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## Causality Implies the Lorentz Group

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Causality is represented by a partial ordering on Minkowski space, and the group of all automorphisms that preserve this partial ordering is shown to be generated by the inhomogeneous Lorentz group and dilatations.

LET  $M$  denote Minkowski space, the real 4-dimensional space-time continuum of special relativity, and let  $Q$  denote the characteristic quadratic form on  $M$ ,

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2,$$

$$x = (x_0, x_1, x_2, x_3) \in M.$$

There is a partial ordering on  $M$  given by  $x < y$  if an event at  $x$  can influence an event at  $y$ ; more precisely,  $x < y$  if  $y - x$  is a time vector,  $Q(y - x) > 0$ , oriented towards the future,  $x_0 < y_0$ . Let  $f: M \rightarrow M$  be a function that is a one-to-one mapping (we make no assumptions that  $f$  is linear or continuous). We call  $f$  a *causal automorphism* if both  $f$  and  $f^{-1}$  preserve the partial ordering; in other words,

$$x < y \Leftrightarrow fx < fy, \quad \text{all } x, y \in M.$$

The causal automorphisms form a group, which we call the *causality group*.

Let  $G$  be the group generated by (i) the orthochronous Lorentz group (linear maps of  $M$  that leave  $Q$  invariant, and preserve time orientation, but possibly reverse space orientation), (ii) translations of  $M$ , and (iii) dilatations of  $M$  (multiplication by a scalar).

*Theorem.* The causality group =  $G$ .

*Remark 1.* The significance of the theorem is that if we interpret the principle of causality mathematically as the set  $M$  together with the partial ordering, then the inhomogeneous Lorentz group appears naturally (with dilatations and spacereversal) as the symmetry group of  $M$ . Therefore the basic invariants of physics, which are the representations of the inhomogeneous Lorentz group, follow naturally from the single principle of causality.

*Remark 2.* It is easy to see that  $G$  is contained in the causality group, since the generators of  $G$  preserve the partial ordering. The converse is not obvious at first sight, because there seems no

reason why a causal automorphism should be linear or even continuous. In fact, the result depends essentially upon space being more than 1-dimensional. If space were 1-dimensional then the causality group would be much larger than  $G$ , and the general causal automorphism would map the space and time axes into curved lines, as is shown by the example below. Thus the typical 2-dimensional picture of Minkowski space to be found in most textbooks is misleading.

*Remark 3.* The condition for  $f$  to be a causal automorphism is a global condition, but is equivalent (by an elementary compactness argument using the transitivity of  $<$ ) to the following local condition: given  $x \in M$ , then there is a neighborhood  $N$  of  $x$  such that

$$y < z \Leftrightarrow fy < fz, \quad \text{all } y, z \in N.$$

Intuitively this means we need only think of the principle of causality acting in our laboratories for a few seconds, rather than between distant galaxies forever, and still we are able to deduce the Lorentz group.

*Remark 4.* There is another relation on  $M$  given by  $x < \cdot y$  if light can go from  $x$  to  $y$ ; more precisely  $x < \cdot y$  if  $y - x$  is a light vector,  $Q(y - x) = 0$ , oriented towards the future,  $x_0 < y_0$ . The relation  $x < \cdot y$  is not a partial ordering because it is not transitive,

$$x < \cdot y < \cdot z \not\Rightarrow x < \cdot z.$$

We shall show in Lemma 1 that, in the definition of causal automorphism, it does not matter whether we use  $<$  or  $< \cdot$  (or both). Intuitively this means that the Lorentz group can be deduced equally well either from causality between heavy particles, or from causality between photons, or from both. Remark 3 also holds for  $< \cdot$ , although the argument is slightly more complicated due to the lack of transitivity.

*Remark 5.* In Remarks 2 and 3, when referring to the "continuity" of  $f$  or the "neighborhood" of  $x$ , we have implicitly assumed a topology on  $M$ , although for the proof of the theorem we assume no topology. It is customary to think of  $M$  as having the topology of real 4-dimensional Euclidean space, but there are reasons why this is wrong. In particular:

(i) Euclidean topology is locally homogeneous whereas  $M$  is not; every point has its associated light cone separating space vectors from time vectors.

(ii) The group of all homeomorphisms of Euclidean space is vast, and of no physical significance.

In a subsequent paper<sup>1</sup> we suggest alternative topologies for  $M$ , which are not homogeneous, and have the property that any homeomorphism maps light cones to light cones. Therefore any homeomorphism preserves or reverses the relation  $<\cdot$ , and so the group of all homeomorphisms of  $M$  with such a topology will be the double cover of  $G$ . Consequently, the topology is physically significant because it implies the Lorentz group.

The existence of such topologies on Minkowski space suggest the possibility of similar topologies on the inhomogeneous Lorentz group, finer than the Lie-group topology. Any representation with the Lie-group topology would *a fortiori* be a representation with a finer topology, but not necessarily conversely. This raises the question: are there some new representations of the inhomogeneous Lorentz group?

#### Example.

Let  $K$  denote 2-dimensional Minkowski space with characteristic quadratic form

$$Q(x) = x_0^2 - x_1^2, \quad x = (x_0, x_1) \in K.$$

Choose new coordinates

$$y_0 = x_0 - x_1, \quad y_1 = x_0 + x_1.$$

Let  $f_0, f_1 : \mathbb{R} \rightarrow \mathbb{R}$  be two arbitrary nonlinear orientation-preserving homeomorphisms of the real line onto itself. Define  $f : K \rightarrow K$  by

$$f(y_0, y_1) = (f_0 y_0, f_1 y_1).$$

Then  $f$  is a causal automorphism, but  $f \notin G$  because  $f$  is nonlinear. In general, the images of the space and time axes will not be straight lines.

*Lemma 1.* Let  $f : M \rightarrow M$  be a function that is a one-to-one mapping. Then  $f, f^{-1}$  preserve the partial ordering  $<$  if and only if they preserve the relation  $<\cdot$ .

<sup>1</sup> E. C. Zeeman, "The topology of Minkowski space" (to be published).

*Proof.* If  $x < y$  implies  $f^{-1}x < f^{-1}y$ , then  $x <\cdot y$  implies  $fx <\cdot fy$ . Therefore if  $f, f^{-1}$  preserve  $<$ , then  $f$  preserves  $<$  and  $<\cdot$ . Now

$$x <\cdot y \Leftrightarrow \begin{cases} x <\cdot y \\ y < z \Rightarrow x < z. \end{cases}$$

Therefore if  $f$  preserves  $<$  and  $<\cdot$ , then  $f$  preserves  $<\cdot$ . Therefore if  $f, f^{-1}$  preserve  $<$ , then  $f, f^{-1}$  preserve  $<\cdot$ . Conversely,

$$x < y \Leftrightarrow \begin{cases} x <\cdot y \\ x <\cdot z <\cdot y, \text{ for some } z. \end{cases}$$

Therefore if  $f$  preserves  $<\cdot$  and  $<\cdot$ , then  $f$  preserves  $<$ , and so if  $f, f^{-1}$  preserve  $<\cdot$ , then  $f, f^{-1}$  preserve  $<$ .

#### Notation.

If  $x \in M$ , let  $C_x$  denote the light cone through  $x$ ,

$$C_x = \{y; x <\cdot y \text{ or } x = y \text{ or } y <\cdot x\}.$$

If  $x <\cdot y$ , we call the line through  $x$  and  $y$  a light ray and denote it by  $R_{x,y}$ . We deduce

$$R_{x,y} = C_x \cap C_y.$$

*Lemma 2.* A causal automorphism maps light rays to light rays.

*Proof:* Let  $f$  be a causal automorphism. By Lemma 1,  $f$  and  $f^{-1}$  preserve  $<\cdot$ , and so  $fC_x = C_{fx}$ . Therefore if  $x <\cdot y$ ,

$$fR_{x,y} = f(C_x \cap C_y) = C_{fx} \cap C_{fy} = R_{fx,fy}.$$

*Lemma 3.* A causal automorphism maps parallel light rays to parallel light rays.

*Proof:* Let  $a_1, a_2$  be parallel light rays, and let  $P$  be the plane through them. There are two cases according to whether or not  $P$  is a tangent to all the light cones with vertex in  $P$ .

Case (1). Suppose  $P$  is not a tangent (this is the usual case). Then  $P$  contains two families  $\{a\}, \{b\}$  of light rays, where  $\{a\}$  consists of all lines parallel to  $a_1$  (and  $a_2$ ), and  $\{b\}$  consists of all lines parallel to another direction. If  $x \in P$ , then the light cone with vertex  $x$  meets  $P$  in the two light rays through  $x$ , one from each family.

Let  $f$  be a causal automorphism. The images  $\{fa\}, \{fb\}$  are families of lines with the property that each  $fa$  meets each  $fb$ , but no two of any one family meet. There are two possibilities; (i)  $fa_1$  and  $fa_2$  are coplanar, or (ii) they are not. We shall show that (ii) leads to a contradiction. For if  $fa_1, fa_2$  are not coplanar they lie in a 3-dimensional subspace  $S$ , say, of  $M$ . Then each  $fb \subset S$ , because

it meets  $fa_1$  and  $fa_2$ , and therefore each  $fa \subset S$ . Therefore the two families are generators of a nondegenerate quadric surface in  $S$ . If this quadric surface is a hyperboloid (meeting  $S_\infty$  in a conic, where  $S_\infty$  denotes the plane at infinity), then each  $fa$  is parallel to some  $fb$  (the unique  $fb$  through  $fa \cap S_\infty$ ), contradicting that  $fa$  meets  $fb$  (in a finite point). Alternatively, if the quadric is a paraboloid (meeting  $S_\infty$  in two lines), then the directions of  $\{fa\}$  are parallel to all the lines in a plane (which meets  $S_\infty$  in one of the lines). But all light rays are parallel to the rays in a single light cone (which meets  $M_\infty$  in a sphere), and so any plane contains at most two lines parallel to light rays (through the points in  $M_\infty$  where the sphere meets the line in  $S_\infty \subset M_\infty$ ), and so again we have a contradiction.

Therefore  $fa_1, fa_2$  must be coplanar, and, since they do not meet, must be parallel.

Case (2). Suppose  $P$  is a tangent to all light cones with vertex in  $P$  (this is the exceptional case). The argument of Case (1) breaks down because  $P$  has the property that it contains only one family of light rays, namely all the lines parallel to  $a_1$  and  $a_2$ . The planes through  $a_1$  with this property span a 3-dimensional subspace  $A_1$ , say, of  $M$  (the tangent prime to the light cones through  $a_1$ ). Similarly the planes through  $a_2$  with the property span  $A_2$ . Choose  $a_3$  parallel to  $a_1$  and  $a_2$ , and not in  $A_1 \cup A_2$ . Then by Case (1),  $a_1$  and  $a_2$  are both parallel to  $a_3$ , and hence parallel to each other. The proof of Lemma 3 is complete.

*Remark.* So far, everything we have done applies to the 2-dimensional example above. As yet we have not proved that  $f$  maps each light ray linearly, nor have we proved that  $f$  maps straight lines other than light rays into straight lines. We prove this in the next lemma, using the fact that the dimension of space is greater than 1.

**Lemma 4.** *A causal automorphism maps each light ray linearly.*

*Proof:* Suppose  $a, a_1$  are parallel light rays, as in Case (1) of Lemma 3. The family  $\{b\}$  of parallel light rays meeting  $a$  and  $a_1$  determine a linear map  $g_1: a \rightarrow a_1$ , and if  $f$  is a causal automorphism the image family  $\{fb\}$  determine a linear map  $e_1: fa \rightarrow fa_1$  such that the diagram

$$\begin{array}{ccc} & f & \\ a & \longrightarrow & fa \\ \downarrow g_1 & f & \downarrow e_1 \\ a_1 & \longrightarrow & fa_1 \end{array}$$

is commutative. If  $a_2$  is also parallel to  $a_1$ , we can define similar maps  $g_2, e_2$  for the pair  $a_1, a_2$ , and maps  $g_3, e_3$  for the pair  $a_2, a$ , provided neither of the pairs is exceptional as in Case (2) of Lemma 3. Composing the three diagrams gives a commutative diagram

$$\begin{array}{ccc} & f & \\ a & \longrightarrow & fa \\ \downarrow g & f & \downarrow e \\ a & \longrightarrow & fa \end{array}$$

where  $g = g_3g_2g_1$  is a translation of  $a$ , and  $e = e_3e_2e_1$  is a translation of  $fa$  ( $g, e$  are translations because they are compositions of parallel displacements).

If Minkowski space were 2-dimensional, then any such translations would have to be the identity. But in higher dimensions—and this is where the difference is essential—we claim that any given translation  $g$  of  $a$  can be obtained in this manner. It suffices to construct an arbitrary translation on one particular light ray, for then the result will be true for all light rays since  $G$  is transitive on the set of all light rays.

Let  $x = (0, 0, 0, 0)$ ,  $y = (0, -t, 0, t)$ ,  $z = (0, 0, 0, 2t)$  and  $x^* = (0, 0, t, t)$ . Let  $a, a_1, a_2$  be the light rays through  $x, y, z$ , respectively, parallel to the direction  $[0, 0, 1, 1]$ .

Then

$$x \xrightarrow{a_1} y \xrightarrow{a_2} z \xrightarrow{a_3} x^*.$$

Therefore  $gx = x^*$ , and the given translation  $g$  can be obtained by suitable choice of the parameter  $t$ .

Let  $r, s$  be coordinates chosen on  $a, fa$  such that  $f(0) = 0$ . Suppose that when  $g$  is the translation  $r \rightarrow r + t$ , then  $e$  (which is uniquely determined by  $g$ ) is the translation  $s \rightarrow s + u$ , where  $u = u(t)$ . Then

$$f(r + t) = fg(r) = ef(r) = f(r) + u(t),$$

for all  $r, t$ . Putting  $r = 0$ , we have

$$f(t) = u(t),$$

and so

$$f(r + t) = f(r) + f(t).$$

Therefore by induction  $f(nt) = nf(t)$ , for positive and negative integers  $n$ . If  $m$  is also an integer, then  $nf[(m/n)t] = f(mt) = mf(t)$ , and so

$$f(rt) = rf(t)$$

for  $r$  rational. But the last equation is also true for  $r$  real, because  $f$  preserves  $<$ , and so is order-

preserving on each light ray. Hence  $f$  is linear on the light ray, and Lemma 4 is proved.

*Lemma 5. A causal automorphism maps parallel equal intervals on light rays to parallel equal intervals.*

*Proof:* Parallel light rays must be mapped with the same linear expansion because the family of parallel light rays meeting them both also remains parallel. (In the exceptional case use a third ray, as in the proof of Lemma 3.) Therefore equal intervals are mapped with the same linear expansion onto equal intervals.

*Proof of the Theorem:* We are given a causal automorphism  $f : M \rightarrow M$ . We can assume the  $f$  keeps the origin fixed, by first composing  $f$  with a translation if necessary. Choose four linearly independent vectors  $v_1, v_2, v_3, v_4$  directed along four light rays through the origin: these form a base for the vector-space structure of  $M$ , and so an arbitrary vector  $x \in M$  can be written

$$x = \sum x_i v_i, \quad x_i \text{ scalar.}$$

Let  $g : M \rightarrow M$  be the linear map given by

$$gx = x_i(fv_i).$$

We shall show that  $f$  is linear by proving that  $f = g$ . For each  $i$ ,  $1 \leq i \leq 4$ , let  $M_i$  denote the  $i$ -dimensional vector subspace spanned by  $v_i$ ,  $1 \leq j \leq i$ . We shall show that  $f = g$  on  $M_i$  by induction on  $i$ .

The induction starts with  $i = 1$  by Lemma 4, and finishes with  $i = 4$ . Assume the induction for  $i - 1$ . Given  $x \in M_i$ , write  $x = y + x_i v_i$ , where  $y \in M_{i-1}$ . Then the interval from  $y$  to  $x$  is parallel and equal in length to the vector  $x_i v_i$ . By Lemma 5 the interval from  $fy$  to  $fx$  is parallel and equal in length to  $f(x_i v_i)$ . Therefore

$$\begin{aligned} fx &= fy + f(x_i v_i) \\ &= gy + g(x_i v_i), \text{ by induction and by Lemma 4,} \\ &= gx, \text{ because } g \text{ is linear.} \end{aligned}$$

This completes the inductive step, and the proof that  $f$  is linear.

Since  $f$  preserves  $< \cdot$ , the light cone,  $Q(x) = 0$ , through the origin is kept fixed. Therefore, multiplying  $f$  by a scalar if necessary, we deduce that  $f$  leaves  $Q$  invariant. In other words, in modulo multiplication by a translation and a dilatation,  $f$  is a time-orientation-preserving element of the Lorentz group. Therefore  $f \in G$ , and the proof of the theorem is complete.

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