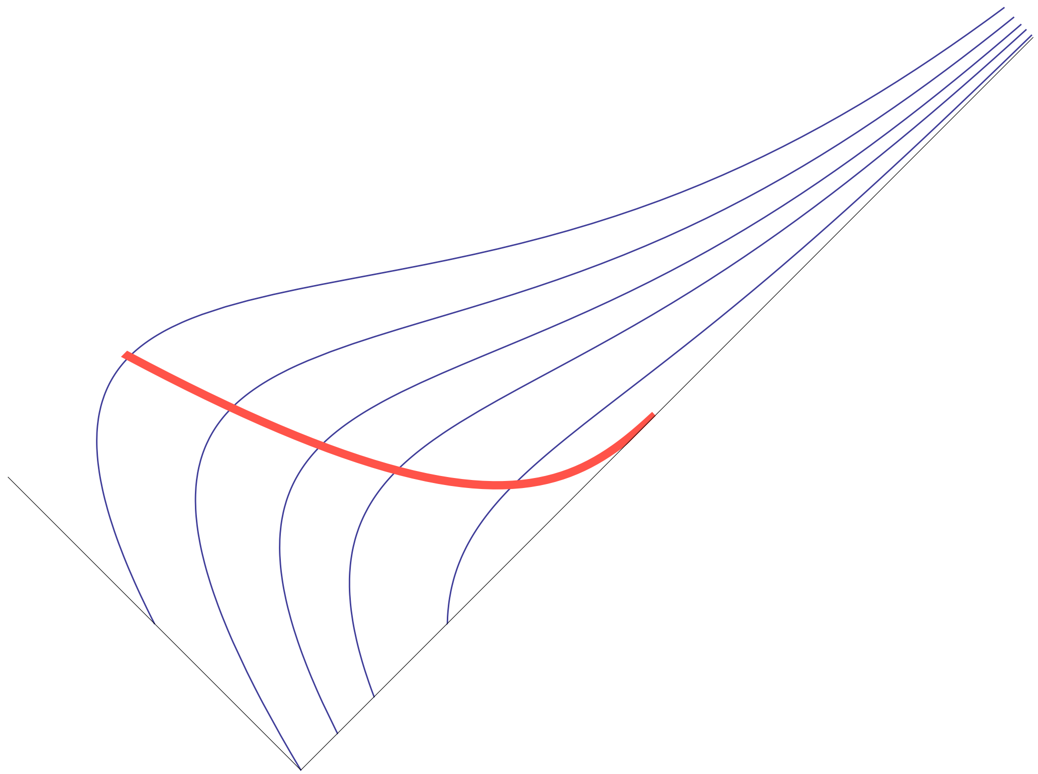
$$\widehat{\varpi}(r) = \varpi_0 - \int_r^{r_+} \widehat{f}(\tilde{r})\, d\tilde{r}.$$

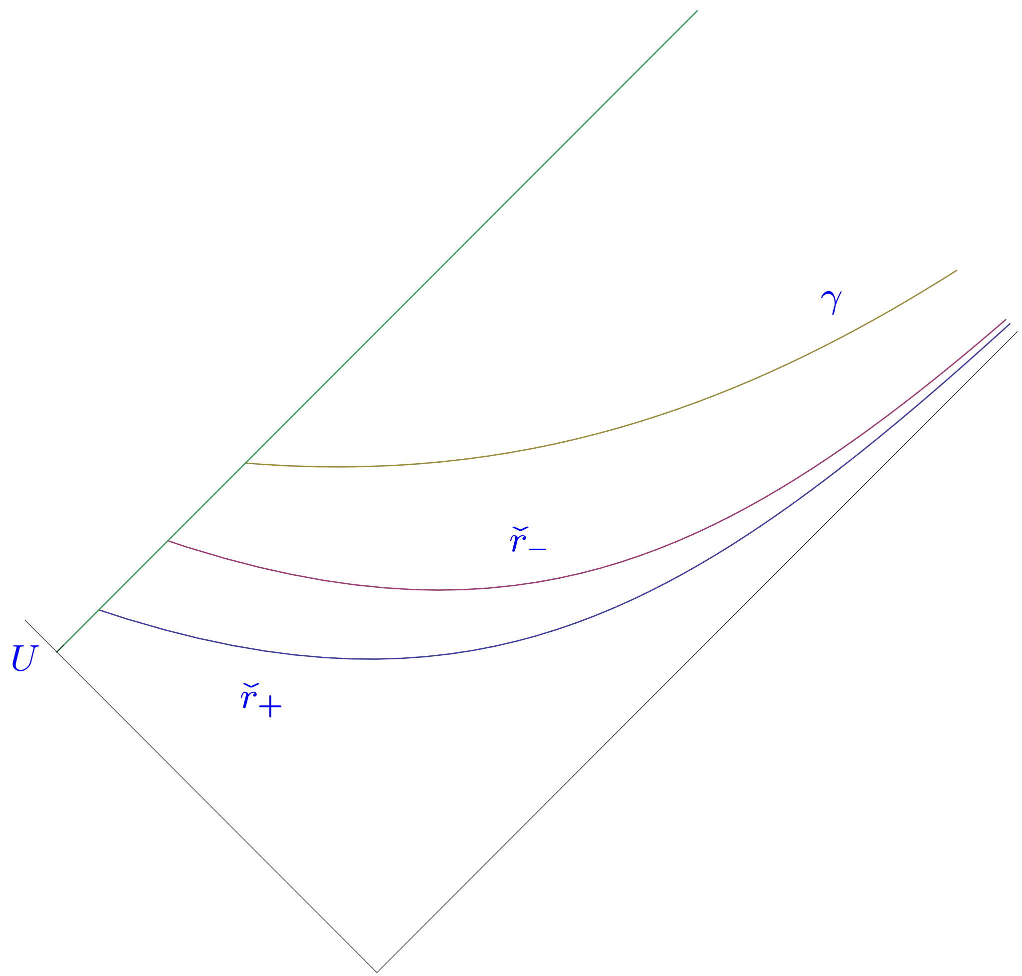
- Assume

$$\lim_{r \to r_+} \widehat{f}(r) = A \in \left[k_+ r_+, +\infty\right].$$

- Exponential decay corresponds to $A \in \left(k_+ r_+, +\infty\right)$.

- Define $s = \frac{A}{k_+ r_+} - 1 \in (0, +\infty)$.





γ
$\check{r}_-$
U
$\check{r}_+$

- Two competing effects: the redshift $e^{-2k_+ v}$ arising from the equation, and the exponential decay $e^{-sk_+ v}$ of the initial data.

- Curve γ probes the region near the Cauchy horizon, where k_- comes into play.