Bernoulli's elastic curves, rockets, and time travel

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Bernoulli's elastic curves

- If $c : [0, L] \to \mathbb{R}^n$ is a curve parameterized by its arclength s then its curvature is

$$k(s) = \left\| \frac{d^2 c}{ds^2}(s) \right\|$$

- If the curve represents an elastic beam then its elastic energy is (proportional to)

$$E[c] = \int_0^L k(s)^2 ds$$

Therefore equilibrium configurations of the beam correspond to minima of E (Bernoulli's elastic curves).

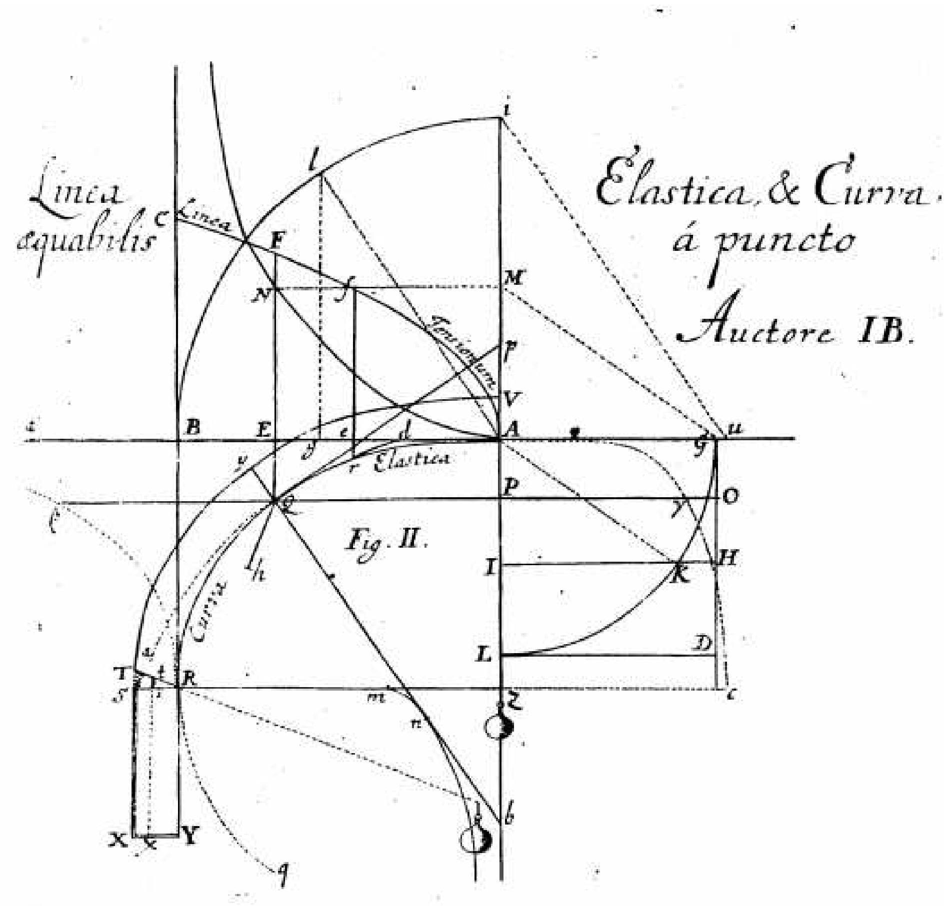


Figure 5.5. Bernoulli's 1694 publication of the elastica.

Tabula III.

Additamentum.

Fig.1.

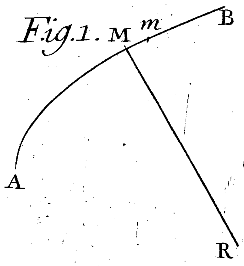


Fig.2.

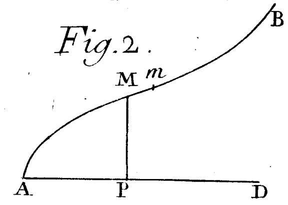


Fig.3.

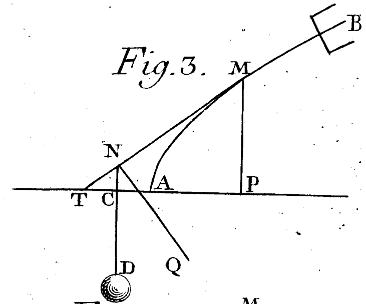


Fig.4.

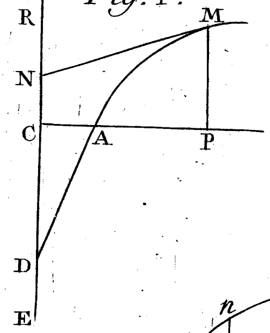


Fig.5.

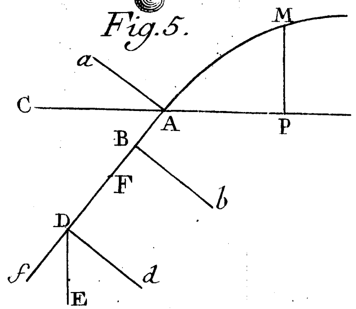
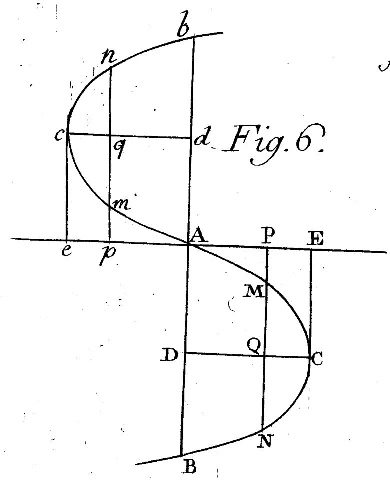


Fig.6.



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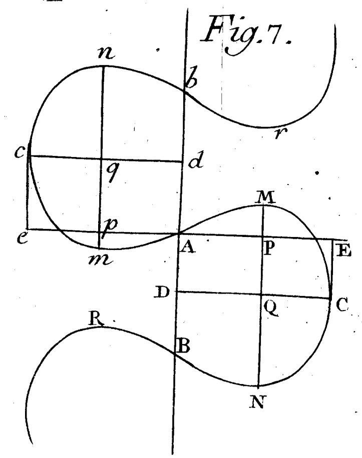


Tabula. IV.

Fig. 8.

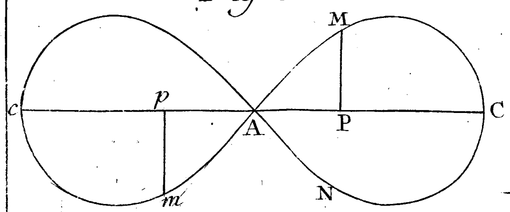


Fig. 9.

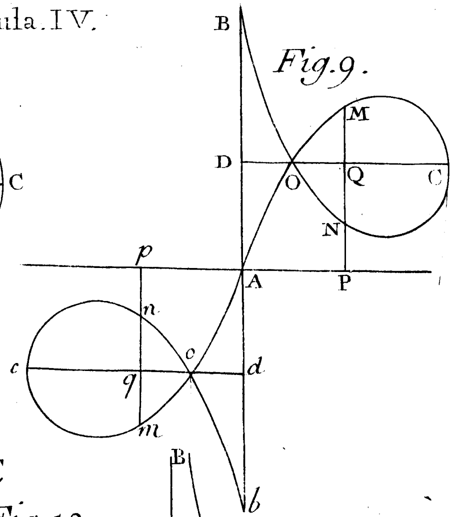


Fig. 11.

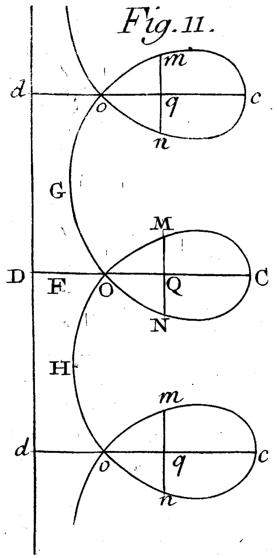


Fig. 12.

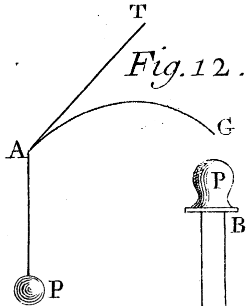


Fig. 13.

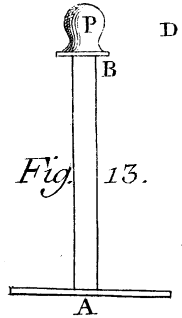


Fig. 10.

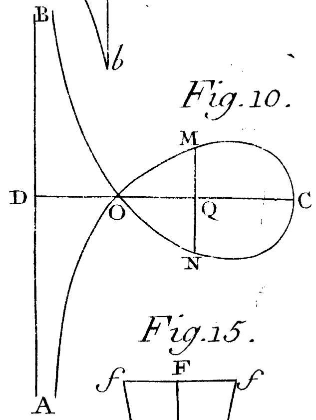


Fig. 14.

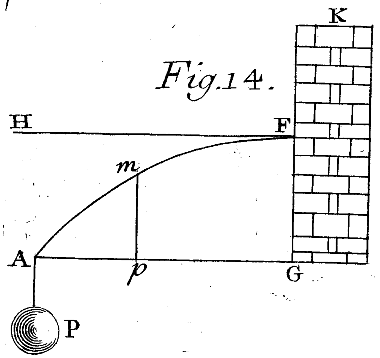
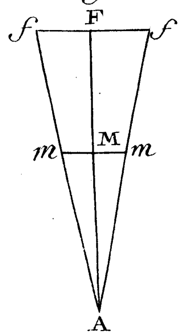


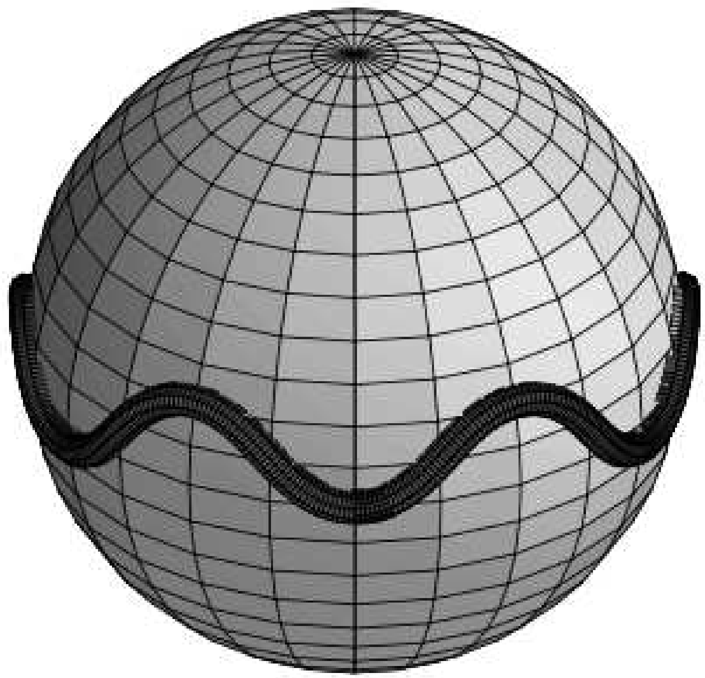
Fig. 15.



- Can be generalized for curves $c : [0, L] \to M$, where $(M, \langle \cdot, \cdot \rangle)$ is a Riemannian manifold.

- In this case one uses the Levi-Civita covariant derivative D/ds to define

$$k(s) = \left\| \frac{D}{ds} \left(\frac{dc}{dt} \right) (s) \right\|$$



General relativity and rocket theory

- In general relativity space-time is modelled as a 4-dimensional pseudo-Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ with signature $(-1, 1, 1, 1)$.

- Trajectories (histories) of material point particles are modelled by curves $c : [0, T] \to M$ such that

$$\left\langle \frac{dc}{d\tau}, \frac{dc}{d\tau} \right\rangle = -1$$

- Arclength τ is interpreted as the particle's proper time.

- Curvature $a(\tau) = \left\| \frac{D}{d\tau} \left(\frac{dc}{d\tau} \right) (\tau) \right\|$ is interpreted as the particle's proper acceleration (so geodesics correspond to free fall).

- Rocket equation: if the trajectory describes a rocket ejecting propellant with velocity v_e then its rest mass $m(\tau)$ satisfies

$$m(T) = m(0) \exp\left(-\frac{1}{v_e}\int_0^T a(\tau)d\tau\right)$$

- The most economic trajectory minimizes

$$TA = \int_0^T a(\tau)d\tau$$

- One has to deal with instantaneous accelerations, unlike for

$$TA_p = \int_0^T a(\tau)^p d\tau$$

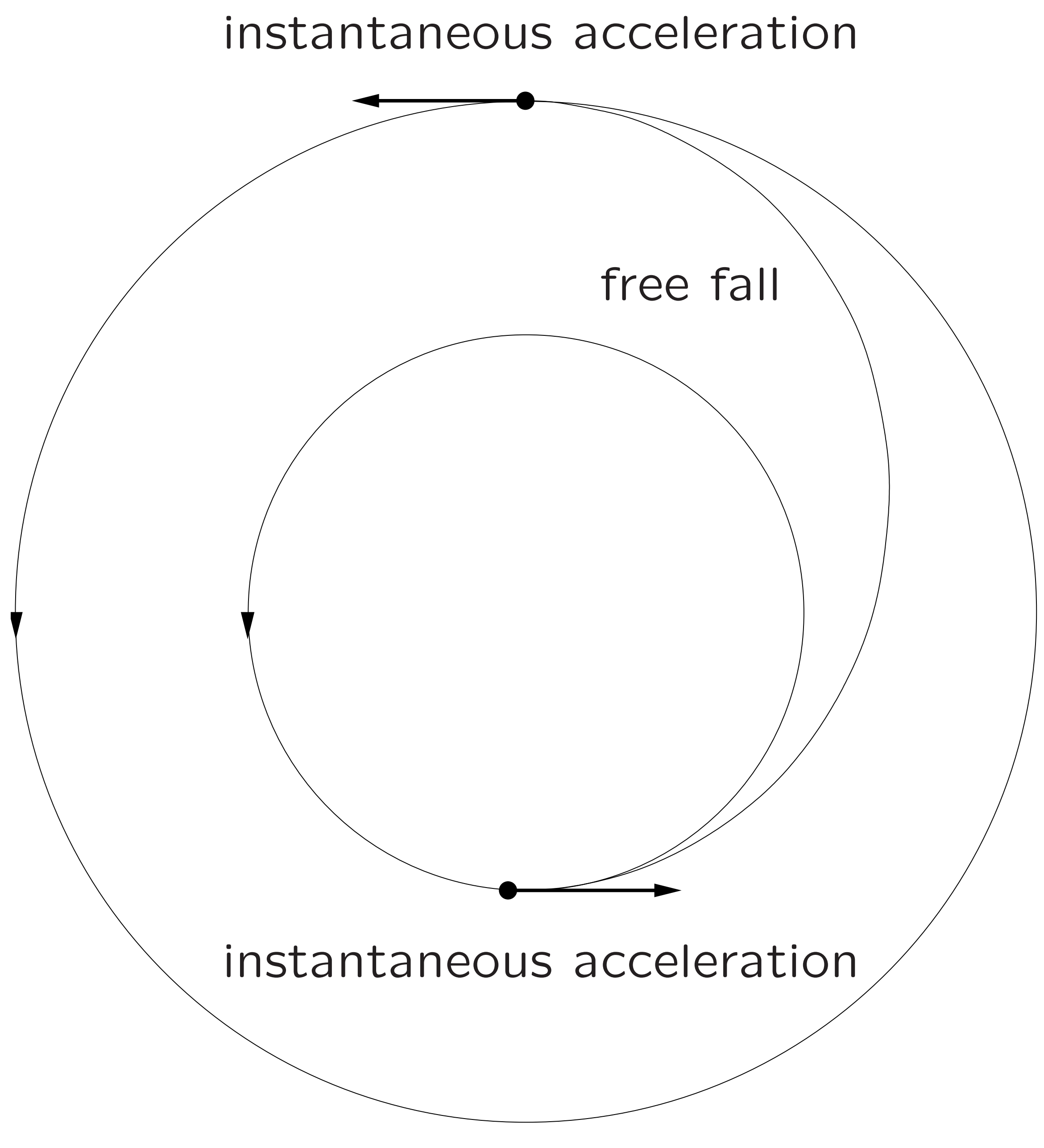
with $p > 1$.

- Rocket theory: the optimal trajectory satisfies ($U = \frac{dc}{d\tau}$)

$$\begin{cases} \dfrac{DU}{d\tau} = aP \\[2ex] \dfrac{DP}{d\tau} = -Q + aU \qquad \text{(Henriques and Natário, 2012)} \\[2ex] \dfrac{DQ}{d\tau} = R(U,P)U \end{cases}$$

- $\|P\| \leq 1$ with $\|P\| = 1$ if $a \neq 0$ and at instantaneous accelerations.

- If the initial/final U is not specified then $P = 0$ at the beginning/end.


instantaneous acceleration
free fall
instantaneous acceleration

Time travel in the Gödel universe

- Gödel (1949) discovered a model $(M, \langle \cdot, \cdot \rangle)$ for the universe containing closed histories $c : [0, T] \to M$ with $c(0) = c(T)$. These represent material particles which visit their own past.

- Malament (1985) proved that for these curves

$$TA \geq \ln(2 + \sqrt{5}) \simeq 1.4436$$

and conjectured that in fact

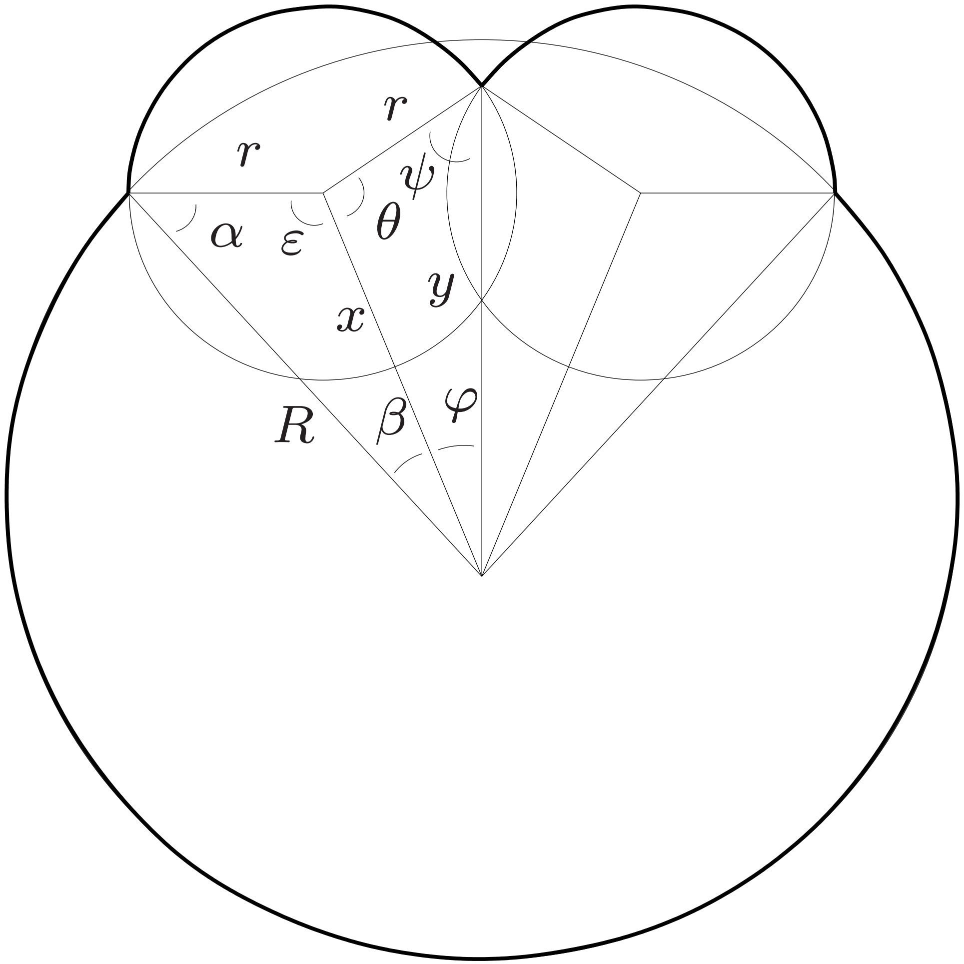
$$TA \geq 2\pi\sqrt{9 + 6\sqrt{3}} \simeq 27.6691$$

This would mean (Gödel, 1949)

$$\frac{m(T)}{m(0)} \gtrsim 10^{12}$$

- Manchak (2011) gave a non-periodic counter-example.

- Using rocket theory we gave the following refinement.



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- Satisfies the minimum conditions
- $TA \simeq 24.9927$
- $TA \simeq 28.6085$ if we make it periodic
- $TA \simeq 27.6691$ for Malament's conjecture

- Is $TA \simeq 24.9927$ the minimum?

- Does Malament's conjecture still hold for periodic CTCs?

Bibliography

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- [3] J. Manchak, *On efficient "time travel" in Gödel spacetime*, Gen. Rel. Grav. **43** (2011), 51–60.

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- [5] J. Natário, *Optimal time travel in the Gödel universe*, Gen. Rel. Grav. **44** (2012), 855–874 (`arXiv:1105.6197`).