Decay of solutions of the wave equation in expanding cosmological spacetimes

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FLRW models, conformal time and scri

$M = [t_0, \infty) \times \mathbb{R}^n, \qquad a : [t_0, \infty) \to \mathbb{R}^+$ nondecreasing

$$g = -dt^2 + a^2(t)\left(\left(dx^1\right)^2 + \ldots + (dx^n)^2\right)$$

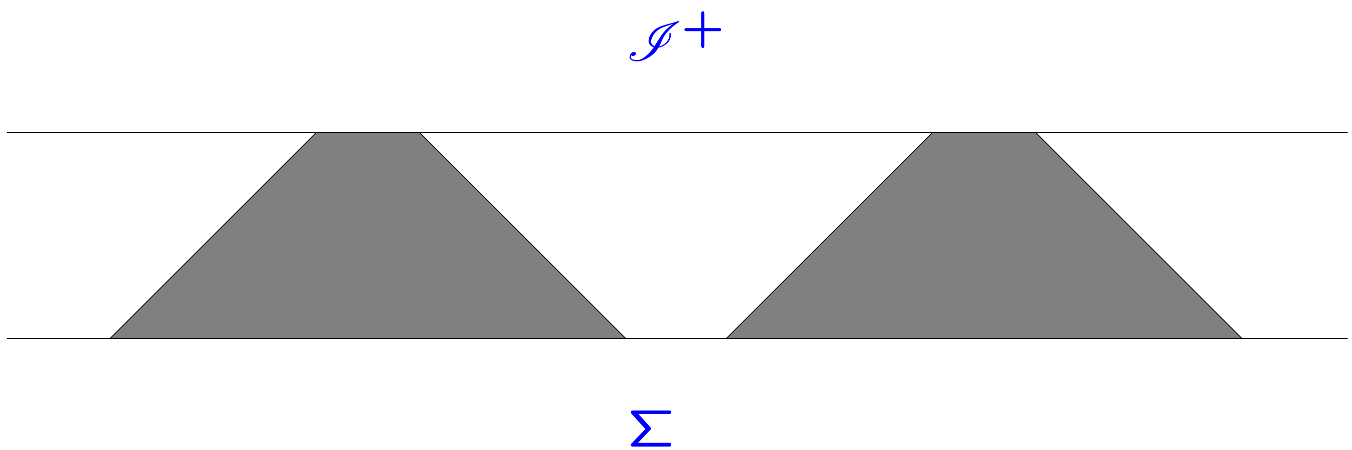
$$\begin{cases} \Box_g \phi = 0 \\ \\ \phi(t_0, x) = \phi_0(x) \\ \partial_t \phi(t_0, x) = \phi_1(x) \end{cases}$$

$$\tau = \int_{t_0}^{t} \frac{dt}{a(t)} \qquad \text{conformal time}$$

$$g = a^2(t)\left(-d\tau^2 + \left(dx^1\right)^2 + \ldots + (dx^n)^2\right)$$

<u>Spacelike scri</u> if $\int_{t_0}^{\infty} \frac{dt}{a(t)} < \infty$

In this case, ϕ approaches a <u>function</u> as $t \to \infty$



Fourier modes

$$M = [t_0, \infty) \times \frac{\mathbb{R}^n}{2\pi \mathbb{Z}^n}, \qquad \phi(t,x) = \sum_{k \in \mathbb{Z}^n} c_k(t) e^{i\langle k, x\rangle}$$

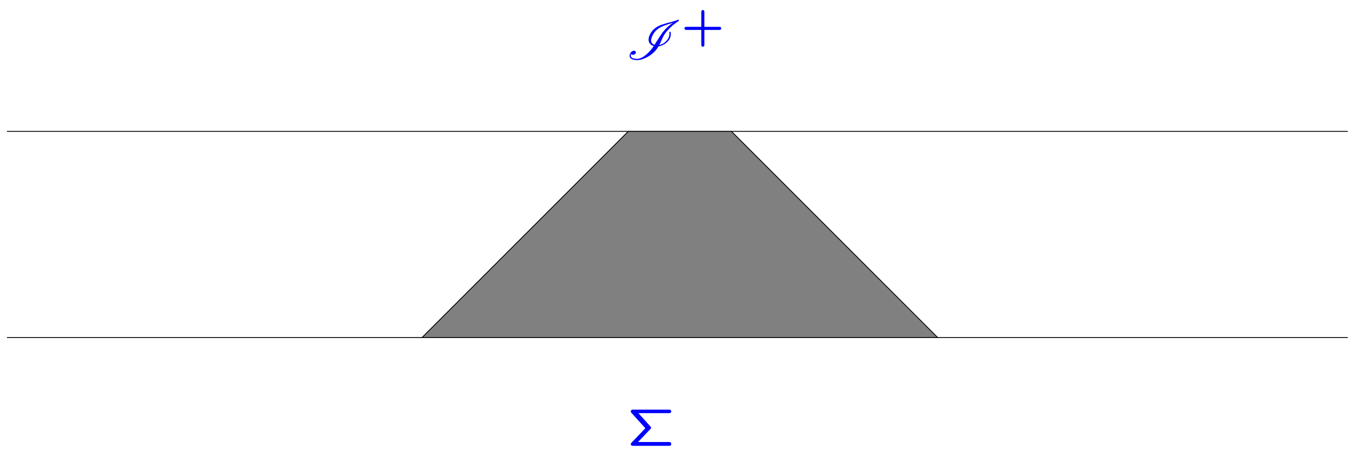
$$\Box_g \phi = 0 \quad \Leftrightarrow \quad \ddot{c}_k + \frac{n\dot{a}}{a}\dot{c}_k + \frac{k^2}{a^2} c_k = 0$$

$$a(t) = t^p, \quad p < 1 \quad \Rightarrow \quad |c_k| \lesssim t^{-\frac{n-1}{2}p}, \quad |\dot{c}_k| \lesssim t^{-\frac{n+1}{2}p}$$

$$a(t) = t^p, \quad p > 1 \quad \Rightarrow \quad c_k \sim C_k + D_k t^{-2p+2}$$

$$a(t) = e^t \quad \Rightarrow \quad c_k \sim C_k + D_k e^{-2t}$$

If scri is spacelike then dispersion does not matter:



The operator trick (Klainerman and Sarnak)

$$\Box_g \phi = 0 \quad \Leftrightarrow \quad \partial_\tau^2 \phi + (n-1)\frac{a'}{a}\partial_\tau \phi = \Delta \phi$$

If $$\hat{O}\left(\partial_\tau^2 + (n-1)\frac{a'}{a}\partial_\tau\right)\phi = \partial_\tau^2 \hat{O}\phi$$

and $\hat{O}\Delta = \Delta\hat{O}$ then $\partial_\tau^2 \hat{O}\phi = \Delta \hat{O}\phi$

$n = 3$ and $a(t) = t^{2/3}$ (dust): $\hat{O}\phi = \tau^2 \partial_\tau \phi + 3\tau\phi$

$$\begin{aligned}\tau^3 \phi(\tau, x) = & \int_{\tau_0}^{\tau} \frac{s}{4\pi(s-\tau_0)^2} \int_{\partial B_{s-\tau_0}(x)} \psi_0(y) dV_2(y) ds \\ & + \int_{\tau_0}^{\tau} \frac{s}{4\pi(s-\tau_0)} \int_{\partial B_{s-\tau_0}(x)} \nabla\psi_0(y) \cdot \frac{y-x}{|y-x|} dV_2(y) ds \\ & + \int_{\tau_0}^{\tau} \frac{s}{4\pi(s-\tau_0)} \int_{\partial B_{s-\tau_0}(x)} \psi_1(y) dV_2(y) ds + {\tau_0}^3 \phi_0(x)\end{aligned}$$

For appropriately decaying initial data, $|\phi| \lesssim t^{-1}$

Closed form expressions also for:

$$a(t) = 1 \quad \text{(Minkowski)}$$

$$a(t) = t^{1/2} \quad \text{(radiation)}$$

$$a(t) = e^t \quad \text{(de Sitter)}$$

In the first two cases, $|\phi| \lesssim t^{-1}$ for appropriately decaying initial data

Expected from Fourier modes: $|\phi| \lesssim t^{-p}$

As p increases redshift helps decay but dispersion becomes harder, and the two effects compensate exactly

Conjecture: If $a(t) = t^p$ with $p \in [0, 1]$ then $|\phi| \lesssim t^{-1}$

$$\left(\quad \text{More generally} \quad |\phi| \lesssim t^{-\frac{n-1}{2}} \quad \right)$$

$$\psi = a^{\frac{n-1}{2}}\phi \quad \Rightarrow \quad \partial_\tau^2\psi - \Delta\psi = \frac{\mu}{\tau^2}\psi$$

$$\mu = \frac{(n-1)p(2p-1)}{2(1-p)^2} + \frac{(n-1)(n-3)p^2}{4(1-p)^2}$$

$$\left(\quad \mu \le 0 \quad \text{for} \quad p \le \frac{2}{n+1} \quad \right)$$

Dispersive estimate (vector field method) $\Rightarrow$ $|\phi| \lesssim \tau^{-\frac{n-1}{2}}$

Therefore $|\phi| \lesssim (\tau a(\tau))^{-\frac{n-1}{2}} \sim t^{-\frac{n-1}{2}}$

$\Big($ Böhme & Reissig (2012): $\|\nabla\phi\|_{L^q} \lesssim t^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$ $\Big)$