An elementary derivation of the Montgomery phase formula

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1ª questão Dado um sistema mecânico de ligações não holonômicas perfeitas: $(M, \langle,\rangle, \Sigma, \mathcal{F})$, com todos os dados de classe C^∞, demonstrar que existe um único campo de forças $R \in \mathcal{F}^\infty_\Sigma$ tal que

(i) $\mu^{-1} R(v_q) \in \Sigma^\perp_q$ para todo $q \in M$ e $v_q \in \Sigma_q$ (μ é o operador massa);

(ii) para cada $q \in M$ e $v_q \in \Sigma_q$, a solução maximal $t \mapsto q(t)$ que satisfaz $\dot{q}(0) = v_q$ e

$$\mu(\nabla_{\dot q}\dot q) = (\mathcal{F} + R)\dot q,$$

é compatível com a ligação Σ

(aqui ∇ é a conexão de Levi-Civita associada à $\langle,\rangle$).

Mostrar além disso que

(iii) o movimento em (ii) é C^∞ e está univocamente determinado por v_q;

(iv) existe um campo de forças $Q \in \mathcal{F}^\infty_\Sigma$ dependendo somente de $(M, \langle,\rangle, \Sigma)$ tal que

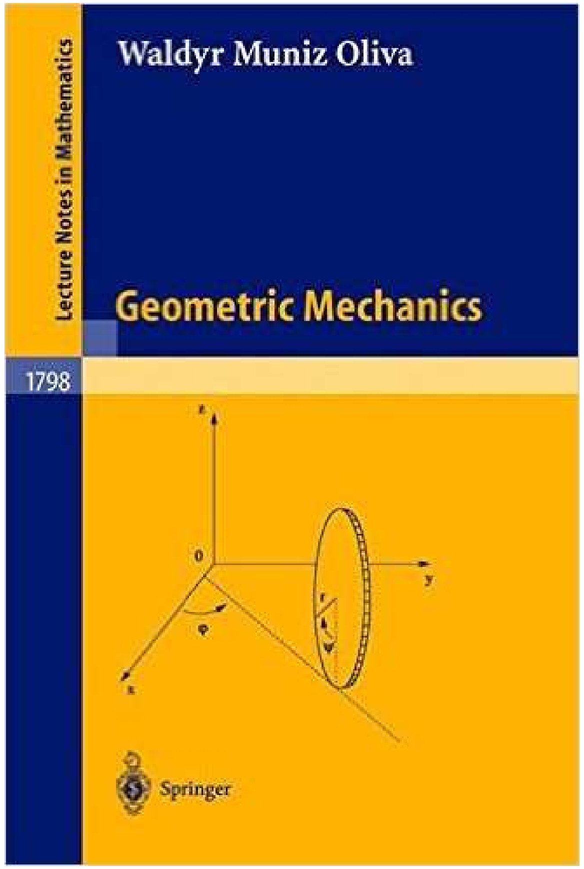
$$R(v_q) + \mu P^\perp \mu^{-1} \mathcal{F}(v_q) = Q(v_q), \quad \forall q \in M \text{ e } v_q \in \Sigma_q,$$

Q dado localmente por

$$Q(v_q) = -\mu\left(\sum_i \langle v_q, \nabla_{v_q} Y_i\rangle Y_i(q)\right),$$

(Y_i) sendo uma base local ortonormada de campos vetoriais de $\Sigma^\perp$ e $P^\perp : TM \to \Sigma^\perp M$ a projeção (na métrica $\langle,\rangle$) de $w_q \in T_qM$ sobre o subespaço $\Sigma^\perp_q$, ortogonal à Σ_q.

2ª questão Com a notação da questão anterior, mostrar que $Q \equiv 0$ se, e somente se, ~~todos os movimentos compatíveis com Σ são geodésicas da métrica $\langle,\rangle$~~. Σ é invariante pelo fluxo geodésico.



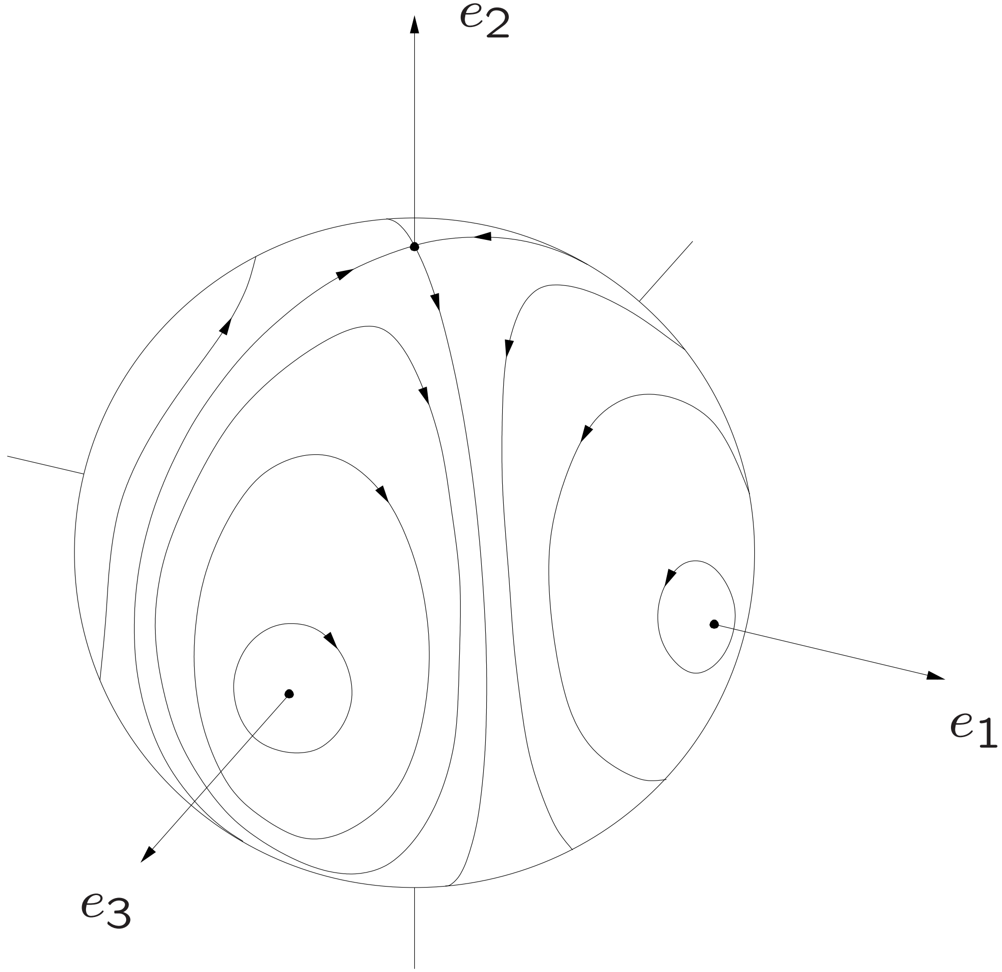
Euler Top

- An Euler top is a rigid body with a fixed point moving freely in an inertial frame.

- Motion given by $S : \mathbb{R} \to SO(3)$, whence $\dot{S}(t) = S(t)A(t)$ whith $A(t) \in \mathfrak{so}(3)$.

- Angular velocity: $\boldsymbol{\omega}(t) = S(t)\boldsymbol{\Omega}(t)$, where $A(t)\boldsymbol{\xi} = \boldsymbol{\Omega}(t) \times \boldsymbol{\xi}$.

- Angular momentum: $\mathbf{p} = S(t)\mathbf{P}(t)$, where $\mathbf{P}(t) = I\boldsymbol{\Omega}(t)$.
- I is the symmetric, positive definite inertia tensor.
- Euler equation: $\dot{\mathbf{p}} = \mathbf{0} \Leftrightarrow \dot{\mathbf{P}} + \boldsymbol{\Omega} \times \mathbf{P} = \mathbf{0}$.
- Conserved quantities: $\|\mathbf{P}\| = \|\mathbf{p}\| = p$ and $K = \frac{1}{2}\langle \mathbf{P}, \boldsymbol{\Omega} \rangle$.



e_2
e_1
e_3

A picture of the motion

- $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ right-handed orthonormal basis fixed in the inertial frame, with $\mathbf{n} = \mathbf{p}/p$.

- $\mathbf{N}(t) = S^{-1}(t)\,\mathbf{n} = \mathbf{P}(t)/p$ unit normal to the sphere $\|\mathbf{P}\| = p$.

- $\mathbf{E}(t) = S^{-1}(t)\,\mathbf{e}$ and $\mathbf{F}(t) = S^{-1}(t)\,\mathbf{f}$ tangent.

- $\dot{\mathbf{E}} + \mathbf{\Omega} \times \mathbf{E} = \mathbf{0} \Rightarrow \dot{\mathbf{E}}^{\top} = -\frac{2K}{p}\mathbf{F}$.

- In other words, $\mathbf{E}(t)$ rotates with constant angular velocity $-2K/p$ with respect to a parallel-transported frame.

Montgomery's formula

- After one period of $\mathbf{P}(t)$, the fixed vector $\mathbf{p}$ as seen in the by the rotating Euler top has returned to the initial position.

- Thus the Euler top has rotated by some angle $\Delta\theta$ around $\mathbf{p}$.

- Montgomery's formula: $\boxed{\Delta\theta = \frac{2KT}{p} - \frac{A}{p^2}}$, where T is the period and A is the area of the region bounded by the orbit.

- Dynamical phase: $\frac{2KT}{p}$ is minus the rotation of the fixed frame about $\mathbf{N}(t)$.

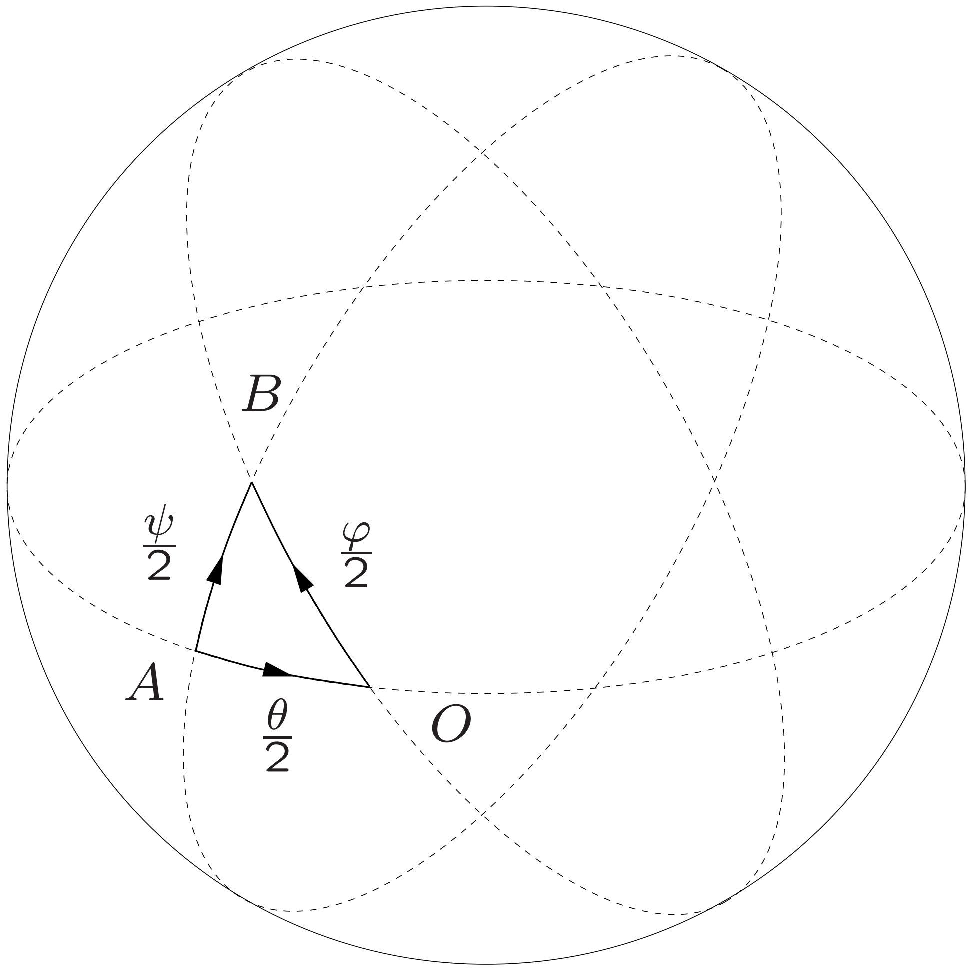
- Geometric phase: $-\frac{A}{p^2}$ is the holonomy of the parallel transport on the sphere.

Interpretation of the geometric phase

- We can give an approximate interpretation of the geometric phase for orbits starting close to the unstable equilibria.

- Heteroclinics are contained in great circles.

- With respect to an auxiliary frame rotating about $\mathbf{p}$ with angular speed $2K/p$, the motion of the Euler top during one period consists of two 180° rotations about certain axes perpendicular to $\mathbf{p}$.



B
A
O
$\frac{\psi}{2}$
$\frac{\varphi}{2}$
$\frac{\theta}{2}$