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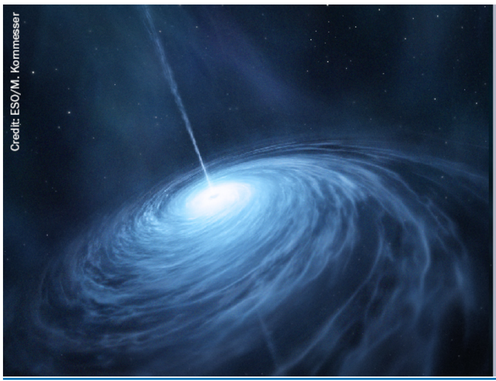

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Dissipative cosmology

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Abstract. A review of dissipative cosmology supports the application of the full causal theory of bulk viscosity. It is shown that truncated versions of this theory lead, under many conditions, to pathological behaviour of the temperature. The thermodynamical consistency of viscous fluid inflation is discussed. Dissipative de Sitter inflationary solutions are given, in which the viscosity is lower than for the non-causal or truncated solutions, and the inflation rate is different. It is possible, in principle, to generate the right amount of entropy without re-heating, given an *inflation of about 60 e-folds at a rate of about* 6×10^{33} s^{-1}.

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1. Introduction

The conventional theory of the evolution of the universe includes a number of dissipative processes, as it must if the current large value of the entropy per baryon is to be accounted for. Some of these processes may be understood by conventional physics—such as the decoupling of neutrinos during the radiation era, or of radiation and matter during the recombination era. Other processes involve the partly speculative physics of the very early universe—such as entropy production via string creation or GUT phase transitions (see [1–3] for further discussions).

It is clearly important that each dissipative process is subject to as detailed an analysis as possible. However, it is also important to develop a robust model of dissipative cosmological processes in general, so that one can analyse the overall dynamics of dissipation without getting lost in the details of particular complex processes.

What are the requirements of such a model? I believe that the model should: (i) be causal and stable, and (ii) provide a consistent relativistic thermodynamics in the 'conventional' post-inflationary regime. In the absence of any well founded theory of non-equilibrium thermodynamics at very high energies or far from equilibrium, the best current option appears to be to apply standard relativistic non-equilibrium thermodynamics in and beyond its own range. This option is not as straightforward as it may sound, since there are difficulties and subtleties involved in standard relativistic thermodynamics. A brief review is given, in order to situate this paper theoretically.

1.1. Relativistic non-equilibrium thermodynamics

The first candidate for a relativistic theory of non-equilibrium thermodynamics was developed rather late—by Eckart in 1940 [4]. (Subsequently an essentially equivalent formulation was given by Landau and Lifshitz [5].) However, after a surprisingly long time, it emerged that the Eckart theory is *not* in fact a satisfactory relativistic theory. The theory is non-causal [6, 7] (since it admits dissipative signals with superluminal velocities), and all its equilibrium states are unstable [8].

The problem arises from the *first-order* nature of the theory, i.e. it considers only first-order deviations from equilibrium. The neglected second-order terms are, in fact, a necessary requirement to prevent non-causal and unstable behaviour. These terms transform the equations governing dissipative quantities from the algebraic first-order type into differential evolution equations. A key feature of the second-order theory is that the equilibrium and dissipative variables are considered on the same footing, so that the theory is well suited to dealing with non-stationary processes, such as would occur in the early universe.

A relativistic second-order theory was found by Israel [7], independently of the non-relativistic second-order theory found earlier by Muller [6], and subsequently developed by Israel and Stewart [9] and by Pavon, Jou and co-workers [10] into what is often called 'transient' or 'extended' irreversible thermodynamics. (See [10, 11] for reviews.) The (Muller–)Israel–Stewart second-order theory was shown by Hiscock and Lindblom [12, 13] to be causal and stable under reasonable conditions. (Alternative theories have been given by Carter [14] and Geroch and Lindblom [15]. The former is shown in [16] to give equivalent results to the Israel–Stewart theory near equilibrium. The latter provides a general framework for non-equilibrium theories, rather than specific evolution equations for the dissipative quantities, although the Muller–Ruggeri theory [11] is a special case which does, and which agrees with the Israel–Stewart results near equilibrium.)

A further subtlety arises within the causal second-order theories. As emphasized by Hiscock and co-workers [12, 17], most versions of the causal evolution equations omit certain divergence terms. The resulting 'truncated' equations can give rise to very different behaviour compared to the full equations. A striking example is given in [17], which considers a Boltzmann gas in an FRW universe. The truncated causal thermodynamics of bulk viscosity leads to pathological behaviour in the late universe (as does the non-causal theory), while the solutions of the full causal theory are well behaved for all times.

In fact, it is possible to reconcile the truncated and full versions, at least near enough to equilibrium, in the following sense [18]. In the full causal theory, the temperature and pressure are taken to be those of a fiducial local equilibrium state, and may be defined via a Gibbs equation of equilibrium form. If instead one uses a non-equilibrium temperature and pressure [10], defined via a modified Gibbs equation (equation (10) below) in which dissipation enters explicitly, then the truncated form of the equation agrees with the non-truncated form (up to second order in dissipative quantities) [18]. Although it can be argued [10, 19] that the generalized quantities are measurable in laboratory experiments, it is not clear how the non-equilibrium temperature and pressure are to be defined for a cosmological fluid, given that even their equilibrium counterparts are often not known. Thus the equivalence does not seem to be of much practical importance in cosmology.

It therefore appears that the best currently available theory for analysing dissipative processes in the universe is the full (i.e. non-truncated) causal thermodynamics of Israel and Stewart. This paper is concerned with a specific aspect: the dynamics of bulk viscosity in cosmology, and in particular viscosity-driven inflation. A critical review of results in this field follows, putting the results of this paper in context.

1.2. Viscous FRW cosmology and inflation

Most of the results on bulk viscous FRW cosmologies are based on the non-causal and unstable first-order thermodynamics (see [3] for a review). One obvious reason for this is the far greater complexity of the causal equations. A less obvious and often unstated reason is the belief that the unstable, non-causal behaviour occurs only in unphysical regimes, so that the first-order theory is reasonable provided one excludes unphysical extremes. However, the results of [17] overturn this belief in the arena of cosmology. (See also [10] for non-relativistic examples, including some with experimental verification.)

Furthermore, one can argue that the neglect of the transient time-derivative term in the evolution equation for bulk viscosity, however small this term may be in particular regimes, fundamentally alters the nature of the equation, its properties and its solutions. This is dramatically illustrated by the analysis in [8], and the solutions in [17, 20–29]. In the extreme conditions of the early universe, significant differences are introduced by the transient terms.

Few studies have used the causal relativistic thermodynamics of second order, although recently the trend has been changing. The pioneer paper was by Belinskii *et al* [20]. This paper and most others used the *truncated* causal theory [21–27]. I am only aware of three papers [17, 28, 29] which have considered the full causal equation for bulk viscosity (equation (4) below).

Hiscock and Salmonson [17] point out that for a dissipative Boltzmann gas the relativistic thermodynamic coefficients and equations of state are known for all temperatures. (This is also true for a radiative fluid [30].) It is therefore possible to develop a dissipative cosmological model which is thermodynamically sound, rather than resting on *ad hoc* assumptions. One of the features of this model with the full causal thermodynamics is that inflationary solutions appear not to arise. Hiscock and Salmonson then conjecture that the inflationary solutions of the truncated causal theory [21, 25, 26] (and by implication the non-causal inflationary solutions [3]) are spurious and an artificial consequence of truncation.

In section 2 the full causal theory is briefly reviewed in a general spacetime. A rough condition for viscous expansion to be non-thermalizing is given (equation (11)), which leads to limits on the Hubble rate (equation (19)). It is shown why in many cases (i.e. not just for an FRW Boltzmann gas) the truncated equation can lead to pathological behaviour. Essentially the temperature can rise with expansion (equation (8)), so that truncated models can lead to heating of the late universe, or rapidly increasing temperature in the early universe. Unless one is able to model the thermodynamics of the cosmological fluid in terms of the generalized non-equilibrium temperature and pressure, it seems necessary to retain the divergence term for physically reasonable models, as argued by Hiscock and Salmonson.

However, their conjecture about inflationary solutions does not hold without further conditions. As shown by Zakari and Jou [28] (and supported by the results of [29]), it is possible in principle to get inflationary solutions, on the basis of *ad hoc* power-law equations of state for the thermodynamical variables. (Although some of the equations in [28] are incorrect, the conclusion is correct.)

The conjecture that viscous inflation is spurious may be correct, since inflationary expansion could imply that the dynamics of the fluid evolves significantly on time-scales that are short compared to the microscopic timescale inherent in the hydrodynamic description that underlies the causal thermodynamics [9, 10]. Certainly inflation is not the main application of causal thermodynamics, but it is of interest. This paper *assumes* that the causal thermodynamics is applicable during inflation, and investigates the consequences of that assumption. One immediate problematic consequence is that the viscous stress is greater than the equilibrium pressure (equation (22)), so that the solutions are not close to equilibrium. *The assumption that causal thermodynamics holds for inflation therefore incorporates an assumption that the theory holds far from equilibrium.* (The same point obviously applies to the non-causal theory.)

In section 3, viscous de Sitter inflation is analysed. It is shown that solutions in the truncated causal theory reduce to solutions of the non-causal theory, while in the full causal theory, the effective viscosity is lower, leading to a *different* inflation rate (equation (24)). This rate is shown to be consistent with the non-thermalizing condition, provided the local equilibrium state is not too close to the non-relativistic regime (equation (27)). The truncated theory is *inconsistent* with the condition.

Entropy production due to causal viscous inflation is derived (equation (28)). The generally accepted value of total entropy may be generated (without re-heating) with about 60 e-folds and inflation rate of about $6 \times 10^{33}\ \mathrm{s}^{-1}$. Finally, it is shown that the full and truncated inflationary solutions are stable for the same range of viscosity (equation (31)).

It must be emphasized that the detailed behaviour of a relativistic dissipative fluid depends strongly on the thermodynamic coefficients and equations of state, which are generally inadequately known [17]. There is insufficient thermodynamical basis for some of the equations of state that are used here and in [28, 29], especially in the conditions of the inflationary universe. At best, these *ad hoc* equations indicate the limits of possible behaviour for restricted (and often unknown) ranges of temperature. Furthermore, a nonlinear generalization of the viscous stress law (equation (4)) may be necessary since inflationary evolution is far from equilibrium. The conclusion is that further work is necessary to settle the question: *do there exist physically reasonable models in which inflation is driven by bulk viscosity?*

2. Bulk viscous cosmologies in general

For a fluid without shear viscosity or heat flow, the energy–momentum tensor, number 4-flux and entropy 4-flux (with Boltzmann's constant $k = 1$) take the form [17, 28]

$$T_{ab} = \rho u_a u_b + (p + \Pi) h_{ab} \qquad N^a = n u^a \qquad S^a = s N^a - \left(\frac{\tau \Pi^2}{2\xi T}\right) u^a \tag{1}$$

where u^a is the particle-frame 4-velocity, ρ the energy density, p the equilibrium pressure, Π the bulk viscous stress, $h_{ab} = g_{ab} + u_a u_b$ the projection tensor, s the specific entropy, n the number density, $T \geqslant 0$ the temperature, $\xi \geqslant 0$ the bulk viscosity coefficient, and $\tau \geqslant 0$ a relaxation coefficient for transient bulk viscous effects. Bulk viscosity arises typically in mixtures (e.g. low- and high-energy particles in a Boltzmann gas, radiation and matter, massive gauge bosons and ultra-relativistic particles [31]), and the viscous stress Π represents the dissipation from microscopic heat flow due to different cooling rates in the mixture [30]. The causal second-order theory is distinguished from the non-causal first-order theory ($\tau = 0$) by the fact that the dissipative quantity Π is an independent variable, on the same footing as the local equilibrium quantities p, s and T.

The fundamental thermodynamic tensors (1) are subject to the dynamical laws of energy–momentum conservation, number conservation and the Gibb's equation:

$$T^{ab}{}_{;b} = 0 \qquad N^a{}_{;a} = 0 \qquad T ds = d\left(\frac{\rho}{n}\right) + p d\left(\frac{1}{n}\right). \tag{2}$$

Then equations (1) and (2) imply

$$TS^a{}_{;a} = -\Pi\left[3H + \frac{\tau}{\xi}\dot{\Pi} + \tfrac{1}{2}T\Pi\left(\frac{\tau}{\xi T}u^a\right)_{;a}\right] \tag{3}$$

where $3H = u^a{}_{;a}$ is the rate of volume expansion (H reduces to the Hubble rate in FRW spacetime). Then H defines a comoving length scale R by $H = \dot{R}/R$ (R reduces to the cosmic scale factor in FRW spacetime). By equations (2) and (3), the simplest way (linear in Π) to satisfy the H-theorem $S^a{}_{;a} \geqslant 0$ leads to the causal evolution equation for bulk viscosity [12, 17, 28]:

$$\Pi + \tau\dot{\Pi} = -3\xi H - \tfrac{\epsilon}{2}\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right) \tag{4}$$

(which corrects equations (4) and (24) in [28]). In equation (4), $\epsilon = 0$ gives the truncated theory, while the full theory has $\epsilon = 1$. The non-causal theory has $\tau = 0$. The viscosity *coefficient ξ determines the magnitude of viscous stress relative to expansion in the limit* $\tau \to 0$. For steady non-equilibrium states, (4) reduces to

$$\Pi = -3\xi_{eff}H \qquad \xi_{eff} \equiv \frac{\xi}{1 + \frac{3}{2}\epsilon\tau H}. \tag{5}$$

Thus *for steady states, the truncated theory reduces to the non-causal theory, while the full theory has a different effective viscosity coefficient.*

From equations (3) and (4), the rate of entropy production is

$$S^a{}_{;a} = \frac{\Pi^2}{\xi T}$$

which leads, together with (1*c*), (2*b*) and (4), to the rate of increase of specific entropy

$$\dot{s} = -\frac{3\Pi H}{Tn}. \tag{6}$$

The total entropy in a comoving volume is $\Sigma = snR^3$. From equation (6) it follows that the growth of total comoving entropy over a proper time interval $t_0 \to t_1$ is

$$\Sigma_1 - \Sigma_0 = -\frac{3}{k}\int_{t_0}^{t_1}\left(\frac{\Pi R^3 H}{T}\right)dt \tag{7}$$

where $nR^3 =$ constant (from equation (2*b*)) has been used, and Boltzmann's constant k has been restored.

The truncated theory implies a drastic condition on the temperature, if one follows the usual simple linear model for satisfying the H-theorem that underlies the deduction of (4) from (3). Setting the bracketed terms on the right of (4) to zero, one gets

$$T = (\text{constant})\frac{\tau}{\xi}R^3. \tag{8}$$

This temperature law is implicitly imposed on the truncated version of standard causal thermodynamics, in any spacetime, and appears not to have been pointed out previously. It is clear that, in general, when the thermodynamic coefficient τ/ξ is an independently determined function of T and ρ, this law cannot be satisfied. In particular, the law (8) implies that *in an expanding fluid spacetime, the temperature rises if τ/ξ does not decrease at least as fast as the volume increases.*

Although it is conceivable that viscous heating could overcome the cooling due to expansion for short periods, in many cases (8) leads to sustained heating in violation of the physical conditions. For a spatially flat FRW Boltzmann gas, where τ/ξ is a very complicated function of T, the numerical integrations of [17] show that (8) forces T to rise as $R \to \infty$, leading to the unphysical heating of the late universe. The phenomenological model

$$\tau = \frac{\xi}{\rho} \tag{9}$$

has been adopted as one way of ensuring that viscous signals do not exceed the speed of light in the truncated theory [20, 25–29]. With equation (9), the law (8) imposes the unphysical behaviour $T \propto R^3/\rho$. This means that temperature rises during any expansion phase—rapidly in the late universe, very rapidly during inflation.

In order to avoid the problems of (8), one should employ the full, non-truncated form (4), which has the additional appeal of directly linking the temperature to the viscosity. The alternative is to use the generalized non-equilibrium temperature T_* and pressure p_*. The variables T, p and s characterize the fiducial local equilibrium state. One may introduce T_*, p_* and s_* as follows [10, 18, 28]: the non-equilibrium specific entropy is given by

$$S^a = s_* N^a \quad \Rightarrow \quad s_* = s - \frac{\tau \Pi^2}{2\xi T n}$$

where the implication follows on comparison with (1c). Then a generalized Gibbs equation is postulated:

$$\mathrm{d}s_* = T_*^{-1}\,\mathrm{d}u + T_*^{-1} p_*\,\mathrm{d}v - \frac{\tau}{\xi T n}\Pi\,\mathrm{d}\Pi \tag{10}$$

where u and v are the specific internal energy and specific volume. From equation (10) it follows that

$$T_*^{-1} = \left(\frac{\partial s_*}{\partial u}\right)_{v,\Pi} = T^{-1} - \tfrac{1}{2}\Pi^2 \frac{\partial}{\partial u}\left(\frac{\tau v}{\xi T}\right)$$

$$T_*^{-1} p_* = \left(\frac{\partial s_*}{\partial v}\right)_{u,\Pi} = T^{-1} p - \tfrac{1}{2}\Pi^2 \frac{\partial}{\partial v}\left(\frac{\tau v}{\xi T}\right).$$

Then $p_* + \Pi_* = p + \Pi$ and Π_* satisfies to second order an equation of form (4) with $\epsilon = 0$ [18], thus avoiding the constraint (8)—provided one knows T_* and p_*.

Dissipative expansion is non-thermalizing. This requires that the rate Γ for some interaction which is crucial to maintaining equilibrium remains lower than the expansion rate H, so that the fluid is unable to adjust sufficiently rapidly to the changes in temperature induced by expansion [1]. In order to get a rough idea of the implication of this condition for viscous cosmology, we assume that the crucial interaction rate of the viscous fluid is determined by the characteristic time τ. This leads to

$$\Gamma \sim \tau^{-1} < H \tag{11}$$

which is a rough consistency condition on causal viscous cosmologies.

The self-gravitating viscous fluid obeys Einstein's field equations (using units such that $8\pi G = 1 = c$):

$$G_{ab}u^a u^b = \rho \qquad G_{cd}h^c{}_a h^d{}_b = (p + \Pi)h_{ab} \qquad G_{cd}u^c h^d{}_a = 0 \tag{12}$$

where G_{ab} is the Einstein tensor. These equations imply the conservation equations (2a), which are

$$\dot\rho + 3(\rho + p + \Pi)H = 0 \qquad (\rho + p + \Pi)\dot u_a + (p + \Pi)_{,b}\,h^b{}_a = 0\,. \tag{13}$$

Thus the bulk viscous cosmology satisfies (4) and (12) (where (13) may be used to replace some of (12)). The system must be closed by equations of state for p and T, and by specifying the thermodynamic coefficients ξ and τ. These are known in closed form from kinetic theory for a Boltzmann gas [17] and a radiative gas [30]. For more general dissipative fluids, the information is at best incomplete. It is standard to assume [3, 20–29] the following *ad hoc* laws: equation (9) for τ; the linear barotropic equation of state

$$p = (\gamma - 1)\rho \tag{14}$$

for p; and

$$\xi = \alpha\rho^q \tag{15}$$

where γ ($1 \leqslant \gamma \leqslant 2$), α ($\geqslant 0$) and q ($\geqslant 0$) are constants. As discussed in section 1, these are without sufficient thermodynamical motivation, but in the absence of better alternatives, I will follow the practice of adopting them in the hope that they will at least provide an indication of the range of possibilities. Similar comments apply to the temperature law [28, 29]

$$T = \beta\rho^r \tag{16}$$

where β (> 0) and r ($\geqslant 0$) are constants, which is the simplest law guaranteeing positive heat capacity [29].

From now on, the metric is taken as that of a spatially flat FRW universe

$$\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = -\mathrm{d}t^2 + R(t)^2[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2]$$

for which (12) reduce to

$$\rho = 3H^2 \qquad p + \Pi = -2\dot{H} - 3H^2 \tag{17}$$

while (13b) is identically satisfied. With equations (14), (12a) become

$$\dot{\rho} + 3(\gamma\rho + \Pi)H = 0\,. \tag{18}$$

By equations (9), (15) and (17), the condition (11) for dissipative expansion implies

$$H^{2q-1} > \frac{3^{1-q}}{\alpha}\,. \tag{19}$$

For $0 \leqslant q < \frac{1}{2}$, (19) imposes an *upper limit* on the Hubble rate, while for $q > \frac{1}{2}$ it imposes a *lower limit*. For $q = \frac{1}{2}$, (19) gives a lower limit on the viscosity constant α. Furthermore, $q = \frac{1}{2}$ implies

$$\xi \propto H \qquad \tau \propto H^{-1}$$

so that the viscosity is determined by the expansion rate, while the relaxation time is determined by the expansion time-scale. In the conditions of the very early universe, where the influence of gravity over matter is strongest, this 'gravito-thermodynamic' law may be a useful approximation if the thermodynamics are unknown.

For general ξ, τ and T but linear barotropic pressure (14), the fundamental dynamical equation for the Hubble rate is given by (4), (17) and (18):

$$\tau\ddot{H} + \tfrac{3}{2}\tau(2\gamma + \epsilon)H\dot{H} + \dot{H} + \tfrac{1}{2}\epsilon\tau\left(\frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right)\dot{H} + \tfrac{9}{4}\epsilon\gamma H^3$$
$$+\tfrac{3}{4}\epsilon\gamma\tau\left(\frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right)H^2 + \tfrac{3}{2}\gamma H^2 - \tfrac{3}{2}\xi H = 0\,. \tag{20}$$

When $\tau = 0$, equation (20) reduces to the non-causal equation (first order in H) [3], while for $\epsilon = 0$ it reduces to the truncated causal equation [20–27]. If one further assumes (9), (15) and (16), then (20) becomes

$$\ddot{H} + \tfrac{3}{2}[\epsilon + (2 - \epsilon - \epsilon r)\gamma]H\dot{H} + 3^{1-q}\alpha^{-1}H^{2-2q}\dot{H}$$
$$-\epsilon(1+r)H^{-1}\dot{H}^2 + \tfrac{9}{4}(\epsilon\gamma - 2)H^3 + \tfrac{1}{2}3^{2-q}\alpha^{-1}\gamma H^{4-2q} = 0\,. \qquad (21)$$

which corrects equation (39) of [28].

3. Causal viscous inflation

Dissipative inflation arises in an 'ordinary' fluid (i.e. with non-negative equilibrium pressure, as opposed to a scalar field), when the viscous stress becomes large enough to create sufficient negative effective pressure ($p_{eff} = p + \Pi$). The condition for inflation, $\ddot{R} > 0$, leads by (17) to

$$-\Pi > p + \tfrac{1}{3}\rho\,. \qquad (22)$$

Thus the viscous stress is greater than the equilibrium pressure. It follows that the fluid is *far from equilibrium*, and one must postulate that the causal thermodynamics holds beyond the near-equilibrium regime assumed in its derivation.

In the case of de Sitter inflation ($\dot{H} = 0$), (17) implies $\Pi = -p - \rho$, which clearly satisfies (22). Suppose that the local equilibrium pressure obeys a barotropic equation of state, i.e. $p = p(\rho)$. It follows from (17) that for de Sitter inflation, ρ, p and Π are constant. The latter implies, by (4): *for a barotropic equation of state $p = p(\rho)$, de Sitter inflationary solutions in the truncated causal theory reduce to solutions of the non-causal theory, while the solutions of the full causal theory have different viscosity.*

If the laws (9) and (14)–(16) are adopted, then all thermodynamic quantities are constant:

$$\begin{array}{lll} \rho = 3H_0^2 & p = 3(\gamma - 1)H_0^2 & \Pi = -3\gamma H_0^2 \\ \xi = 3^q\alpha H_0^{2q} & \tau = 3^{q-1}\alpha H_0^{2q-2} & T = T_0\,. \end{array} \qquad (23)$$

If the non-thermalizing condition (11) is assumed, then (5) shows that ξ_{eff} is at most 40% of ξ. This leads to a change in the inflation rate H_0: by (21)

$$q \neq \tfrac{1}{2} \;\Rightarrow\; H_0 = \left[3^q\alpha\left(\frac{2-\epsilon\gamma}{2\gamma}\right)\right]^{1/(1-2q)} \qquad (24)$$

$$q = \tfrac{1}{2} \;\Rightarrow\; H_0 \text{ arbitrary} \qquad \alpha = \tfrac{1}{3}\sqrt{3}\left(\frac{2\gamma}{2-\epsilon\gamma}\right). \qquad (25)$$

It follows that in the full theory ($\epsilon = 1$), $\gamma \neq 2$, so that *there is no stiff-fluid viscous inflation*, unlike in the truncated or non-causal theories.

The form of H_0 in (24) corrects the statement in [28] that the inflation rate is the same as in the truncated theory. For $0 \leqslant q < \frac{1}{2}$, H_0 is *lower* than in the truncated or non-causal solutions; for $q > \frac{1}{2}$, H_0 is *greater*. For $q = \frac{1}{2}$, (25) shows that the viscosity constant α for full-theory inflation must be *greater* than in the truncated or non-causal solutions. For all q, equations (23)–(25) imply

$$\tau^{-1} = 3\left(\frac{2-\epsilon\gamma}{2\gamma}\right)H_0 \qquad (26)$$

so that the condition (11) for vsicous expansion to be non-thermalizing requires

$$\gamma > \frac{6}{2+3\epsilon}\,. \qquad (27)$$

For the full theory, this implies $\gamma > \frac{6}{5}$, so that the local equilibrium state must be hot enough to be at least mildly relativistic. The condition (27) cannot be met in the truncated theory (unless (9) is modified).

The constant temperature implied by (16) means that viscous heating balances the cooling due to inflation. This heating produces entropy *during* inflation, so that any re-heating at the close of inflation does not need to generate *all* the entropy as in standard (non-dissipative) inflation. In principle, causal viscous inflation can generate all the necessary entropy without re-heating. In the simplified models of this paper, inflation is assumed to start and end instantaneously, without re-heating. The physical processes underlying the onset of inflation, exit from inflation and possible associated re-heating effects, are not considered (see, for example, [31] for further discussion).

Taking t_0 as the time that inflation begins and t_1 as the time of exit to the radiation era, (7) gives the increase in total entropy in the comoving volume $R(t)^3$ as

$$\Sigma_1 - \Sigma_0 = \left(\frac{3c^2\gamma}{8\pi Gk}\right)\left(\frac{R_0^3 H_0^2}{T_0}\right)\left[e^{3H_0(t_1-t_0)} - 1\right] \tag{28}$$

where c and G have been restored and (14), (17) have been used. Typical parameters of inflation [1,31] may be taken as $t_0 \approx 10^{-35}$ s and $t_1 \approx 10^{-32}$ s, so that 60 e-folds require $H_0 \approx 6 \times 10^{33}$ s^{-1}; $R_0 \approx ct_0$; $T_0 \approx 10^{28}$ K. Assuming that the universe is dominated by ultra-relativistic particles before and after inflation (as is the case for a GUT phase transition [31]), $\gamma = \frac{4}{3}$. The total entropy in the observable universe that inflated from R_0 is then approximately Σ_1, assuming that nearly all entropy production occurs via inflation. Since $\Sigma_0 \ll \Sigma_1$, equation (28) gives

$$\Sigma_1 \approx 2.2 \times 10^{88} \tag{29}$$

which is in agreement with the accepted value [1], i.e. *the full causal thermodynamics of bulk viscosity in principle accounts for the generally accepted entropy production via dissipative inflation, without re-heating.*

For the above value of H_0 and $\gamma = \frac{4}{3}$, (23) and (26) give

$$\rho \approx 5.8 \times 10^{94}\ \text{erg cm}^{-3} \qquad p \approx 1.9 \times 10^{94}\ \text{erg cm}^{-3}$$
$$\Pi \approx -7.7 \times 10^{94}\ \text{erg cm}^{-3} \qquad \tau \approx 2.2 \times 10^{-34}\ \text{s}$$

while equation (5) gives

$$\xi \approx 1.3 \times 10^{61}\ \text{erg s cm}^{-3}\,.$$

Finally, consider the stability of the causal inflationary solutions within spatially flat FRW spacetime. Suppose that the solution $H = H_0$ of (17) is perturbed

$$H = H_0[1 + h(t)] \qquad |h| \ll 1$$

keeping the thermodynamic relations (9), (14)–(16) unchanged, so that to first order

$$\rho \approx \rho_0(1+2h) \qquad p \approx p_0(1+2h) \qquad \Pi \approx \Pi_0(1+2h) - 2H_0\dot{h}$$
$$\xi \approx \xi_0(1+2qh) \qquad \tau \approx \tau_0[1+2(q-1)h] \qquad T \approx T_0(1+2rh)\,.$$

Linearizing (21) and using (23) and (24):

$$\ddot{h} + \left(\frac{3H_0}{2\gamma}\right)\left[2 + (2-\epsilon-\epsilon r)\gamma^2\right]\dot{h} + \left[\tfrac{9}{4}H_0^2(\epsilon\gamma-2)(2q-1)\right]h \approx 0\,. \tag{30}$$

The solution of (30) is

$$h \approx h_+ e^{\lambda_+ t} + h_- e^{\lambda_- t}$$

where $h_\pm$ are small constants and

$$\lambda_\pm = \frac{3H_0}{4\gamma}\left[-\Lambda \pm \sqrt{\Lambda^2 - 4\gamma^2(2-\epsilon\gamma)(1-2q)}\right] \qquad \Lambda \equiv 2 + (2-\epsilon-\epsilon r)\gamma^2 .$$

Stability requires that the real parts of $\lambda_\pm$ are negative. For the truncated theory, this gives

$$0 \leqslant q \leqslant \tfrac{1}{2} . \tag{31}$$

The full theory has the same stability condition (31)—provided also that

$$r \leqslant \tfrac{3}{2} \tag{32}$$

to ensure $\Lambda > 0$ for all $\gamma < 2$. In the very early universe, the local equilibrium temperature would usually be taken as the radiation temperature, for which $r = \frac{1}{4}$, so that (32) would readily be satisfied. These results are consistent with [29], which shows that if the de Sitter solution in Bianchi I spacetime is asymptotically stable, then (31) holds for both the full and truncated theories.

4. Concluding remarks

This paper has presented a case for using the full Israel–Stewart causal theory of bulk viscosity in cosmology, based on a critical review of previous results, and further supported by the result (8), which shows that truncated versions of the theory can often lead to pathological temperature evolution. The theoretical consistency of viscous inflation was discussed, including the point that such expansion must be far from equilibrium. This may necessitate a non-linear generalization of (4), or indicate that dissipative fluid inflation is inconsistent.

Assuming that the causal theory may be applied to inflation, it was shown that de Sitter inflation in the truncated theory reduces to non-causal de Sitter inflation and fails to satisfy the non-thermalizing condition (11). The non-truncated solutions do satisfy this condition, due to their different inflation rate (24). In principle they can account for the total entropy production, without the need for re-heating.

Further work is in progress to determine how the full theory affects the non-causal [32, 33] and truncated [34] results on entropy production in the radiation and recombination eras, where the possible problems associated with inflation do not arise.

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