Solutions of the wave equation bounded at the Big Bang

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Introduction

- Alho, Fournodavlos and Franzen, arXiv:1805.12558: solutions of the wave equation in flat FLRW and in Kasner generically blow up at the Big Bang, but many do not.

- Same behavior has been shown to occur in Bianchi models (Ringström, arXiv:1808.00786), in Schwarzschild's black hole region (Fournodavlos and Sbierski, arXiv:1804.01941) and near compact Cauchy horizons (Petersen, arXiv:1802.10057).

- Our goal: prove the existence of a large class of non-generic solutions of the wave equation bounded at the Big Bang of general n-dim FLRW. Possibly important for Conformal Cyclic Cosmology and singular bouncing cosmologies.

The theorem

Let $(M,g) = (\mathbb{R}\times\Sigma, -dt^2 + a^2(t)h)$ be an expanding FLRW model, with $\Sigma = \mathbb{R}^n, \mathbb{S}^n$ or $\mathbb{H}^n$ and h the standard unit constant curvature metric, and assume that $a(t) \sim t^p$ with $0 < p < 1$. Given $A \in H^3(\Sigma)$, there exists a unique solution of the wave equation $\Box_g \phi = 0$ in $C^0\left((0,T], H^1(\Sigma)\right) \cap C^1\left((0,T], L^2(\Sigma)\right)$ such that

$$\lim_{t\to 0^+} \|\phi(t,\cdot\,) - A(\,\cdot\,)\|_{H^1(\Sigma)} = 0$$

and

$$\lim_{t\to 0^+} \left(a(t)\,\|\partial_t\phi(t,\cdot\,)\|_{L^2(\Sigma)}\right) = 0$$

The proof: preliminaries

- Defining the conformal time

$$\tau = \int_0^t \frac{ds}{a(s)}$$

the metric becomes

$$g = a^2(\tau)\left(-d\tau^2 + h_{ij}dx^i dx^j\right)$$

and the wave equation

$$\Box_g \phi = 0 \Leftrightarrow \partial_\tau\left(a^{n-1}\partial_\tau\phi\right) = a^{n-1}\Delta_h\phi$$

- The energy-momentum tensor associated to the wave equation is

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2}(\partial_\alpha \phi \, \partial^\alpha \phi) \, g_{\mu\nu}$$

Choosing the multiplier vector field

$$X = a^{1-n} \frac{\partial}{\partial \tau}$$

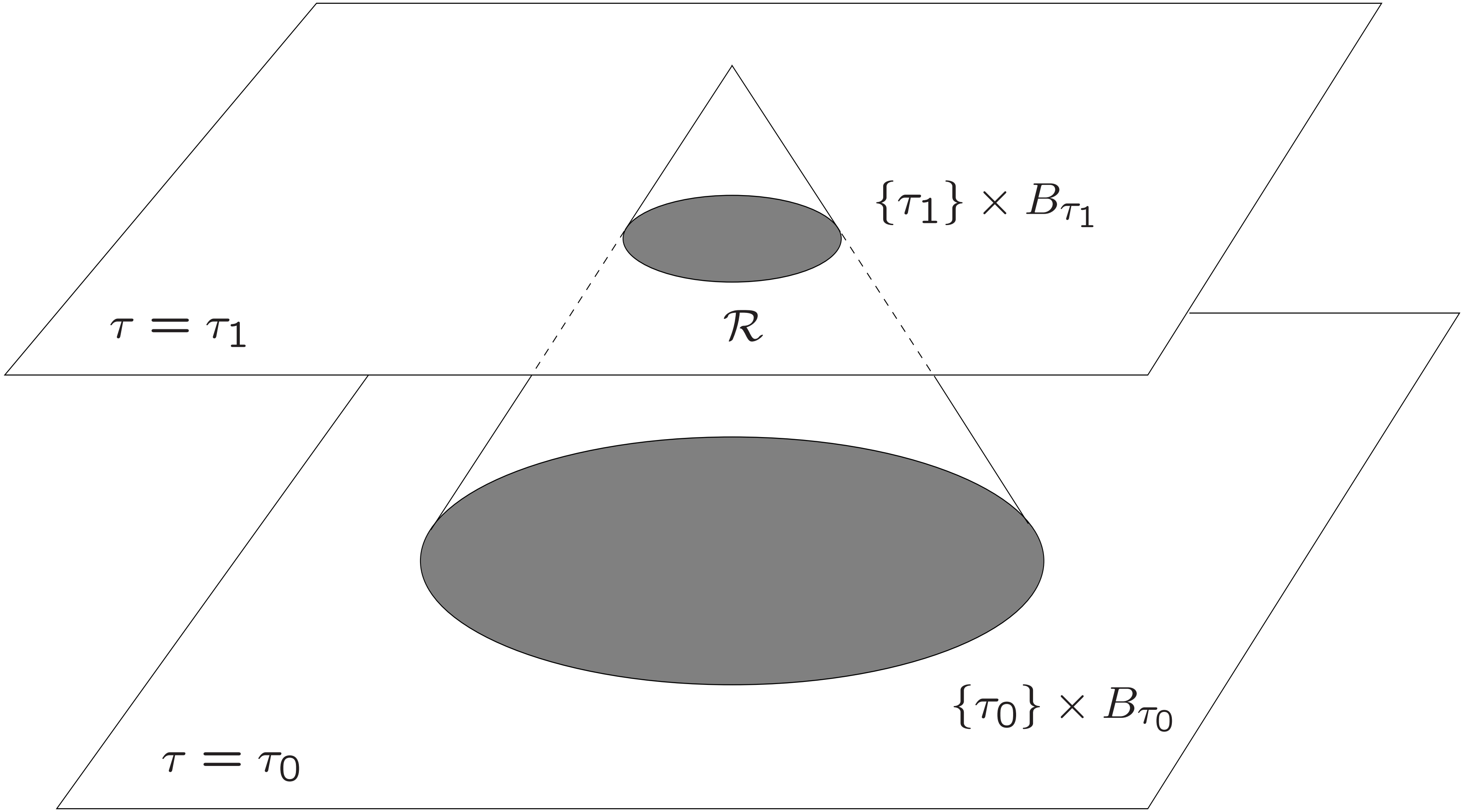
and a geodesic ball $B \subset \Sigma$ we can define the energy

$$E(\tau) = \int_{\{\tau\} \times B_\tau} T_{\mu\nu} X^\mu N^\nu \, a^n dV_\Sigma = \frac{1}{2} \int_{B_\tau} \left[(\partial_\tau \phi)^2 + |\nabla_h \phi|^2 \right] dV_\Sigma$$

and prove

$$E(\tau_1) \leq E(\tau_0)$$

for smooth solutions of the wave equation.



$\{\tau_1\} \times B_{\tau_1}$
$\tau = \tau_1$
$\mathcal{R}$
$\{\tau_0\} \times B_{\tau_0}$
$\tau = \tau_0$

- By the fundamental theorem of calculus,

$$\|\phi(\tau,\cdot)\|_{L^2(B_\tau)} \le \|\phi(\tau_0,\cdot)\|_{L^2(B_{\tau_0})} + (\tau-\tau_0)\sqrt{2E(\tau_0)}$$

for $\tau \ge \tau_0$, and so

$$\|\phi(\tau,\cdot)\|_{H^1(B_\tau)} \le C(1+\tau-\tau_0)\Big(\|\phi(\tau_0,\cdot)\|_{H^1(B_{\tau_0})} + \|\partial_\tau\phi(\tau_0,\cdot)\|_{L^2(B_{\tau_0})}\Big)$$

The proof: approximate smooth solutions

- Conditions in the theorem can be written as

$$\lim_{\tau \to 0^+} \|\phi(\tau,\,\cdot\,) - A(\,\cdot\,)\|_{H^1(\Sigma)} = 0$$

and

$$\lim_{\tau \to 0^+} \|\partial_\tau \phi(\tau,\,\cdot\,)\|_{L^2(\Sigma)} = 0$$

- Assume $A \in C^\infty(\Sigma) \cap H^3(\Sigma)$ and let $\phi^{\tau_0} \in C^\infty(M)$ be the unique solution of

$$\begin{cases} \Box_g \phi = 0 \\ \phi(\tau_0,\,\cdot\,) = A(\,\cdot\,) \\ \partial_\tau \phi(\tau_0,\,\cdot\,) = 0 \end{cases}$$

- Fix $\tau_* \in (0,1)$ and assume that $0 < \tau_0 \le \tau_1 \le \tau_*$.

- From the wave equation,

$$\partial_\tau \phi^{\tau_0}(\tau_1, \cdot\,) = \frac{1}{a^{n-1}(\tau_1)} \int_{\tau_0}^{\tau_1} a^{n-1}(\tau) \Delta_h \phi^{\tau_0}(\tau, \cdot\,)\, d\tau$$

- Commuting with Killing vector fields,

$$\|\Delta_h \phi^{\tau_0}(\tau, \cdot\,)\|_{H^1(B_\tau)} \le C \|A\|_{H^3(B)}$$

  and so

$$\|\partial_\tau \phi^{\tau_0}(\tau_1, \cdot\,)\|_{H^1(B_{\tau_1})} \le C \|A\|_{H^3(B)}\, \tau_1$$

- Integrating,

$$\|\phi^{\tau_0}(\tau_1, \cdot\,) - A(\cdot\,)\|_{H^1(B_{\tau_1})} \le C \|A\|_{H^3(B)}\, {\tau_1}^2$$

- In other words,

$$\|\phi^{\tau_0}(\tau_1, \cdot\,) - \phi^{\tau_1}(\tau_1, \cdot\,)\|_{H^1(B_{\tau_1})} \le C\|A\|_{H^3(B)}\,{\tau_1}^2$$

and

$$\|\partial_\tau\phi^{\tau_0}(\tau_1, \cdot\,) - \partial_\tau\phi^{\tau_1}(\tau_1, \cdot\,)\|_{H^1(B_{\tau_1})} \le C\|A\|_{H^3(B)}\,\tau_1$$

- Since $\phi^{\tau_0} - \phi^{\tau_1}$ is a solution of the wave equation,

$$\|\phi^{\tau_0}(\tau, \cdot\,) - \phi^{\tau_1}(\tau, \cdot\,)\|_{H^1(B_\tau)} \le C\|A\|_{H^3(B)}\,\tau_1\,(1+\tau)$$

and

$$\|\partial_\tau\phi^{\tau_0}(\tau, \cdot\,) - \partial_\tau\phi^{\tau_1}(\tau, \cdot\,)\|_{L^2(B_\tau)} \le C\|A\|_{H^3(B)}\,\tau_1$$

for $\tau \ge \tau_1$.

The proof: taking limits

- All previous estimates can be extended from geodesic balls to Σ.

- For $A \in H^1(\Sigma)$ and $\mathcal{T} > \tau_0$ there is a unique weak solution

$$\phi^{\tau_0} \in C^0\left([\tau_0, \mathcal{T}], H^1(\Sigma)\right) \cap C^1\left([\tau_0, \mathcal{T}], L^2(\Sigma)\right)$$

  depending continuously on A for these norms.

- Since $C^\infty(\Sigma) \cap H^3(\Sigma)$ is dense in $H^3(\Sigma)$ and the injection $H^3(\Sigma) \subset H^1(\Sigma)$ is continuous, for each $A \in H^3(\Sigma)$ there exists a unique weak solution satisfying all estimates above.

- For $A \in H^3(\Sigma)$ define the sequence $\phi_n = \phi^{\tau_n}$ with $\tau_n = \frac{\tau_*}{n}$.

- This is a Cauchy sequence of weak solutions of the wave equation in the Banach space

$$C^0\left([\epsilon, \mathcal{T}], H^1(\Sigma)\right) \cap C^1\left([\epsilon, \mathcal{T}], L^2(\Sigma)\right)$$

  for any given $\epsilon \in (0, \mathcal{T})$; therefore so is its limit ϕ.

- From

$$\|\phi^{\tau_n}(\tau, \cdot) - A(\cdot)\|_{H^1(\Sigma)} \leq C\|A\|_{H^3(\Sigma)}\,\tau^2$$

  and

$$\|\partial_\tau \phi^{\tau_n}(\tau, \cdot)\|_{L^2(\Sigma)} \leq C\|A\|_{H^3(\Sigma)}\,\tau$$

for any $\tau_n < \tau < \tau_*$ we obtain

$$\|\phi(\tau,\cdot) - A(\cdot)\|_{H^1(\Sigma)} \leq C\|A\|_{H^3(\Sigma)}\,\tau^2$$

and

$$\|\partial_\tau\phi(\tau,\cdot)\|_{L^2(\Sigma)} \leq C\|A\|_{H^3(\Sigma)}\,\tau$$

for any $0 < \tau < \tau_*$.

- Therefore ϕ and $\partial_\tau\phi$ have the correct limits as $\tau \to 0$. Moreover, we can now guarantee that

$$\phi \in C^0\Big((0,\mathcal{T}],H^1(\Sigma)\Big) \cap C^1\Big((0,\mathcal{T}],L^2(\Sigma)\Big)$$

- Uniqueness follows from the energy inequality.