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Kerr–Newman black holes cannot be over-charged or over-spun

Robert M. Wald

Enrico Fermi Institute and Department of Physics,
University of Chicago, Chicago, Illinois 60637, USA
rmwa@uchicago.edu

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I describe research done in collaboration with J. Sorce showing that one cannot over-charge and/or over-spin an initially slightly nonextremal Kerr–Newman black hole via the type of gedanken experiments proposed by Hubeny and others, assuming that the nonelectromagnetic stress-energy tensor of the matter entering the black hole satisfies the null energy condition. Analysis of such gedanken experiments requires that we calculate all effects on the final mass of the black hole that are second-order in the charge and the angular momentum carried into the black hole. We do so using Lagrangian methods, and our formula for the second-order correction to mass, $\delta^2 M$, is obtained by generalizing the canonical energy analysis of Hollands and Wald to the Einstein–Maxwell case. Our formula for $\delta^2 M$ automatically includes all self-force and finite size effects.

Keywords: Black holes; cosmic censorship.

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1. Introduction

The Kerr–Newman family of metrics are the unique stationary, asymptotically flat black hole solutions of the Einstein–Maxwell equations in four spacetime dimensions. The Kerr–Newman metrics comprise a three-parameter family of solutions parameterized by mass M, charge Q, and angular momentum $J = Ma$. However, these solutions describe black holes only for a limited region of this parameter space, characterized by the inequality

$$M^2 \geq \left(\frac{J}{M}\right)^2 + Q^2. \tag{1}$$

When this inequality is not satisfied, the spacetime contains a naked singularity, i.e. the singularity is visible from infinity.

The above facts give rise to a possible means of testing the weak cosmic censorship conjecture, which states that all singularities arising from gravitational collapse must be hidden within black holes, so that no physical process can give rise to a naked singularity. Suppose that we start with a Kerr–Newman black hole satisfying (1). Now throw/drop matter into the black hole carrying energy E, angular momentum ℓ, and charge q, so that the final state will have mass $M + E$, angular momentum $J+\ell$, and charge $Q+q$. Then, if ℓ and/or q can be made sufficiently large compared with E, the inequality (1) would be violated, resulting in a contradiction with the final state being a black hole.

The most obvious case to consider for an attempt to destroy a black hole in this manner would be to start with an extremal black hole, satisfying $M^2 = (J/M)^2 + Q^2$, and to throw in particle matter. I analyzed this case in 1974[1] and I showed that no violations of (1) can occur by throwing particle matter into an extremal Kerr–Newman black hole. The nature of this result is well illustrated by considering the special case of attempting to "over-charge" an extremal Reissner–Nordstrom ($Q = M$) black hole. Let ξ^a denote the horizon Killing field, which, for a Reissner–Nordstrom black hole, coincides with the static Killing field $(\partial/\partial t)^a$. A test particle with mass m and charge q in this spacetime has energy given by

$$E = -p_a\xi^a = -(mu_a + qA_a)\xi^a. \tag{2}$$

Since ξ^a is null on the horizon, the first term $-mu_a\xi^a$ is nonnegative on the horizon, although it can be made arbitrarily small. Thus, the energy of a particle that crosses the horizon is bounded below by the electromagnetic potential energy term

$$E \geq q\Phi_H, \tag{3}$$

where $\Phi_H = (-A_a\xi^a)|_H$ is the electromagnetic potential evaluated on the horizon. However, $\Phi_H = 1$ for an extremal Reissner–Nordstrom black hole, so any particle that enters the black hole must satisfy

$$E \geq q. \tag{4}$$

Consequently, we have $M+E \geq Q+q$, so (1) holds. In other words, any particle with sufficiently large charge q as compared with E to produce a violation of (1) for the final state would be repelled by the electric field of the black hole and cannot enter it. As shown in Ref. 1, similar results hold for attempting to over-charge and/or over-spin a general extremal Kerr–Newman black hole using particle matter.

Nevertheless, in 1999 Hubeny[2] proposed that violations of (1) might still occur if one suitably added matter to a slightly nonextremal black hole. To see this, consider a slightly nonextremal Reissner–Nordstrom black hole. It is useful to introduce the dimensionless parameter

$$\epsilon = \frac{\sqrt{M^2 - Q^2}}{M}, \tag{5}$$

so that $\epsilon \to 0$ in the extremal limit. For $\epsilon \ll 1$, we have

$$\Phi_H = \frac{Q}{r_+} \approx 1 - \epsilon, \tag{6}$$

where $r_+ = M + \sqrt{M^2 - Q^2}$ is the horizon radius. In place of (4), we now obtain

$$E \geq q(1 - \epsilon). \tag{7}$$

Consequently, we have

$$(M + E) - (Q + q) \approx -\epsilon q + \frac{M\epsilon^2}{2}. \tag{8}$$

Thus, it might appear that we can obtain a violation of (1) by taking $q > \epsilon M/2$ (but still keeping $q \ll Q$).

The main difficulty with Hubeny's argument is that for $q \sim \epsilon M$, the violation of (1) given by (8) is of order $\epsilon q \sim q^2/M$. Consequently, to determine if one truly can obtain a violation of (1), the quantities appearing in (8) must all be calculated consistently to the appropriate order. Specifically, the energy, E, of the matter must be calculated to order q^2. However, formula (2) applies only to "test matter" and is valid only to linear order in q; it does not take into account the contributions of electromagnetic self-energy (which require consideration of bodies of finite size) or the energy contributed by self-force effects, both of which enter at order q^2. In particular, it is possible that self-force effects could contribute to a repulsion of the body from the black hole, requiring that the body be given additional energy at order q^2 in order to enter the black hole.

Similar potential violations of (1) have been found in other cases.[3–8] However, just as in Hubeny's argument, in order to determine whether these potential violations actually occur, one needs to calculate all contributions to energy that are quadratic order in the relevant parameters of the particle. This would appear to require a complete analysis of self-force effects as well as finite size effects and any other effects that might enter at this order. Unfortunately, the analytic computation of electromagnetic and gravitational self-force effects on the motion of bodies near a Kerr–Newman black hole is well beyond present capabilities. Thus, the main results that have been obtained prior to our work[9] came from numerical calculations,[10–12] which gave indications that violations do not occur.

In this paper, I will describe results that Sorce and I recently obtained[9] showing that no over-charging or over-spinning can occur in gedanken experiments of the Hubeny type. The first step was to obtain a general expression — first derived in[13] — expressing δM in terms of the flux of charge and angular momentum carried into the black hole together with the nonelectromagnetic energy flux. Assuming only that the nonelectromagnetic contribution to the stress energy tensor satisfies the null energy condition, this formula enables one to prove[9,14] that for arbitrary processes involving matter falling into an exactly extremal Kerr–Newman black hole, no violation of (1) can occur at linear order in the perturbation.

We then considered the possible Hubeny-type violations that might occur for slightly nonextremal black holes. As discussed above, this requires a complete analysis — valid to second-order — of the contributions to the mass of a black hole for arbitrary matter that enters a black hole. We performed this analysis by expressing the second-order change in mass, $\delta^2 M$, of the black hole in terms of the canonical energy of the first-order perturbation. To proceed further, it was necessary for us to make the additional assumption that the *nonextremal* black hole is stable under *linear* perturbations, so that the first-order perturbation decays to a stationary final state. We were then able to evaluate the canonical energy in terms of a positive flux contribution through the horizon and a contribution from the final stationary perturbation. The resulting formula gives rise to an inequality on $\delta^2 M$, which is precisely what is needed to show no violations of the Hubeny type can ever occur. Remarkably, we were able to derive this inequality — which automatically takes account of all self-force and finite size effects — without having to explicitly calculate these effects themselves.

Our analysis differed from most previous analyses in the following three key respects: (1) We considered completely general matter rather than particle matter. Of course, "particle matter" makes sense in general relativity only when considered to be a limiting case of general matter as described in Refs. 15 and 16, so the general results derived in this paper also automatically hold for physically realizable particle matter. (2) Rather than analyzing the motion of bodies to determine what trajectories will or will not enter the black hole, we simply restrict consideration to the case where all matter that is initially present enters the black hole, and we compute the second-order variation of the mass for this case. This allows us to derive the desired inequality without having to calculate the motion of bodies. (3) Most importantly, we obtain an exact expression for the full second-order effects on the mass of a black hole. This allows us to obtain the above-mentioned inequality on $\delta^2 M$.

In Sec. 2, we review general variational formulas arising from the Lagrangian formulation that we will need for our analysis. We analyze gedanken experiments to destroy an extremal black hole in Sec. 3. The Hubeny-type gedanken experiments to destroy a slightly nonextremal black hole are analyzed in Sec. 4.

2. Variational Formulas

The Lagrangian for a diffeomorphism-covariant theory on an n-dimensional spacetime is given by an n-form $\mathbf{L}$ on spacetime, which is a local function of the metric, g_{ab}, its curvature, and symmetrized covariant derivatives of the curvature, and which may also depend on other tensor fields, ψ, and their symmetrized covariant derivatives. We refer to the full field configuration as $\phi = (g_{ab}, \psi)$. We vary the Lagrangian by considering a one-parameter family of field configurations, $\phi(\lambda)$, and taking derivatives of $\mathbf{L}$ with respect to λ. The first-order variation of the Lagrangian can be written as

$$\frac{d\mathbf{L}}{d\lambda} = \mathbf{E}(\phi) \cdot \frac{d\phi}{d\lambda} + d\boldsymbol{\theta}\left(\phi, \frac{d\phi}{d\lambda}\right), \tag{9}$$

where $\mathbf{E}$ is locally constructed from the fields ϕ and their derivatives, while $\boldsymbol{\theta}$ is locally constructed from $\phi, d\phi/d\lambda$, and their derivatives; $\boldsymbol{\theta}$ corresponds to the "boundary term" one would obtain by putting the variation of $\mathbf{L}$ under an integral sign and integrating by parts to remove all spacetime derivatives from $d\phi/d\lambda$. The Euler–Lagrange equations of motion of the theory are simply

$$\mathbf{E}(\phi) = 0. \tag{10}$$

The *symplectic current* $(n-1)$-form $\boldsymbol{\omega}$ is defined in terms of a second variation of $\boldsymbol{\theta}$. For a two-parameter family of field configurations $\phi(\lambda_1, \lambda_2)$, we define

$$\boldsymbol{\omega}\left(\phi; \frac{\partial\phi}{\partial\lambda_1}, \frac{\partial\phi}{\partial\lambda_2}\right) = \frac{\partial}{\partial\lambda_1}\boldsymbol{\theta}\left(\phi, \frac{\partial\phi}{\partial\lambda_2}\right) - \frac{\partial}{\partial\lambda_2}\boldsymbol{\theta}\left(\phi, \frac{\partial\phi}{\partial\lambda_1}\right). \tag{11}$$

The Noether current associated with an arbitrary vector field X^a is defined as

$$\boldsymbol{\mathcal{J}}_X(\phi) = \boldsymbol{\theta}(\phi; \mathscr{L}_X\phi) - \iota_X\mathbf{L}(\phi), \tag{12}$$

where $\iota_X\mathbf{L}$ denotes contraction of X^a into the first index of the differential form $\mathbf{L}$. A simple calculation[17] shows that the first variation of $\boldsymbol{\mathcal{J}}_X$ can be written as

$$\frac{d\boldsymbol{\mathcal{J}}_X}{d\lambda} = -\iota_X\left(\mathbf{E}(\phi)\cdot\frac{d\phi}{d\lambda}\right) + \boldsymbol{\omega}\left(\phi; \frac{d\phi}{d\lambda}, \mathscr{L}_X\phi\right) + d\left[\iota_X\boldsymbol{\theta}\left(\phi, \frac{d\phi}{d\lambda}\right)\right]. \tag{13}$$

On the other hand, it was shown in Ref. 18 that the Noether current can be written in the form

$$\boldsymbol{\mathcal{J}}_X = \mathbf{C}_X + d\mathbf{Q}_X, \tag{14}$$

where $\mathbf{Q}_X$ is called the *Noether charge* and $\mathbf{C}_X \equiv X^a\mathbf{C}_a$ are the constraints of the theory, so that $\mathbf{C}_a = 0$ when the equations of motion are satisfied. In particular, $d\boldsymbol{\mathcal{J}} = 0$ when the equations of motion are satisfied, as can be shown directly from the definition (12) of $\boldsymbol{\mathcal{J}}$.

By differentiating equation (14) with respect to λ and comparing it to Eq. (13), we obtain the fundamental identity

$$d\left[\frac{d\mathbf{Q}_X}{d\lambda} - \iota_X\boldsymbol{\theta}\left(\phi, \frac{d\phi}{d\lambda}\right)\right] = \boldsymbol{\omega}\left(\phi; \frac{d\phi}{d\lambda}, \mathscr{L}_X\phi\right) - \iota_X\left(\mathbf{E}(\phi)\cdot\frac{d\phi}{d\lambda}\right) - \frac{d\mathbf{C}_X}{d\lambda}. \tag{15}$$

Now, assume that $\phi(\lambda)$ is globally hyperbolic with Cauchy surface Σ. Evaluating (15) at $\lambda = 0$ and integrating the resulting equation over Σ, we obtain

$$\int_{\partial\Sigma}[\delta\mathbf{Q}_X - \iota_X\boldsymbol{\theta}(\phi, \delta\phi)] = \int_\Sigma \boldsymbol{\omega}(\phi; \delta\phi, \mathscr{L}_X\phi) - \int_\Sigma \iota_X(\mathbf{E}(\phi)\cdot\delta\phi) - \int_\Sigma \delta\mathbf{C}_X. \tag{16}$$

A Hamiltonian h_X associated with a vector field X^a is a functional of ϕ such that if and only if ϕ satisfies the equations of motion, then under all variations $\delta\phi$, we have

$$\delta h_X = \int_\Sigma \boldsymbol{\omega}(\phi; \delta\phi, \mathscr{L}_X\phi). \tag{17}$$

If the spacetime is asymptotically flat and there is no "interior boundary" to Σ, then a Hamiltonian, h_X, conjugate to X^a must satisfy

$$\delta h_X = \int_\infty [\delta \mathbf{Q}_X - \iota_X \boldsymbol{\theta}(\phi, \delta\phi)] + \int_\Sigma \delta \mathbf{C}_X, \tag{18}$$

where "$\int_\infty$" denotes the limit to spatial infinity of integration over a suitable family of spacelike $(n-2)$-spheres. This motivates the following definition[a] of the ADM conserved quantity H_X conjugate to an asymptotic symmetry X^a for asymptotically flat solutions: H_X (if it exists) is the quantity such that, for all one-parameter families of solutions, we have

$$\delta H_X = \int_\infty [\delta \mathbf{Q}_X - \iota_X \boldsymbol{\theta}(\phi, \delta\phi)]. \tag{19}$$

Finally, let us restrict consideration to the case where (i) $\phi_0 = \phi(\lambda = 0)$ is a globally hyperbolic, asymptotically flat solution of the equations of motion, $\mathbf{E} = 0$, and (ii) ϕ_0 possesses a Killing field ξ^a that is also a symmetry of the matter fields ψ, so that $\mathscr{L}_\xi \phi_0 = 0$. Then (16) yields

$$\int_{\partial\Sigma} [\delta \mathbf{Q}_\xi - \iota_\xi \boldsymbol{\theta}(\phi, \delta\phi)] = -\int_\Sigma \delta \mathbf{C}_\xi. \tag{20}$$

The case of greatest interest for us is where ϕ_0 represents the exterior of a stationary black hole, and ξ^a is the horizon Killing field

$$\xi^a = t^a + \Omega_H \varphi^a, \tag{21}$$

where t^a is the timelike Killing field of ϕ_0, φ^a is the axial Killing field of ϕ_0, and Ω_H is the angular velocity of the horizon. The contribution to the boundary integral from infinity is then just

$$\int_\infty [\delta \mathbf{Q}_\xi - \iota_\xi \boldsymbol{\theta}(\phi, \delta\phi)] = \delta H_\xi = \delta M - \Omega_H \delta J, \tag{22}$$

where M is the ADM mass and J is the ADM angular momentum. If the spacetime represents the exterior of a black hole, then there will be a contribution from the "internal boundary" as well.

Let us now continue to restrict consideration to the case where $\phi_0 = \phi(\lambda = 0)$ is a globally hyperbolic solution of the equations of motion that possesses a Killing field ξ^a that is also a symmetry of the matter fields ψ, so that $\mathscr{L}_\xi \phi_0 = 0$. Again, we do *not* require that the perturbation $\delta\phi = (d\phi/d\lambda)|_{\lambda=0}$ satisfy the linearized equations of motion. Let Σ be a Cauchy surface. We define the *canonical energy* of

[a] We assume here that the matter fields fall off at infinity rapidly enough so as not to contribute to the surface integral on the right side of (19). Otherwise, these matter fields may make contributions of the form "potential times varied charge" that would need to be subtracted to obtain the conventional definition of ADM conserved quantities.

the perturbation $\delta\phi$ on Σ by

$$\mathcal{E}_\Sigma(\phi_0;\delta\phi) \equiv \int_\Sigma \boldsymbol{\omega}(\phi;\delta\phi,\mathscr{L}_\xi\delta\phi). \tag{23}$$

Following Ref. 19, we can obtain an extremely useful expression for canonical energy by differentiating (15) with respect to λ and evaluating the resulting expression at $\lambda = 0$. We obtain

$$d[\delta^2\mathbf{Q}_\xi - \iota_\xi\delta\boldsymbol{\theta}(\phi,\delta\phi)] = \boldsymbol{\omega}(\phi;\delta\phi,\mathscr{L}_\xi\delta\phi) - \iota_\xi(\delta\mathbf{E}\cdot\delta\phi) - \delta^2\mathbf{C}_\xi, \tag{24}$$

Here, the meaning of the "δ's" in the expression $\delta\boldsymbol{\theta}(\phi,\delta\phi)$ is that both derivatives in this term are to be evaluated simultaneously, i.e.

$$\delta\boldsymbol{\theta}(\phi,\delta\phi) \equiv \left.\left[\frac{d}{d\lambda}\boldsymbol{\theta}\left(\phi,\frac{d\phi}{d\lambda}\right)\right]\right|_{\lambda=0}. \tag{25}$$

Integrating (24) over Σ, we obtain

$$\mathcal{E}_\Sigma(\phi;\delta\phi) = \int_{\partial\Sigma}[\delta^2\mathbf{Q}_\xi - \iota_\xi\delta\boldsymbol{\theta}(\phi,\delta\phi)] + \int_\Sigma \iota_\xi(\delta\mathbf{E}\cdot\delta\phi) + \int_\Sigma \delta^2\mathbf{C}_\xi. \tag{26}$$

The case we are most interested in here is one where ϕ_0 corresponds to a stationary black hole, ξ^a is the horizon Killing field, and Σ is a Cauchy surface for the exterior of the black hole. In that case, it follows from (19) that the contribution to the boundary term in (26) from infinity is

$$\int_\infty [\delta^2\mathbf{Q}_\xi - \iota_\xi\delta\boldsymbol{\theta}(\phi,\delta\phi)] = \delta^2 M - \Omega_H\delta^2 J. \tag{27}$$

We will evaluate the interior boundary term at the end of the next subsection.

We now consider Einstein–Maxwell theory in four spacetime dimensions and provide explicit expressions for many of the above quantities. The Einstein–Maxwell Lagrangian is given by

$$\mathbf{L} = \frac{1}{16\pi}(R - F^{ab}F_{ab})\boldsymbol{\epsilon}, \tag{28}$$

where $\boldsymbol{\epsilon}$ is the volume element associated with the metric. For this Lagrangian, the field configuration consists of the metric and the vector potential, $\phi = (g_{ab}, A_a)$. We will treat A_a as a one-form on spacetime. The symplectic potential, Noether charge, equations of motion, and constraints for this Lagrangian were computed in Ref. 13. The symplectic potential can be written as

$$\theta_{abc}\left(\phi,\frac{d\phi}{d\lambda}\right) = \theta^{GR}_{abc} + \theta^{EM}_{abc}, \tag{29}$$

where

$$\theta^{GR}_{abc}\left(\phi,\frac{d\phi}{d\lambda}\right) = \frac{1}{16\pi}\epsilon_{dabc}g^{de}g^{fg}\left(\nabla_g\frac{dg_{ef}}{d\lambda} - \nabla_e\frac{dg_{fg}}{d\lambda}\right), \tag{30}$$

$$\theta^{EM}_{abc}\left(\phi,\frac{d\phi}{d\lambda}\right) = -\frac{1}{4\pi}\epsilon_{dabc}F^{de}\frac{dA_e}{d\lambda}. \tag{31}$$

The Noether charge is given by

$$(Q_X)_{ab} = (Q_X^{GR})_{ab} + (Q_X^{EM})_{ab}, \tag{32}$$

where

$$(Q_X^{GR})_{ab} = -\frac{1}{16\pi}\epsilon_{abcd}\nabla^c X^d, \tag{33}$$

$$(Q_X^{EM})_{ab} = -\frac{1}{8\pi}\epsilon_{abcd}F^{cd}A_e X^e. \tag{34}$$

The equations of motion and constraints are given by

$$\mathbf{E}(\phi)\cdot\frac{d\phi}{d\lambda} = -\boldsymbol{\epsilon}\left[\frac{1}{2}T^{ab}\frac{dg_{ab}}{d\lambda} + j^a\frac{dA_a}{d\lambda}\right], \tag{35}$$

$$C_{bcda} = \epsilon_{ebcd}[T_a{}^e + A_a j^e]. \tag{36}$$

Here, we have written $T_{ab} \equiv G_{ab} - 8\pi T_{ab}^{EM}$ so that T_{ab} corresponds to the non-electromagnetic part of the stress-energy tensor, and $j^a = (1/4\pi)\nabla_b F^{ab}$ so that j^a corresponds to the electromagnetic charge-current. Note that in the absence of sources, when both T_{ab} and j_a are zero, the constraints (36) vanish and the Euler–Lagrange equations of motion (35) are satisfied.

We now restrict attention to the case where $\phi_0 = \phi(\lambda = 0)$ is a stationary black hole solution to the Einstein–Maxwell equations (i.e. $T^{ab} = j^a = 0$ at $\lambda = 0$) with horizon Killing field ξ^a, and we let Σ be a Cauchy surface for the exterior region. In fact, by the black hole uniqueness theorems,[20] ϕ_0 must be a Kerr–Newman solution, but we need not make use of this fact here. We work in a gauge where $\mathscr{L}_\xi A_a(\lambda = 0) = 0$ and $A_a(\lambda = 0) \to 0$ at infinity. In this gauge, $A_a(\lambda = 0)$ will, in general, be singular at the horizon, but this does not cause any difficulties. Furthermore, the variations δA_a and $\delta^2 A_a$ may be assumed to be smooth (as can be justified by working in the principal bundle framework of Prabhu[21]).

By definition, for a *nonextremal* black hole the horizon will be of bifurcate type, and Σ will terminate at the bifurcation surface B. For a nonextremal black hole, we now evaluate the boundary contribution to (20) arising from B. Since $\xi^a = 0$ on B, we have

$$\int_B [\delta\mathbf{Q}_\xi^{GR} - \iota_\xi\boldsymbol{\theta}^{GR}(\phi,\delta\phi)] = \int_B \delta\mathbf{Q}_\xi^{GR} = \frac{\kappa}{8\pi}\delta A_B, \tag{37}$$

where A_B is the area of B and κ is the surface gravity of the event horizon. To evaluate the electromagnetic contribution to the boundary term[b] at B, we note that by (31), $\boldsymbol{\theta}^{EM}$ is smooth at B (since δA_a is smooth), so $\iota_\xi\boldsymbol{\theta}^{EM} = 0$. However, by (34), we have

$$\delta\mathbf{Q}_\xi^{EM} = -\frac{1}{8\pi}[\xi^e A_e \delta(\epsilon_{abcd}F^{cd}) + \xi^e(\delta A_e)\epsilon_{abcd}F^{cd})]. \tag{38}$$

[b]We assume that $A_a t^a$ and $A_a\varphi^a$ fall off as $1/r$ and F_{ab} falls off as $1/r^2$ at infinity, so there is no electromagnetic contribution to the boundary term at infinity.

Again, the second term vanishes at B on account of the smoothness of δA_a and the vanishing of ξ^a. However, the quantity

$$\Phi_H \equiv -[\xi^e A_e(\lambda)]|_{\mathscr{H}} \tag{39}$$

is, in general, nonvanishing at B. Since Φ_H must be constant on the horizon at $\lambda = 0$ (see theorem 1 of Ref. 21 for a general proof for Yang–Mills fields), we find that the electromagnetic contribution to the boundary term at B is

$$\int_B [\delta \mathbf{Q}_\xi^{EM} - \iota_\xi \boldsymbol{\theta}^{EM}(\phi, \delta\phi)] = \frac{1}{8\pi}\Phi_H \int_B \delta(\epsilon_{abcd}F^{cd}) = \Phi_H \delta Q_B, \tag{40}$$

where Q_B is the electric charge flux integral over B.

The ingredients are now in place to write out (20) explicitly for a nonextremal black hole. We previously evaluated the boundary term from infinity in (22), and, in the previous paragraph, we have evaluated the boundary term from B. Using (36) and the fact that $T_{ab} = j^a = 0$ in the background spacetime (since ϕ_0 is a solution), we see that the remaining term $\delta \mathbf{C}_\xi$ takes the form

$$\delta C_{bcda}\xi^a = \epsilon_{ebcd}[\delta T_a{}^e + A_a \delta j^e]\xi^a. \tag{41}$$

Thus, we see that (20) takes the explicit form

$$\delta M - \Omega_H \delta J - \frac{\kappa}{8\pi}\delta A_B - \Phi_H \delta Q_B = -\int_\Sigma \epsilon_{ebcd}[\delta T_a{}^e + A_a \delta j^e]\xi^a. \tag{42}$$

For source free perturbations, $\delta T_{ab} = \delta j_a = 0$, this yields the usual first law of black hole mechanics of Einstein–Maxwell theory.

It should be emphasized that (42) holds only for nonextremal black holes. In this paper, we will be concerned with both nonextremal and extremal black holes. However, it is clear from the derivation that (42) (with $\delta A_B = \delta Q_B = 0$) also holds for extremal black holes in the special case where Σ is not a Cauchy surface but rather an asymptotically flat hypersurface with one boundary at spatial infinity and the other boundary on the horizon at an early time such that the perturbation vanishes in a neighborhood of this internal boundary. In this case, there clearly will be no boundary contribution from the internal boundary of Σ. We will use (42) in this form for extremal black holes in Sec. 3.

The canonical energy may also be split into gravitational and electromagnetic contributions

$$\mathcal{E}_\Sigma(\phi; \delta\phi) = \mathcal{E}_\Sigma^{GR} + \mathcal{E}_\Sigma^{EM}. \tag{43}$$

Explicit formulas for these parts can be obtained from the definition (23), but these formulas are quite complicated and will not be written out explicitly here. Fortunately, we will need to evaluate the canonical energy integral only over (a portion of) the horizon (where its form simplifies considerably) and for stationary perturbations (where it can be evaluated straightforwardly).

We may now explicitly evaluate the other terms appearing in (26) for Einstein–Maxwell theory, in exact parallel with our above evaluation of the terms appearing in (20). For a nonextremal black hole, we obtain[c]

$$\delta^2 M - \Omega_H \delta^2 J - \Phi_H \delta^2 Q_B - \frac{\kappa}{8\pi}\delta^2 A_B = \mathcal{E}_\Sigma(\phi;\delta\phi) - \int_\Sigma \iota_\xi(\delta\mathbf{E}(\phi)\cdot\delta\phi) - \int_\Sigma \delta^2 \mathbf{C}_\xi. \tag{44}$$

Again, this equation (with $\delta^2 A_B = \delta^2 Q_B = 0$) will hold for an extremal black hole if we restrict consideration to the case where both the first- and second-order perturbations vanish in a neighborhood of the horizon at the internal boundary of Σ. In Sec. 4, we will evaluate the right side of (44) in the context relevant to our calculations.

3. Gedanken Experiments to Destroy an Extremal Black Hole

Consider an extremal Kerr–Newman black hole,

$$M^2 = \left(\frac{J}{M}\right)^2 + Q^2. \tag{45}$$

We wish to see if we can cause the inequality (1) to be violated into the black hole. By a simple calculation, it can be seen that (1) will be violated — and a contradiction with cosmic censorship obtained — if we can perturb the black hole by throwing/dropping charged and/or rotating matter so that

$$\delta M < \frac{a}{M^2+a^2}\delta J + \frac{QM}{M^2+a^2}\delta Q, \tag{46}$$

where $a = J/M$.

To analyze whether it is possible to produce such a perturbation, let Σ_0 be an asymptotically flat hypersurface that terminates on the future horizon and extends to spatial infinity.

We consider a perturbation $\delta\phi$ whose initial data on Σ_0 for the fields δg_{ab} and δA_a vanishes in a neighborhood of $\Sigma_0 \cap \mathcal{H}$, as shown in Fig. 1. We assume that the matter sources δT_{ab} and δj^a are nonvanishing only in a compact region of Σ_0, as shown. Physically, this corresponds to considering a perturbation that is induced by bringing matter in from infinity in such a way that the disturbance to the black hole at very early advanced times is negligibly small. If we evolve the perturbation, in general, some of the matter will go into the black hole and some will go out to infinity or remain in orbit around the black hole. The matter that does not fall into the black hole is of no interest to us. Therefore, we can greatly simplify our analysis by restricting to the case where all of the matter goes into the black hole. Note that this also saves us the trouble of analyzing the motion of bodies outside of the black hole; we do not care about the details of how the matter managed to get into the black hole as long as it does get in.

[c]It should be noted that since we take ξ^a to be fixed, the quantities Ω_H and κ do not vary. This means that if we perturb toward another stationary black with different values of Ω_H or κ, then ξ^a cannot be the horizon Killing field of the perturbed black hole. See Ref. 19 for further discussion.

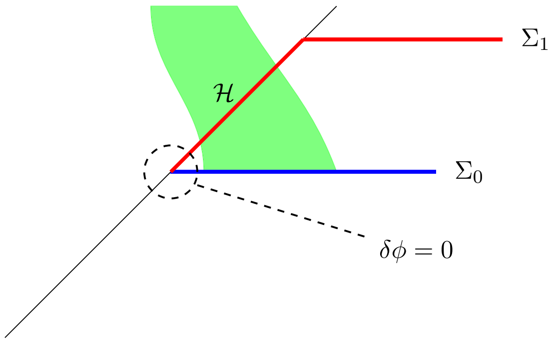

Fig. 1. Charged matter, occupying the shaded region, falls entirely through the event horizon of an extremal black hole. The perturbed initial data on Σ_0 vanishes in a neighborhood of the horizon.

Thus, we wish to consider a one-parameter family where δT_{ab} and δj^a are nonvanishing only in a region like the shaded region of Fig. 1. Let Σ be a hypersurface

$$\Sigma = \mathcal{H} \cup \Sigma_1 \tag{47}$$

labeled in red in Fig. 1. We now use (42) (with $\delta A_B = \delta Q_B = 0$) for this choice of Σ. The integrand on the right side of (42) is nonvanishing only on $\mathcal{H}$. Thus, we obtain,

$$\delta M - \Omega_H \delta J = -\int_{\mathcal{H}} \epsilon_{ebcd} \xi_a \delta T^{ae} - \int_{\mathcal{H}} \xi_a A^a \delta(\epsilon_{ebcd} j^e). \tag{48}$$

Since $\Phi_H = -\xi^a A_a$ is constant on $\mathcal{H}$, we may pull it out of the integral. The integral $\int_{\mathcal{H}} \delta(\epsilon_{ebcd} j^e)$ is just the total flux of electromagnetic charge through the horizon, δQ_{flux}. Since all of the charge added to the spacetime falls through the horizon, this flux is just equal to the total perturbed charge of the black hole, $\delta Q_{\text{flux}} = \delta Q$. Combining these observations yields the following formula relating the perturbed parameters of the black hole spacetime:

$$\delta M - \Omega_H \delta J - \Phi_H \delta Q = -\int_{\mathcal{H}} \epsilon_{ebcd} \xi_a \delta T^{ae}. \tag{49}$$

This result was first derived in Ref. 13. On the horizon, we may write

$$\epsilon_{ebcd} = -4 k_{[e} \tilde{\epsilon}_{bcd]}, \tag{50}$$

where k^a is the future-directed normal to the horizon and $\tilde{\epsilon}_{bcd}$ is the corresponding volume element on the horizon. The right side of (49) can be written as

$$-\int_{\mathcal{H}} \epsilon_{ebcd} \xi_a \delta T^{ae} = \int_{\mathcal{H}} \tilde{\epsilon}_{bcd} \delta T^{ae} \xi_a k_e. \tag{51}$$

Since $\xi^a \propto k^a$, the right side is nonnegative provided only that the nonelectromagnetic stress energy tensor δT_{ab} satisfies the null energy condition, so that $\delta T_{ab} k^a k^b \geq 0$. Thus, (49) yields the inequality

$$\delta M - \Omega_H \delta J - \Phi_H \delta Q \geq 0, \tag{52}$$

which holds for all perturbations of an extremal Kerr–Newman black hole resulting from charged-matter entering the black hole.

For a general (not necessarily extremal) Kerr–Newman black hole, we have

$$\Omega_H = \frac{a}{r_+^2 + a^2} \tag{53}$$

and

$$\Phi_H = \frac{Q r_+}{r_+^2 + a^2}, \tag{54}$$

where r_+ is the horizon radius

$$r_+ = M + \sqrt{M^2 - \left(\frac{J}{M}\right)^2 - Q^2}. \tag{55}$$

For an extremal black hole, we have $r_+ = M$, so (52) yields

$$\delta M \geq \frac{a}{M^2 + a^2}\delta J + \frac{QM}{M^2 + a^2}\delta Q. \tag{56}$$

Thus, (46) cannot be satisfied, and an extremal black hole cannot be destroyed by dropping/throwing matter into it. This generalizes the results of Ref. 1 to arbitrary matter, provided only that the nonelectromagnetic contribution to the stress-energy tensor satisfies the null energy condition. This argument that (49) implies that one cannot over-charge or over-spin an extremal black hole was previously given in Ref. 14.

4. Gedanken Experiments to Destroy a Slightly Nonextremal Black Hole

In the spirit of Hubeny,[2] let us now repeat the gedanken experiment of the previous section starting with a slightly nonextremal Kerr–Newman black hole. The relevant spacetime diagram for this case is shown the same as Fig. 1, except that Σ_0 and Σ are now taken to terminate at the bifurcation surface, B, of the bifurcate Killing horizon. This does not affect the analysis of the first-order perturbation given in the previous section, since the perturbation is assumed to vanish on the horizon at sufficiently early advanced times. Since we will need to calculate second-order effects in this section, we further assume that the second-order perturbation also vanishes in a neighborhood of B, and all of the matter sources go into the black hole at second-order, so that $\delta^2 T_{ab} = \delta^2 j^a = 0$ on Σ_1.

An exact repetition of the analysis of the previous section yields

$$\begin{aligned} \delta M &= \Omega_H \delta J + \Phi_H \delta Q - \int_{\mathcal{H}} \epsilon_{ebcd} \xi_a \delta T^{ae} \\ &\geq \Omega_H \delta J + \Phi_H \delta Q \\ &= \frac{a}{r_+^2 + a^2}\delta J + \frac{Q r_+}{r_+^2 + a^2}\delta Q. \end{aligned} \tag{57}$$

As already noted in the Introduction for the special case of a nearly extremal Reissner–Nordstrom black hole, this equation admits the possibility of violating (1). However, as discussed in the Introduction, in order to determine whether violations of (1) really occur, it is necessary to calculate the second-order corrections, $\delta^2 M$, to the mass of the black hole.

In order to proceed further with our analysis of the second-order corrections to mass, we will make the following additional assumption:

Additional Assumption: The (slightly) nonextremal, unperturbed Kerr–Newman black hole we are considering is linearly stable to perturbations, i.e. any source-free solution to the linearized Einstein–Maxwell equations approaches a perturbation towards another Kerr–Newman black hole at sufficiently late times.

It should be emphasized that this linear stability assumption is entirely compatible with having an instability associated with over-charging or over-spinning the black hole, i.e. we are not assuming what we wish to show. A *finite* perturbation is required to over-charge or over-spin a nonextremal black hole, so one would not expect an instability associated with a violation of (1) to show up at linear order. Indeed, if a nonextremal black hole were linearly unstable, there would be no need to attempt to over-charge or over-spin it in order to destroy it.

In view of this assumption, we may choose Σ so the horizon portion, $\mathcal{H}$, extends to sufficiently late times that it enters the late time stationary era of the perturbation. We may then take Σ_1 so that it extends far from the black hole while remaining in the stationary region. The quantities $\delta^2 M$ and $\delta^2 J$ arising in the boundary term (27) on Σ will then have the interpretation of being the second-order corrections to the mass and angular momentum of the perturbed black hole.

We now consider our gedanken experiment to destroy the slightly nonextremal black hole. We assume that our first-order perturbation has been done optimally (see (57)), so that

$$\delta M = \Omega_H \delta J + \Phi_H \delta Q = \frac{a}{r_+^2 + a^2}\delta J + \frac{Q r_+}{r_+^2 + a^2}\delta Q. \tag{58}$$

As can be seen from (57), this requires vanishing nonelectromagnetic energy flux through the horizon, i.e. $\delta T_{ab} k^a k^b = 0$, as should be (nearly) achievable if the matter is lowered (nearly) to the horizon or is (nearly) at a turning point of its orbit just before entering the black hole.

The second-order change in mass is given by (44) with $\delta^2 Q_B = \delta^2 A_B = 0$ (since the second-order perturbation has been assumed to vanish in a neighborhood of B). We have

$$\delta^2 M - \Omega_H \delta^2 J = \mathcal{E}_\Sigma(\phi; \delta\phi) - \int_{\mathcal{H}} \iota_\xi(\delta \mathbf{E}(\phi) \cdot \delta\phi) - \int_{\mathcal{H}} \delta^2 \mathbf{C}_\xi. \tag{59}$$

Here, the integrals in the last two terms extend only over $\mathcal{H}$ rather than over all of $\Sigma = \mathcal{H} \cup \Sigma_1$ because $\delta \mathbf{E}$ and $\delta^2 \mathbf{C}_\xi$ vanish on Σ_1 by the assumption that there are no sources outside the black hole at late-times.

We now evaluate the last two terms appearing on the right side of (59). From (35), we have

$$(\iota_\xi(\delta\mathbf{E}(\phi)\cdot\delta\phi))_{abc} = -\xi^d\epsilon_{dabc}\left[\frac{1}{2}\delta T^{ef}\delta g_{ef} + \delta j^e\delta A_e\right]. \tag{60}$$

Since ξ^a is tangent to the horizon, the pullback to $\mathcal{H}$ of this term vanishes, so it does not contribute to (59). From (36), we have

$$(\delta^2\mathbf{C}_\xi)_{abc} = \delta^2(\epsilon_{eabc}T_d{}^e\xi^d) + \delta^2(\epsilon_{eabc}A_d j^e\xi^d). \tag{61}$$

We impose the gauge condition $\xi^a\delta A_a = 0$ on $\mathcal{H}$. We see that on $\mathcal{H}$, the second term is

$$\delta^2(\epsilon_{eabc}A_d j^e\xi^d) = -\Phi_H\delta^2(\epsilon_{eabc}j^e) \tag{62}$$

and therefore

$$\delta^2\left[\int_{\mathcal{H}}\xi_a A^a\epsilon_{ebcd}j^e\right] = -\Phi_H\delta^2 Q_{\text{flux}} = -\Phi_H\delta^2 Q, \tag{63}$$

where $\delta^2 Q$ is the second-order change in charge of the black hole. On the other hand, using our assumption that the first-order process was done optimally and thus there was vanishing nonelectromagnetic stress-energy flux through the horizon at first-order, we have

$$\delta^2(\epsilon_{eabc}T_d{}^e\xi^d) = \epsilon_{eabc}\xi^d\delta^2 T_d{}^e. \tag{64}$$

Putting this together, we obtain

$$\delta^2 M - \Omega_H\delta^2 J - \Phi_H\delta^2 Q = \mathcal{E}_\Sigma(\phi;\delta\phi) - \int_{\mathcal{H}}\xi^a\epsilon_{ebcd}\delta^2 T_a{}^e. \tag{65}$$

The last term in this equation is positive provided that the nonelectromagnetic stress-energy tensor satisfies the null energy condition.

What remains is to compute the canonical energy $\mathcal{E}_\Sigma(\phi;\delta\phi)$. Since $\Sigma = \mathcal{H}\cup\Sigma_1$, we have

$$\mathcal{E}_\Sigma(\phi;\delta\phi) = \int_{\mathcal{H}}\boldsymbol{\omega}(\phi,\delta\phi,\mathscr{L}_\xi\delta\phi) + \int_{\Sigma_1}\boldsymbol{\omega}(\phi,\delta\phi,\mathscr{L}_\xi\delta\phi). \tag{66}$$

Let us first calculate the horizon contribution. We have

$$\int_{\mathcal{H}}\boldsymbol{\omega} = \int_{\mathcal{H}}\boldsymbol{\omega}^{GR} + \int_{\mathcal{H}}\boldsymbol{\omega}^{EM}, \tag{67}$$

where $\boldsymbol{\omega}^{GR}$ and $\boldsymbol{\omega}^{EM}$ are the gravitational and electromagnetic contributions to the symplectic current. The integral over $\mathcal{H}$ of the gravitational part of the canonical energy density was computed in Ref. 19, and is given by

$$\int_{\mathcal{H}}\boldsymbol{\omega}^{GR}(g;\delta g,\mathscr{L}_\xi\delta g) = \frac{1}{4\pi}\int_{\mathcal{H}}(\xi^a\nabla_a u)\delta\sigma_{bc}\delta\sigma^{bc}\boldsymbol{\epsilon} + \frac{1}{16\pi}\int_S(\xi^a\nabla_a u)\delta g^{bc}\delta\sigma_{bc}\boldsymbol{\epsilon}, \tag{68}$$

where $\delta\sigma_{ab}$ denotes the perturbed shear of the horizon generators, u is an affine parameter along the future horizon, and $S = \mathcal{H}\cap\Sigma_1$ is the 2-surface formed by the intersection of $\mathcal{H}$ and Σ_1. By our additional assumption above, the perturbation is physically stationary at S, so $\delta\sigma_{ab} = 0$ on S. Thus, we obtain

$$\int_{\mathcal{H}} \boldsymbol{\omega}^{GR}(\phi; \delta\phi, \mathscr{L}_\xi \delta\phi) = \frac{1}{4\pi} \int_{\mathcal{H}} (\xi^a \nabla_a u) \delta\sigma_{bc} \delta\sigma^{bc} \boldsymbol{\epsilon} \geq 0. \tag{69}$$

We may interpret this horizon flux contribution from $\boldsymbol{\omega}^{GR}$ as representing the total flux of gravitational wave energy into the black hole.

Next, the horizon flux contribution from $\boldsymbol{\omega}^{EM}$ is given by

$$\begin{aligned}(\omega^{EM})_{abc}(\phi; \delta\phi, \mathscr{L}_\xi \phi) = {}& \frac{1}{4\pi} \epsilon_{dabc} [\delta A_e \mathscr{L}_\xi \delta F^{de} - \delta F^{de} \mathscr{L}_\xi \delta A_e] \\ & + \frac{1}{4\pi} [(\mathscr{L}_\xi \delta\epsilon_{dabc}) F^{de} \delta A_e - (\delta\epsilon_{dabc}) F^{de} \mathscr{L}_\xi \delta A_e]. \end{aligned} \tag{70}$$

The last two terms on the right side of this equation involve the background electromagnetic field strength F^{ab}. However, on $\mathscr{H}$, F^{ab} must take the form

$$F^{ab} = v^{[a} k^{b]} + w^{ab}, \tag{71}$$

where w^{ab} is purely tangential to the horizon. By (71) together with our gauge condition $\xi^a \delta A_a = 0$ on $\mathcal{H}$, it can be seen that the last two terms in (70) vanish. The first term in (70) can be written as

$$\epsilon_{dabc} \delta A_e \mathscr{L}_\xi \delta F^{de} = \mathscr{L}_\xi [\epsilon_{dabc} \delta A_e \delta F^{de}] - \epsilon_{dabc} \delta F^{de} \mathscr{L}_\xi \delta A_e. \tag{72}$$

When pulled back to $\mathcal{H}$, $\epsilon_{dabc} \delta A_e \delta F^{de}$ is a three-form $\boldsymbol{\eta}$, on a 3-dimensional surface, so when pulled back to $\mathcal{H}$, we have

$$\mathscr{L}_\xi \boldsymbol{\eta} = \iota_\xi d\boldsymbol{\eta} + d(\iota_\xi \boldsymbol{\eta}) = d(\iota_\xi \boldsymbol{\eta}), \tag{73}$$

where the pullback of $\iota_\xi d\boldsymbol{\eta}$ vanishes since ξ^a is tangent to $\mathcal{H}$. Thus, the integral over $\mathcal{H}$ of the first term on the right side of (72) will merely contribute a boundary term at $S = \mathcal{H} \cap \Sigma_1$. However, since the perturbation is assumed to be stationary at S, the electromagnetic energy flux must vanish there, so δF_{ab} must be of the form (71). Using this fact together with our gauge condition $\xi^a \delta A_a = 0$ on $\mathcal{H}$, it can be seen that this boundary term vanishes. Finally, the second term on the right side of (72) combines with the second term of (70). This term can be further simplified by noting that

$$\mathscr{L}_\xi \delta \boldsymbol{A} = \iota_\xi d\delta \boldsymbol{A} + d(\iota_\xi \delta \boldsymbol{A}) = \iota_\xi \delta \boldsymbol{F}, \tag{74}$$

where we have again used our gauge condition $\xi^a \delta A_a = 0$ on $\mathcal{H}$. Putting everything together, we find that

$$\int_{\mathcal{H}} \boldsymbol{\omega}^{EM}(\phi; \delta\phi, \mathscr{L}_\xi \delta\phi) = -\frac{1}{2\pi} \int_{\mathcal{H}} \epsilon_{dabc} \xi^e \delta F^{df} \delta F_{ef}. \tag{75}$$

The right side of this equation is nonnegative and can be interpreted as the total flux of electromagnetic energy into the black hole.

All that remains now is to calculate the contribution to canonical energy from Σ_1

$$\mathcal{E}_{\Sigma_1}(\phi; \delta\phi) = \int_{\Sigma_1} \boldsymbol{\omega}(\phi, \delta\phi, \mathscr{L}_\xi \delta\phi). \tag{76}$$

Since we have assumed that the perturbation is stationary on Σ_1, it might be thought that $\mathscr{L}_\xi \delta\phi = 0$ on Σ_1 and thus this contribution to the canonical energy vanishes. However, this is not the case because our conditions $\delta\xi^a = 0$ as well as our gauge condition $\xi^a \delta A_a = 0$ preclude our writing the perturbation in a gauge where $\mathscr{L}_\xi \delta g_{ab} = 0$ and $\mathscr{L}_\xi \delta A_a = 0$; see Ref. 19 for further discussion. Nevertheless, we can calculate $\mathcal{E}_{\Sigma_1}(\phi; \delta\phi)$ as follows. First, since, by assumption, $\delta\phi$ is equal to a perturbation $\delta\phi^{KN}$ to another Kerr–Newman black hole on Σ_1, we obviously may replace $\delta\phi$ by $\delta\phi^{KN}$ (written in our gauge) on the right side of (76)

$$\mathcal{E}_{\Sigma_1}(\phi; \delta\phi) = \mathcal{E}_{\Sigma_1}(\phi; \delta\phi^{KN}) = \int_{\Sigma_1} \boldsymbol{\omega}(\phi, \delta\phi^{KN}, \mathscr{L}_\xi \delta\phi^{KN}). \tag{77}$$

However, as can be seen from our analysis above, $\delta\phi^{KN}$ has no flux of canonical energy through $\mathcal{H}$, i.e. there is no flux of gravitational or electromagnetic energy through the horizon for a Kerr–Newman perturbation. Thus, we may replace Σ_1 by Σ in (77). Finally, we may evaluate $\mathcal{E}_\Sigma(\phi; \delta\phi^{KN})$ using (44). Consider the one-parameter family, $\phi^{KN}(\alpha)$, where each field configuration in the family is a Kerr–Newman black hole with parameters given by

$$M^{KN}(\alpha) = M + \alpha\delta M, \tag{78}$$

$$Q^{KN}(\alpha) = Q + \alpha\delta Q, \tag{79}$$

$$J^{KN}(\alpha) = J + \alpha\delta J, \tag{80}$$

where δM, δQ, and δJ are chosen to agree with the corresponding values for our first-order perturbation $\phi(\lambda)$. Then, for this family, we have $\delta^2 M = \delta^2 J = \delta^2 Q_B = 0$, as well as $\delta\boldsymbol{E} = \delta^2 \boldsymbol{C}_\xi = 0$. Thus, we obtain

$$\mathcal{E}_\Sigma(\phi; \delta\phi^{KN}) = -\frac{\kappa}{8\pi}\delta^2 A_B^{KN}, \tag{81}$$

where $\delta^2 A_B^{KN}$ denotes the second-order change in the area of the horizon for the one-parameter family (78)–(80).

We have now computed all of the terms appearing in (59). Using the positivity of the gravitational, electromagnetic, and nonelectromagnetic stress-energy fluxes through the horizon, we have thereby derived the following inequality involving the second-order change of the mass of the black hole:

$$\delta^2 M - \Omega_H \delta^2 J - \Phi_H \delta^2 Q \geq -\frac{\kappa}{8\pi}\delta^2 A_B^{KN}. \tag{82}$$

A direct evaluation of the right side of (82) yields[9]

$$\begin{aligned} &\delta^2 M - \Omega_H \delta^2 J - \Phi_H \delta^2 Q \\ &\quad \geq \frac{M}{(M^4 + J^2)^2} \\ &\qquad \times [M^4(\delta J)^2 + (M^6 + J^2Q^2 + M^2J^2)(\delta Q)^2 - 2JM^2Q\delta J\delta Q] + O(\epsilon), \end{aligned} \tag{83}$$

where we have used $\delta M = \Omega_H \delta J + \Phi_H \delta Q$ (see (58)) to eliminate δM from the expression, and where we have defined

$$\epsilon = \frac{r_+}{M} - 1 = \frac{\sqrt{M^2 - Q^2 - \left(\frac{J}{M}\right)^2}}{M} \tag{84}$$

so $\epsilon \to 0$ in the extremal limit. A direct calculation[9] then shows that this is precisely what is needed to keep (1) satisfied to second-order, thereby establishing that gedanken experiments of the Hubeny type can never succeed in over-charging or over-spinning the black hole.

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