

Optimal time travel in the Gödel universe

José Natário

(Instituto Superior Técnico, Lisbon)

Based on [arXiv:1105.6197](https://arxiv.org/abs/1105.6197)

Porto, January 2013

Outline

- Newtonian cosmology
- Rotating Newtonian universe
- Newton-Cartan theory
- The Gödel universe
- Total integrated acceleration
- Rocket theory
- Optimal time travel

Newtonian cosmology

- Choose a galaxy as the center of the universe; all other galaxies are at $\mathbf{r} = a(t)\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^3$.

- Equation of motion:

$$\ddot{\mathbf{r}} = -\frac{4\pi}{3}r^3 \left(\rho(t) - \frac{\Lambda}{4\pi} \right) \frac{\mathbf{r}}{r^3} \quad \Leftrightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi\rho}{3} + \frac{\Lambda}{3}$$

- Could have chosen any other galaxy to be the center of the universe: $\mathbf{r} = a(t)(\mathbf{x} - \mathbf{x}_0)$ leads to the same equation.

Rotating Newtonian universe

- Choose an axis $\mathbb{R}\mathbf{e}_z$ to be the rotation axis of the universe; all galaxies move with velocity $\dot{\mathbf{r}} = \omega \mathbf{e}_z \times \mathbf{r}$.

- Equation of motion:

$$\omega^2 \mathbf{e}_z \times (\mathbf{e}_z \times \mathbf{r}) = 2\pi \left(\rho - \frac{\Lambda}{4\pi} \right) \mathbf{e}_z \times (\mathbf{e}_z \times \mathbf{r}) \quad \Leftrightarrow \quad \boxed{2\omega^2 = 4\pi\rho - \Lambda}$$

- Could have chosen any other parallel axis to be the rotation axis of the universe: $\dot{\mathbf{r}} = \omega \mathbf{e}_z \times (\mathbf{r} - \mathbf{r}_0)$ leads to the same equation.

Newton-Cartan theory

- What are inertial frames in Newtonian cosmology?
- Newtonian gravity does not respect the principle of equivalence – must introduce Coriolis field \mathbf{H} .
- Equation of motion for a free-falling particle:

$$\ddot{\mathbf{r}} = \mathbf{G} + \dot{\mathbf{r}} \times \mathbf{H}$$

- Field equations:

$$\left\{ \begin{array}{l} \nabla \times \mathbf{G} = -\frac{\partial \mathbf{H}}{\partial t} \\ \nabla \cdot \mathbf{H} = 0 \\ \nabla \cdot \mathbf{G} = \frac{1}{2} \mathbf{H}^2 - 4\pi\rho + \Lambda \\ \nabla \times \mathbf{H} = 0 \end{array} \right.$$

- Rotating universe in co-rotating frame: $\mathbf{G} = 0$, $\mathbf{H} = 2\omega \mathbf{e}_z$.

The Gödel universe

- $ds^2 = \frac{1}{2\omega^2} \{-[dt - \sqrt{2}(\cosh(r) - 1)d\varphi]^2 + dr^2 + \sinh^2(r)d\varphi^2 + dz^2\}$
- Solution of the Einstein equations

$$\text{Ric} - \frac{1}{2}Rg + \Lambda g = 8\pi T$$

with

$$\Lambda = -\omega^2, \quad T = \frac{\omega^2}{4\pi} U \otimes U$$

- $ds^2 = \frac{1}{2\omega^2} \{-[dt - \sqrt{2}(\cosh(r) - 1)d\varphi]^2 + dr^2 + \sinh^2(r)d\varphi^2 + dz^2\}$

- Solution of the Einstein equations

$$\text{Ric} - \frac{1}{2}Rg + \Lambda g = 8\pi T$$

with

$$\Lambda = -\omega^2, \quad T = \frac{\omega^2}{4\pi} U \otimes U$$

- $ds^2 = -[dt - \sqrt{2}(\cosh(r) - 1)d\varphi]^2 + dr^2 + \sinh^2(r)d\varphi^2$

- Solution of the Einstein equations

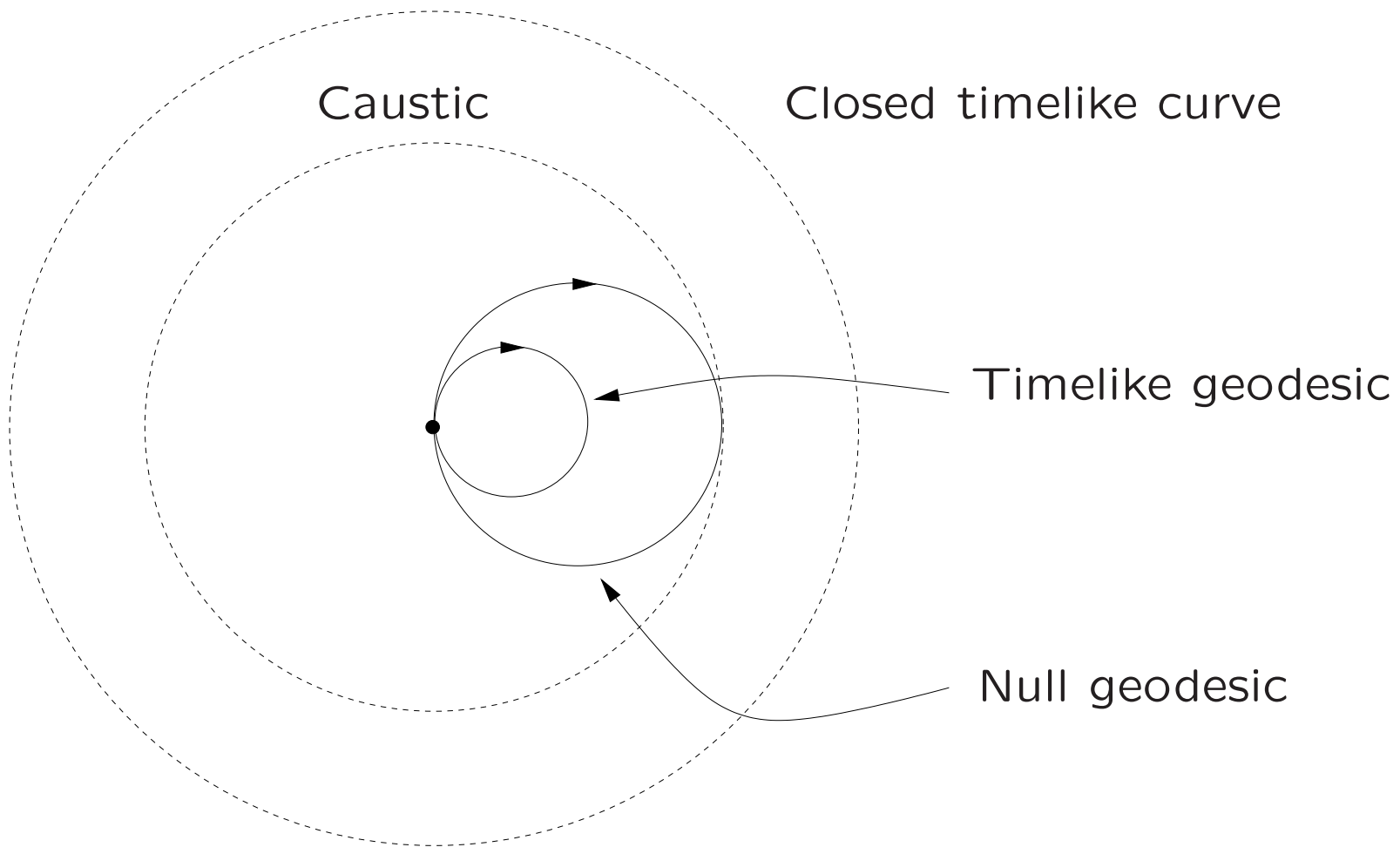
$$Ric - \frac{1}{2}Rg + \Lambda g = 8\pi T$$

with

$$\Lambda = -\frac{1}{2}, \quad T = \frac{1}{8\pi} U \otimes U$$

- $$ds^2 = -[dt - \underbrace{\sqrt{2}(\cosh(r) - 1)d\varphi}_{\theta}]^2 + \underbrace{dr^2 + \sinh^2(r)d\varphi^2}_{\text{hyperbolic plane}}$$

- $$d\theta = \sinh(r) dr \wedge d\varphi = dA$$



Caustic

Closed timelike curve

Timelike geodesic

Null geodesic

- $\Delta t = \oint_{\gamma} dt = \oint_{\gamma} \sqrt{2} \theta + \sqrt{d\tau^2 + dl^2} = \sqrt{2} A + \oint_{\gamma} \frac{dl}{v}$

- So the projection γ of a closed timelike curve must be traversed **clockwise** and satisfy

$$l < \sqrt{2}|A|$$

and so (using the isoperimetric inequality)

$$|A| > 4\pi, \quad l > 4\pi\sqrt{2}$$

and

$$v_{max} > \frac{\sqrt{2}}{2}$$

Total integrated acceleration

- Rocket equation: $m(\tau_1) = m(\tau_0) \exp\left(-\frac{1}{v_e} \int_{\tau_0}^{\tau_1} a(\tau) d\tau\right)$
- $\frac{m(\tau_0)}{m(\tau_1)} \geq \exp(TA), \quad TA = \int_{\tau_0}^{\tau_1} a(\tau) d\tau$

- Malament (1985) proved that

$$TA \geq \ln(2 + \sqrt{5}) \simeq 1.4436$$

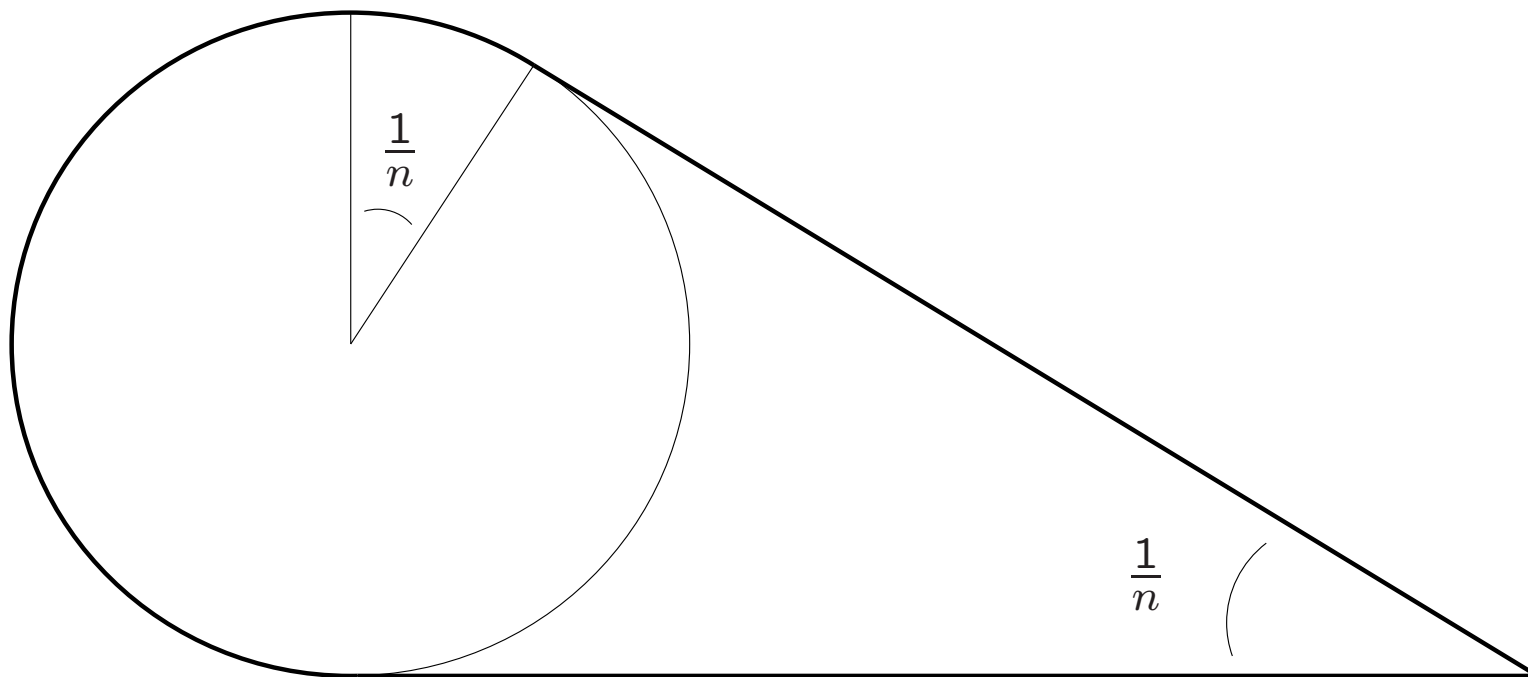
for any CTC, and conjectured that in fact

$$TA \geq 2\pi\sqrt{9 + 6\sqrt{3}} \simeq 27.6691$$

This would mean (Gödel, 1949)

$$\frac{m(\tau_0)}{m(\tau_1)} \gtrsim 10^{12}$$

- Manchak (2011) gave a **non-periodic** counter-example.



Rocket theory

- Problem: minimize $TA = \int_{\tau_0}^{\tau_1} a(\tau) d\tau$
- Issue: $L = a = |\nabla_U U|$ is not differentiable along geodesics
- Issue: one has to deal with **instantaneous accelerations**, unlike for, say, $L = a^2$ (elastic curves)

- $$\begin{cases} \nabla_U U^\mu = a P^\mu \\ \nabla_U P^\mu = -q^\mu + a U^\mu \\ \nabla_U q_\mu = R_{\mu\alpha\beta\gamma} U^\alpha P^\beta U^\gamma \end{cases} \quad (\text{Henriques and Natário, 2012})$$
- $P^\mu P_\mu \leq 1$ with $P^\mu P_\mu = 1$ if $a \neq 0$ and at instantaneous accelerations
- If the initial/final U^μ is not specified then $P^\mu = 0$ at the beginning/end

- $$\begin{cases} \nabla_U U^\mu = a P^\mu \\ \nabla_U P^\mu = -q^\mu + a U^\mu \\ \nabla_U q_\mu = R_{\mu\alpha\beta\gamma} U^\alpha P^\beta U^\gamma \end{cases}$$

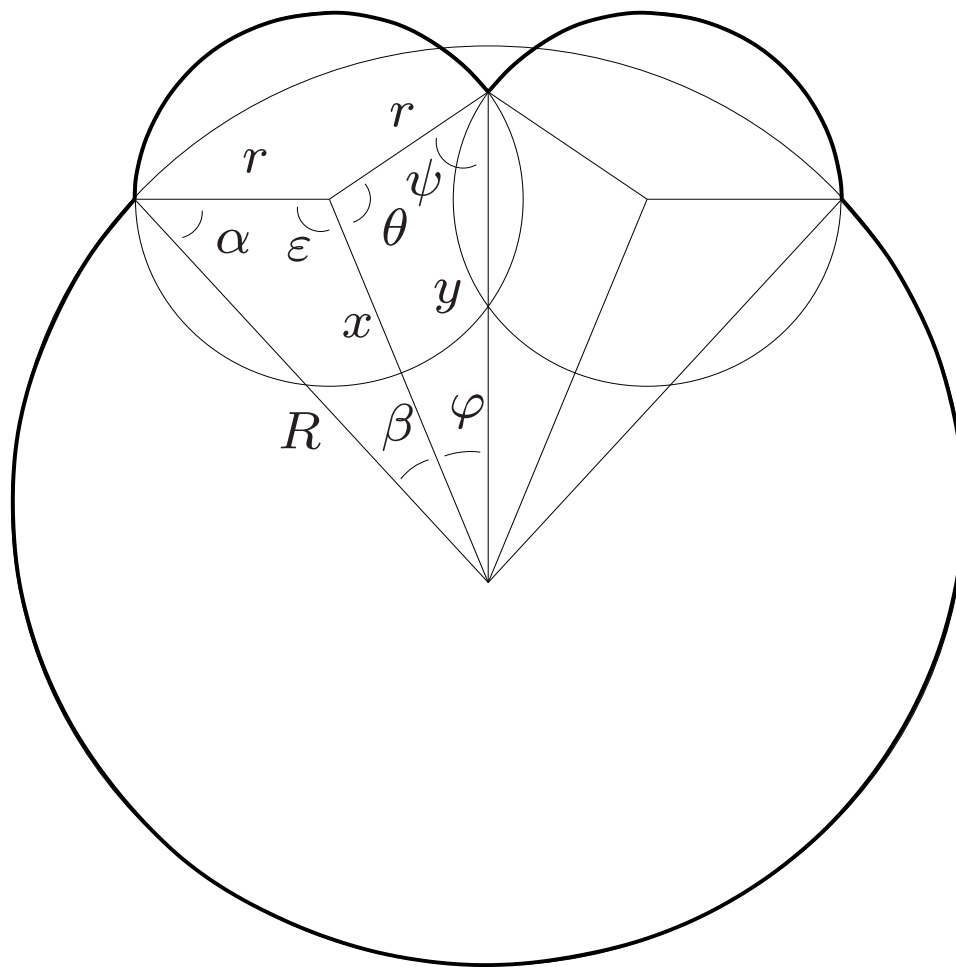
- $P^\mu P_\mu \leq 1$ with $P^\mu P_\mu = 1$ if $a \neq 0$ and at instantaneous accelerations

- If the initial/final U^μ is not specified then $P^\mu = 0$ at the beginning/end

- $$\begin{cases} \nabla_U U^\mu = 0 \\ \nabla_U P^\mu = -q^\mu \\ \nabla_U q_\mu = R_{\mu\alpha\beta\gamma} U^\alpha P^\beta U^\gamma \end{cases}$$
- $P^\mu P_\mu \leq 1$ with $P^\mu P_\mu = 1$ if $a \neq 0$ and at instantaneous accelerations
- If the initial/final U^μ is not specified then $P^\mu = 0$ at the beginning/end

Optimal time travel

- $P^\mu = 0$ at the beginning/end means that we must begin/end with a geodesic arc
- Start with an ansatz depending on finitely many parameters, (numerically) minimize TA with respect to those parameters



- Satisfies the minimum conditions
- $TA \simeq 24.9927$
- $TA \simeq 28.6085$ if we make it periodic
- $TA \simeq 27.6691$ for Malament's conjecture

- Is $TA \simeq 24.9927$ the minimum?
- Does Malament's conjecture still hold for **periodic** CTCs?

Bibliography

- [1] K. Gödel, *A remark about the relationship between relativity theory and idealistic philosophy*, in Albert Einstein: Philosopher-Scientist, Open Court (1949), 560–561.
- [2] D. Malament, *Minimal acceleration requirements for "time travel" in Gödel space-time*, J. Math. Phys. **26** (1985), 774–777.
- [3] J. Manchak, *On efficient "time travel" in Gödel spacetime*, Gen. Rel. Grav. **43** (2011), 51–60.
- [4] P. Henriques and J. Natário, *The rocket problem in general relativity*, J. Optim. Theory Appl. **154** (2012), 500–524 (arXiv:1105.5235).
- [5] J. Natário, *Optimal time travel in the Gdel universe*, Gen. Rel. Grav. **44** (2012), 855–874 (arXiv:1105.6197).