Huygens' Principle and Hadamard's Conjecture

Paul Günther

In this article I shall try to report on a problem connected with hyperbolic differential equations that was posed by Jacques Hadamard in his 1923 Yale Lectures ([11], p. 236). In spite of its age, the problem is still far from being completely solved.

The Wave Equation

Let us begin with a discussion of the wave equation for one unknown function u of $m = n + 1$ independent variables. If f denotes a given real function and c a positive number, then the operator $\square_m$ and the second-order differential equation (e_m) are defined as follows:

$$\square_m u := \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \cdots - \frac{\partial^2 u}{\partial x_n{}^2} = f. \qquad (e_m)$$

The variable t represents the time and $(x_1, \ldots, x_n)$ determines a point x of $\mathbf{R}^n$; consequently, we can think of c as having the dimensions of a velocity (length/time). We call each pair $(t,x) \in \mathbf{R}^m$ an *event* and interpret the line segment between two events (t,x), (τ,y) with $\tau > t$ as the straight, uniform movement of a particle; its velocity v is given by

$$v = r(x,y)/(\tau - t),\ r(x,y) = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2}.$$

The reader may think of a graphic time table with $m = 2$, say, for Phileas Fogg during his 80-day voyage ([20], Chap. 7); see Figure 1. The open set $D_+(t,x) \subseteq \mathbf{R}^m$ of those events that can be reached from (t,x) with a velocity $v < c$ is called the *future* of (t,x); on the other hand, the *past* $D_-(t,x)$ is the set of those events (τ,y) from which (t,x) can be reached with a velocity $v < c$; in other words, $(\tau,y) \in D_+(t,x)$ if and only if $(t,x) \in D_-(\tau,y)$. The boundary of $D_+(t,x)$ (resp., $D_-(t,x)$) is called the *forward* (resp., *backward*) *characteristic half cone* $C_+(t,x)$ (resp., $C_-(t,x)$). Using the function

$$\Gamma(t,x;\ \tau,y) := c^2(t - \tau)^2 - r^2(x,y) \qquad (1)$$

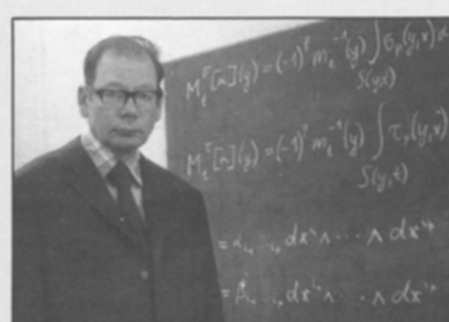
we have

$$D_+(t,x) \cup D_-(t,x) = \{(\tau,y) | \Gamma(t,x;\ \tau,y) > 0\}, \qquad (2)$$

$$C_+(t,x) \cup C_-(t,x) = \{(\tau,y) | \Gamma(t,x;\ \tau,y) = 0\}. \qquad (3)$$

The double cone $C_+(t,x) \cup C_-(t,x)$ looks like an hour glass—an old symbol of transitoriness; see Figure 2.

Paul Günther



Paul Günther is a mathematics professor at Karl-Marx-Universität in Leipzig, Federal Republic of Germany. He is also a member of the *Deutsche Akademie der Naturforscher Leopoldina*.

In order to form an idea about what kind of physical processes are described by equation (e_4), we consider sound waves in the air. Sound waves are small but fast variations of pressure p and density ρ. They are generated by certain sound sources, e.g., a musical instrument, and propagate into the still air. For the density we put $\rho(t,x) = \rho_0(1 + \epsilon u(t,x))$ with a small parameter ϵ and a constant ρ_0, which gives the density for the state of rest. We assume that the pressure is a unique function of the density, $p = \psi(\rho)$. From Euler's equations and the equation of continuity one finds that u satisfies (e_4) with $c^2 = (d\psi/d\rho)(\rho_0)$. If at the event (t,x) none of the sound sources acts, then we have $f(t,x) = 0$; otherwise we have, generally, $f(t,x) \neq 0$. In the sequel we assume (for the sake of simplicity) that f is a given C^∞-function on $\mathbf{R}^4$ having compact support, which we abbreviate by writing $f \in C_0^\infty(\mathbf{R}^4)$; the *support* of a real or complex function f defined on some manifold M is the smallest closed subset $K \subseteq M$ such that f vanishes on $M \setminus K$. One writes $K = \operatorname{supp} f$.

Equation (e_4) also occurs in the theory of electromagnetism. In a vacuum each coordinate of the electric and the magnetic fields satisfies equation (e_4) with $f = 0$ (the homogeneous equation). The number c depends on the dielectric constant and the magnetic permeability; it equals the velocity of light. If charges and currents are present, then one obtains equation (e_4) with $f \neq 0$.

Some Solutions of (e_4)

We give first a useful identity. For a fixed $x = (x_1,x_2,x_3) \in \mathbf{R}^3$ one defines the accelerated function $\{\phi\}^+$ and the retarded function $\{\phi\}^-$ of a function $\phi:\mathbf{R}^4 \to \mathbf{R}$ by the formulas

$$\{\phi\}^\pm(t,y_1,y_2,y_3) := \phi(t \pm r(x,y)/c, y_1,y_2,y_3).$$

If ϕ is differentiable and $x \neq y$, we put

$$A^i[\phi](t,y) := \frac{1}{r}\left\{\frac{\partial\phi}{\partial y_i}\right\}^-(t,y) + \frac{y_i - x_i}{r^2}\left(\frac{1}{r}\{\phi\}^-(t,y) + \frac{1}{c}\left\{\frac{\partial\phi}{\partial t}\right\}^-(t,y)\right),\ i = 1,2,3.$$

Confirmation of the following identity is a simple exercise (see also [1]):

$$\frac{1}{r}\{\Box_4\phi\}^-(t,y) = -\sum_{i=1}^{3}\frac{\partial A^i[\phi]}{\partial y_i}(t,y)\quad \forall\phi \in C^2(\mathbf{R}^4). \tag{4}$$

We integrate for fixed (t,x) over $y \in \mathbf{R}^3 \setminus K(\epsilon,x)$, where $K(\epsilon,x)$ is the ball around x with radius ϵ. Applying the divergence theorem and passing to the limit $\epsilon \to 0$, we obtain for every $\phi \in C_0^\infty(\mathbf{R}^4)$:

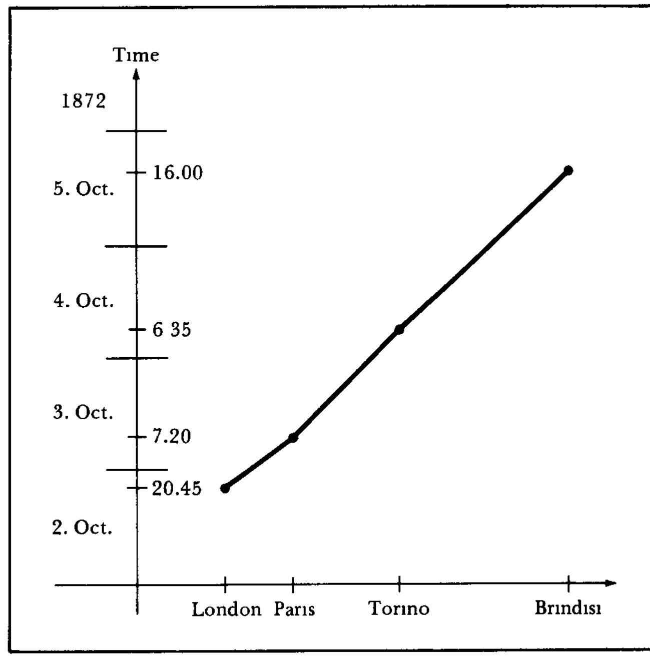


Figure 1. Graphic timetable for Mr. Fogg.

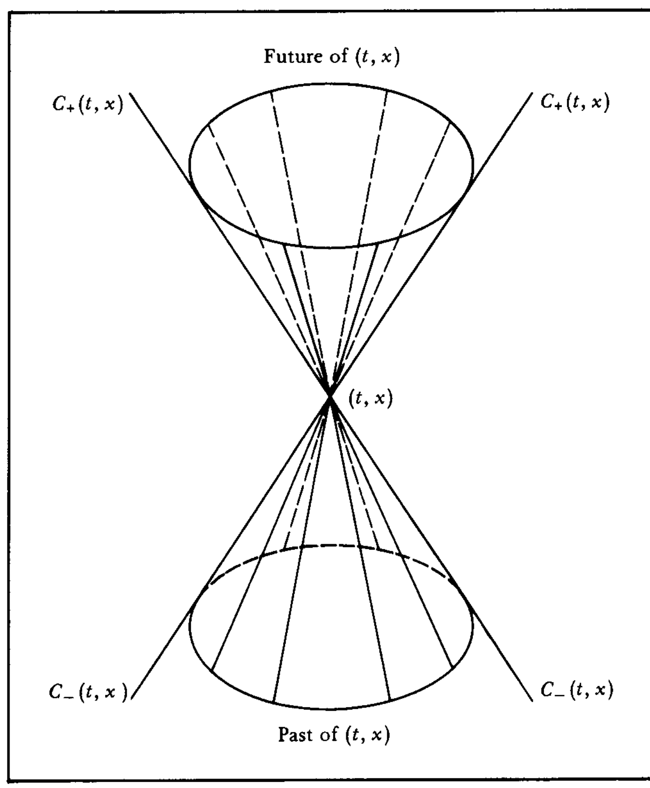


Figure 2. Structure of space-time.

$$\frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{1}{r(x,y)}\{\Box_4\phi\}^-(t,y)dy_1dy_2dy_3 = \phi(t,x). \tag{5}$$

If we equip $C_\pm(t,x)$ with the measure element $dy_1dy_2dy_3$, then we can rewrite (5) as

$$\frac{1}{4\pi}\int_{C_-(t,x)} \frac{1}{r}\,\Box_4\phi\, dy_1 dy_2 dy_3 = \phi(t,x), \tag{6}$$

and this equality is the integrated form of the identity (4). Of course, the same formula holds with $C_+(t,x)$ instead of $C_-(t,x)$.

A function ϕ defined in $\mathbf{R}^m$ is called a *forward function* (resp., *backward function*) if $\operatorname{supp}\phi \cap \overline{D}_-(t,x)$ (resp., $\operatorname{supp}\phi \cap \overline{D}_+(t,x)$) is compact for every event (t,x). (Example: If $\phi(t,x) = 0$ for all events (t,x) with $t < a$, then ϕ is a forward function.) We can now easily see that the identity (6) is valid not only for functions ϕ with compact support, but also for forward functions $\phi \in C^2(\mathbf{R}^4)$. We therefore obtain for a forward solution u_+ of (e_4) the representation

$$u_+(t,x) = \frac{1}{4\pi}\int_{C_-(t,x)} \frac{1}{r} f\, dy_1 dy_2 dy_3, \tag{7}$$

where $f \in C_0^\infty(\mathbf{R}^4)$. The analogous formula with $C_+(t,x)$ instead of $C_-(t,x)$ holds for a backward solution u_- of (e_4). On the other hand, if for a given $f \in C_0^\infty(\mathbf{R}^4)$ the function u_+ is defined by (7), then u_+ is a solution of (e_4). The proof of this assertion can be established again by means of (6), this time applied to f. Consequently, there exists exactly one forward (backward) solution of (e_4). The solution u_+ gives the sound wave that arises from a source described by f.

An event (t,x) belongs to the support of u_+ only if $C_-(t,x) \cap \operatorname{supp} f \neq \emptyset$. This can be reformulated as

$$\operatorname{supp} u_+ \subseteq \cup\, \{C_+(\tau,y) \mid (\tau,y) \in \operatorname{supp} f\}. \tag{8}$$

Let us give another interpretation of the formulas (6) and (7). For any fixed event (t,x) we associate to every function $\phi \in C_0^\infty(\mathbf{R}^4)$ the real number

$$G_+(t,x)[\phi] := \frac{1}{4\pi}\int_{C_+(t,x)} \frac{1}{r}\,\phi\, dy_1 dy_2 dy_3. \tag{9}$$

A linear mapping of a vector space $\mathcal{F}$ into $\mathbf{R}$ is usually called a linear functional on $\mathcal{F}$; equation (9) defines just such a functional $G_+(t,x)$ on $C_0^\infty(\mathbf{R}^4)$. Analogously one defines a functional $G_-(t,x)$ by (9) with $C_-(t,x)$ replacing $C_+(t,x)$. Formula (7) can now be rewritten as

$$u_\pm(t,x) = G_\mp(t,x)[f]. \tag{10}$$

One calls the functional $G_+(t,x)$ (resp., $G_-(t,x)$) the *forward* (resp., *backward*) *fundamental solution* of (e_4) belonging to (t,x). A reader who is familiar with distributions realizes at once from (9) and (6) that $G_\pm(t,x)$ are those distributions that satisfy

$$\Box_4 G_\pm(t,x) = \delta(t,x), \tag{11}$$

where $\delta(t,x)$ is the Dirac measure concentrated at the event (t,x); one can say that $G_+(t,x)$ describes an ideal sound wave, whose source works only at the event (t,x). We disregard these more technical details and say only a few words about the supports of the functionals $G_\pm(t,x)$. The relevant definition is as follows. Let M be a manifold and Φ: $C_0^\infty(M) \to \mathbf{R}$ be a functional. A point $\xi \in M$ belongs to $M \setminus \operatorname{supp}\Phi$ if there exists a neighbourhood $U(\xi)$ such that $\Phi[\phi] = 0$ for every $\phi \in C_0^\infty(M)$ with $\operatorname{supp}\phi \subseteq U(\xi)$. This definition, when applied to $G_\pm(t,x)$, leads to the statement

$$\operatorname{supp} G_+(t,x) = C_+(t,x), \tag{12}$$

which is, so to speak, the limiting case of (8) if $\operatorname{supp} f$ reduces to the single event (t,x).

Hadamard's "Minor Premise"

The general notion "Huygens' Principle" comprises several ideas, which are related to wave propagation (diffraction and interference problems included). J. Hadamard formulated some of them in a famous syllogism; in the present paper we are concerned with its "minor premise":

$(B)_+$ "If at the instant $t = t_0$, or more precisely, in the short interval $t_0 - \epsilon \leq t \leq t_0 + \epsilon$, we produce a sound disturbance localized at the immediate neighbourhood of a point x_0, the effect at the subsequent instant $t = t'$ is localized in a very thin spherical shell with center x_0 and radius $c(t' - t_0)$, where c is the velocity of sound" ([11], p. 54).

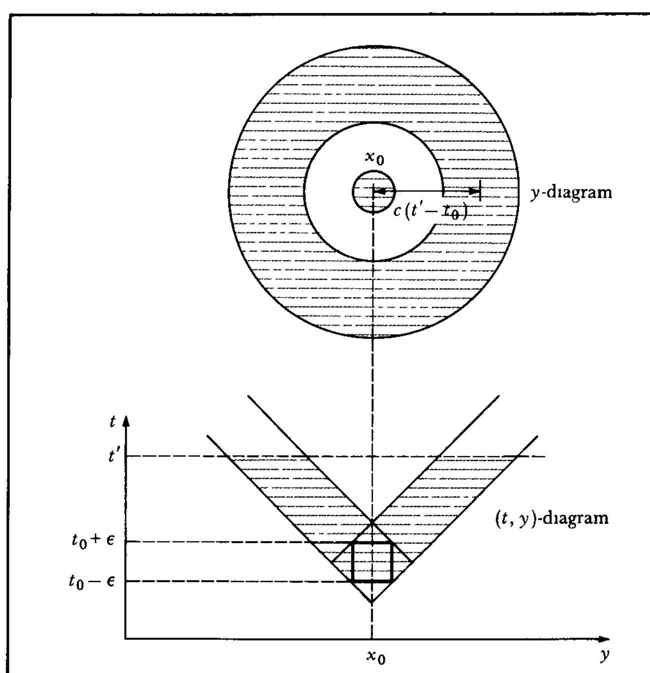


Figure 3. Hadamard's minor premise. The cross-hatched area represents supp u.

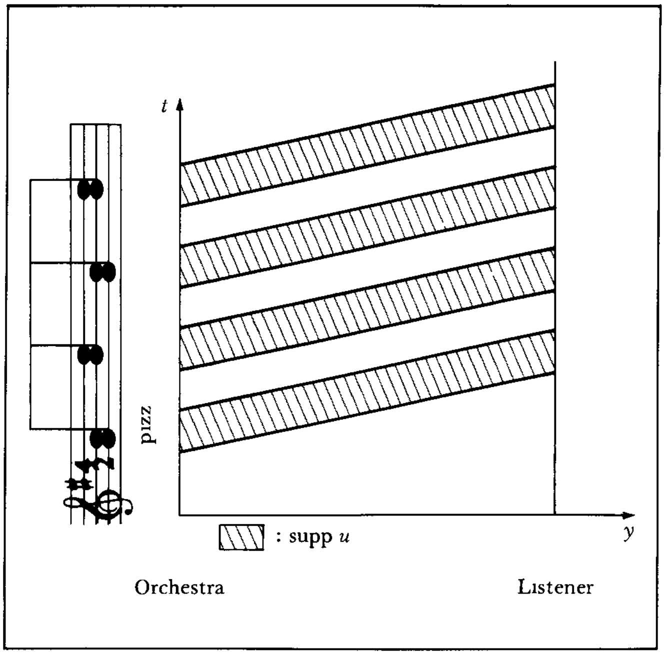


Figure 4. Music.

As a consequence of $(B)_+$ we state: if we produce at some locus several sound impulses, interrupted by proper intervals of silence, then (because of the missing aftereffects) a listener at another locus also receives separated sound impulses and not a confusing mixture. Clearly, $(B)_+$ is fundamental for a suitable acoustic signal transmission and its artistic shape, which we call music. (See Figures 3 and 4.)

Let us again consider the forward solution u_+ of (e_4) with an f whose support is a neighbourhood of the event (t_0,x_0). According to (8) supp u_+ is contained in a neighbourhood of $C_+(t_0,x_0)$. For a fixed $t' > t_0$ the support of $u_+(t', \cdot)$, considered as a function in $\mathbf{R}^3$, is thus contained in a neighbourhood of the sphere $S(c[t' - t_0], x_0)$ around x_0 with radius $c[t' - t_0]$. If supp f shrinks to (t_0, x_0), then supp $u_+(t', \cdot)$ shrinks to $S(c[t' - t_0], x_0)$. These considerations show not only the validity of $(B)_+$ for the forward solutions u_+, but also prove that the number c occurring in equation (e_4) equals the velocity of sound.

Of course, we can give a formulation $(B)_-$ of the minor premise, which is then fulfilled by the backward solutions u_- of (e_4). The validity of $(B)_\pm$ is the reason why we call $\Box_4$ a Huygens operator.

The Equation (e_3)

Let us consider equation (e_3). We shall point out that the minor premise $(B)_\pm$ is not satisfied for its forward and backward solutions; therefore $\Box_3$ is not a Huygens operator. We start from the somewhat modified formula (7) with a function f that does not depend on the third spatial variable and write

$$u_+(t,x_1,x_2) = \frac{1}{4\pi}\int_{\mathbf{R}^3}\frac{1}{\rho}f(t-\rho/c, x_1+z_1, x_2+z_2)dz_1dz_2dz_3,$$

$$\rho = \{z_1^2 + z_2^2 + z_3^2\}^{1/2}.$$

Clearly, $u_+(t,x)$ also does not depend on x_3 and, consequently, solves equation (e_3). The integrand is an even function of z_3; for $z_3 > 0$ we make the substitution

$$y_1 = x_1 + z_1, y_2 = x_2 + z_2, \tau = t - \rho/c$$

and arrive at

$$u_+(t,x) = \frac{1}{2\pi}\int_{D_-(t,x)}\frac{f(\tau,y_1,y_2)}{\sqrt{\Gamma(t,x;\tau,y)}}dy_1dy_2d\tau, \quad (13)$$

$$\Gamma(t,x;\tau,y) = c^2(t-\tau)^2 - (x_1-y_1)^2 - (x_2-y_2)^2.$$

For every $f \in C_0^\infty(\mathbf{R}^3)$ formula (13) gives the only forward solution u_+ of (e_3); replacing $D_-(t,x)$ by $D_+(t,x)$, one obtains the backward counterpart. The forward (backward) fundamental solution $G_+(t,x)$ (resp., $G_-(t,x)$) is defined for fixed (t,x) as the functional

$$C_0^\infty(\mathbf{R}^3) \ni \phi \to G_\pm(t,x)[\phi]:$$

$$= \frac{1}{2\pi}\int_{D_\pm(t,x)}\frac{\phi(\tau,y_1,y_2)}{\sqrt{\Gamma(t,x;\tau,y)}}dy_1dy_2d\tau. \quad (14)$$

We can again write

$$u_+(t,x) = G_-(t,x)[f].$$

Contrary to formula (9) the domain of integration in (14) is an open set, which suggests that the function $G_0(t,x)$ defined by

$$D_+(t,x) \cup D_-(t,x) \ni (\tau,y) \to G_0(t,x)(\tau,y)$$

$$= \frac{1}{2\pi}\Gamma(t,x;\tau,y)^{-1/2}$$

also solves equation (e_3) with respect to (τ,y). An easy calculation confirms this.

Now let us choose a function $f \in C_0^\infty(\mathbf{R}^3)$ which is positive in a neighbourhood $U_\epsilon(t,x)$ of an event (t,x) and zero elsewhere; taking

$$U_\epsilon(t,x) = \{(\tau,y) \in D_+(t-\epsilon,x) | \tau \in (t-\epsilon, t+\epsilon)\}$$

we find from (13) that

$$\text{supp } u_+ = D_+(t-\epsilon,x) \cup C_+(t-\epsilon,x).$$

The support of $u_+(t', \cdot)$ for any $t' > t$ is now given by

$$\operatorname{supp} u_+(t', \cdot) = K(c[t' - t + \epsilon], x) \subset \mathbf{R}^2.$$

In the limit $\epsilon \to 0$ this set shrinks to the ball $K(c[t' - t], x)$ and not to its boundary as demanded in $(B)_+$. We also have

$$\operatorname{supp} G_+(t,x) = D_+(t,x) \cup C_+(t,x).$$

Of course, analogous results are valid for the backward solutions. Consequently, $(B)_\pm$ is not satisfied by the solutions of (e_3). In a world with only two spatial dimensions an annoying noise produced somewhere and at any time never comes to an end. What luck that we have three dimensions at hand!

The General Hyperbolic Equation

Let us consider a second-order differential equation for one unknown function u:

$$L[u]: = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x_i}\left(\sqrt{\gamma} g^{ij} \frac{\partial u}{\partial x_j}\right) + 2A^i \frac{\partial u}{\partial x_i} + Cu = f. \quad (15)$$

The independent variables are $x = (x_0, x_1, \ldots, x_n)$, $m = n + 1$; moreover, the sum convention for dispensing with the summation symbol Σ is used. The coefficients g^{ij}, A^i, C and f are smooth functions of x. We assume that L is a hyperbolic operator; i.e., the quadratic form $g^{ij}(x)\xi_i\xi_j$ has signature $(+,-,-,\ldots,-)$ for every x. Finally, the positive number γ equals the absolute value of the determinant of the inverse matrix $(g_{ij}) = g$ of (g^{ij}). Just as in general relativity, (g_{ij}) determines a pseudo-Riemannian metric.

We restrict our considerations to certain open subsets of the x-space that are called *causal domains*. (For an exact definition, see [5], sec. 4.4.) In such a domain a satisfactory integration theory of equation (15) can be established. (See [5] or [9], Chap. IV.)

Let Ω be a fixed causal domain. We first procure a function $\Omega \times \Omega \ni (x,y) \to \Gamma(x,y) \in \mathbf{R}$ that generalizes our former Γ of formula (1). For that purpose we solve (using the Hamilton-Jacobi theory) the first-order differential equation

$$g^{ij}(x) \frac{\partial \Gamma}{\partial x_i}(x,y) \frac{\partial \Gamma}{\partial x_j}(x,y) = 4\Gamma(x,y)$$

with the initial conditions

$$\Gamma(y,y) = 0, \frac{\partial \Gamma}{\partial x_i}(y,y) = 0, \frac{\partial^2 \Gamma}{\partial x_i \partial x_j}(y,y) = 2g_{ij}(y).$$

For any fixed $x \in \Omega$ the set $\{y \in \Omega | \Gamma(x,y) > 0\}$ decomposes naturally into two open subsets of Ω; one of them we call the future $D_+(x)$ and the other one the past $D_-(x)$ of x. The characteristic half conoids $C_\pm(x)$ are defined as the boundary sets of $D_\pm(x)$, respectively.

The integration theory guarantees the existence of exactly one forward solution u_+ and exactly one backward solution u_- of equation (15) provided that $f \in C_0^\infty(\Omega)$. Both solutions are smooth; for fixed $x \in \Omega$ we write

$$u_\pm(x) = G_\mp(x)[f]$$

thus obtaining linear functionals $G_\pm(x)$ over $C_0^\infty(\Omega)$, which we call the forward and backward fundamental solutions of the differential operator L. For their supports we find

$$\operatorname{supp} u_+ \subseteq \cup\{D_+(x) \cup C_+(x) | x \in \operatorname{supp} f\},$$

$$\operatorname{supp} G_+(x) \subseteq D_+(x) \cup C_+(x).$$

There seems to be no incisive difference between the cases m even and m odd. Far from it!

Let us consider the case m even. Then each of the fundamental solutions splits into the sum of two functionals over $C_0^\infty(\Omega)$ according to

$$G_\pm(x) = G_\pm^{\text{sing}}(x) + G_\pm^{\text{reg}}(x),$$

where

$$\operatorname{supp} G_\pm^{\text{sing}}(x) = C_\pm(x)$$

and

$$G_\pm^{\text{reg}}(x)[f]: = \int_{D_\pm(x)} T(x,y) f(y) \sqrt{\gamma} dy_0 \cdots dy_n \quad (16)$$

with a smooth kernel function $T \in C^\infty(\Omega \times \Omega)$. One can say, apart from the very harmless integral (16), the fundamental solutions are concentrated on the characteristic conoids. The function T is called a "tail term" and corresponds with the "logarithmic term" in J. Hadamard's original version of the integration theory (see [11]). In the case m odd, such a decomposition with a smooth tail term is impossible; for $L = \Box_3$ we see this from (13), where the denominator of the integrand vanishes on $C_+(x)$.

Definition: Assume $x \in \Omega$, $f \in C_0^\infty(\Omega)$, and denote by $u_\pm$ the forward and backward solutions of $L[u] = f$; the operator L is called a *Huygens operator* if $u_+(x) = 0$ for every pair (x,f) with $C_-(x) \cap \operatorname{supp} f = \varnothing$ and $u_-(x) = 0$ for every pair (x,f) with $C_+(x) \cap \operatorname{supp} f = \varnothing$.

If for a Huygens operator L the function f vanishes outside a neighbourhood of $x_0 \in \Omega$, then $u_+(x)$ vanishes outside a neighbourhood of $C_+(x_0)$ in accordance with the minor premise $(B)_+$.

J. Hadamard ([11], p. 236) gave the following famous criterion.

Hadamard's criterion: The operator L is a Huygens operator, if and only if for every $x \in \Omega$ we have

$$\operatorname{supp} G_{\pm}(x) = C_{\pm}(x).$$

This is equivalent to $m \geq 4$, m even and $T(x,y) = 0$ for $x \in \Omega$, $y \in D_{\pm}(x)$.

Hadamard's Conjecture

When studying Huygens operators we must take into consideration certain simple transformations, which do not change the Huygensian character of an operator. Let L be a Huygens operator; L' is also a Huygens operator if it arises from L by one of the following transformations:

(a) Non-singular transformations of the independent variables.

(b) $L'[u] = \lambda^{-1}L[\lambda u]$ for some positive, smooth function λ in Ω (gauge transformation).

(c) $L'[u] = \sigma L[u]$ for some positive, smooth function σ in Ω (conformal transformation).

The wave operator $\Box_m$, $m \geq 4$ even, is a Huygens operator (for $m = 4$ we have seen this above). A Huygens operator that arises from $\Box_m$ by (a), (b), and (c) is called a trivial Huygens operator.

Hadamard's Conjecture: Every Huygens operator is trivial.

In their well-known book *Methoden der Mathematischen Physik*, Vol. 2, Richard Courant and David Hilbert treated the validity of Huygens' principle for $\Box_m$ and formulated the above conjecture. It is doubtful whether J. Hadamard himself explicitly stated the conjecture; nevertheless it served as a guideline for the investigation of the topic. Nowadays non-trivial Huygens operators are known; therefore it is better to speak of

Hadamard's problem: Find all Huygens operators. What do their coefficients look like?

Hadamard's criterion is indeed an elegant characterization of Huygens operators, but the functional relationship between the tail term and the coefficients of an operator is mysterious. Hadamard himself writes that his criterion gives "one answer but not the answer" (see [11], p. 236). The following result is a little step forward (see [9], Chap. VI, §5). By means of a certain differentiation process and passing to the limit $y \to x$ one can derive from the tail term $T(x,y)$ an infinite sequence of symmetric, trace-free tensors, which we call the moments of order $0, 1, 2, \ldots, k, \ldots$ of the operator L:

$$I(x), I_{i_1}(x), I_{i_1 i_2}(x), \ldots, I_{i_1 i_2 \cdots i_k}(x), \ldots.$$

These moments are invariant under the transformations (b) and (c), covariant under (a). More precisely, the moments arise by means of the usual tensor operations from (i), the metric tensor g and its curvature tensor; (ii) the tensor $H_{ij}: = \partial A_j/\partial x_i - \partial A_i/\partial x_j$, where $A_i = g_{ij}A^j$; (iii) the so-called Cotton invariant

$$\mathfrak{C}: = C - \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x_i}(\sqrt{\gamma}A^i) - A_i A^i + \frac{m-2}{4(m-1)}R$$

(R being the scalar curvature of g); and (iv) the covariant derivatives of (i), (ii), (iii). For a Huygens operator the moments must vanish; if the coefficients of L are analytic this condition is even sufficient. In principle the moments can be calculated by use of suitable algorithms, but in practice the matter is complicated. If some moment $I \ldots$ is known as a function of the tensors (i)–(iv), then one can try to get information about the coefficients of Huygens operators from the equations $I \ldots = 0$. This procedure is the "method of moments."

Some Results Concerning Hadamard's Problem

In 1938 E. Hölder proved that for a Huygens operator (15) with $A^i = 0$, $C = 0$, and $m = 4$, the scalar curvature R of the metric g vanishes. For the case under consideration this is indeed equivalent to the vanishing of the zero-order moment [13].

In 1939 the Polish mathematician Myron Mathisson calculated the moments of order at most 2 for an operator of the form

$$L[u] \equiv \frac{\partial^2 u}{\partial x_0^2} - \sum_{\ell=1}^{3} \frac{\partial^2 u}{\partial x_\ell^2} + 2A^i(x)\frac{\partial u}{\partial x_i} + C(x)u$$

with $m = 4$ independent variables and a pseudo-euclidean metric. These assumptions simplify the matter considerably. On the other hand, one can say that Mathisson created the method of moments. He proved that a Huygens operator of the type under consideration is trivial [15].

In 1952 the author expanded Mathisson's calculations to the case of a general operator (15) with $m = 4$ and curved metric; he gave the complete expressions of the moments of order at most 3 [6].

In 1955 Karl L. Stellmacher found the first examples of non-trivial Huygens operators for even $m \geq 6$, namely operators of the Euler-Poisson-Darboux type in the domains $x_i \neq 0$:

$$L[u] \equiv \left(\frac{\partial^2}{\partial x_0^2} + \frac{1-\mu_0^2}{4x_0^2}\right)u - \sum_{\ell=1}^{n}\left(\frac{\partial^2}{\partial x_\ell^2} + \frac{1-\mu_\ell^2}{4x_\ell^2}\right)u$$

with odd integers $\mu_i \geq 1$, such that

$$m < \sum_{i=0}^{n} \mu_i \leq 2m - 4, \; m = n + 1.$$

This result disproves Hadamard's conjecture for $m \geqslant 6$ [19].

In 1957 the author considered "static" operators of the form

$$L[u] \equiv \frac{\partial^2 u}{\partial x_0^2} - L_1[u],$$

where L_1 is a linear elliptic operator on a 3-dimensional manifold M_3. He confirmed Hadamard's conjecture for this operator class under additional assumptions (e.g., compactness of M_3, or constant scalar curvature of the metric associated to L_1) [7].

In a world with only two spatial dimensions an annoying noise produced somewhere and at any time never comes to an end. What luck that we have three dimensions at hand!

In 1965 the author gave examples of non-trivial Huygens operators for every even $m = n + 1 \geqslant 4$, thus ultimately disproving Hadamard's conjecture. Let $(a^{\alpha\beta})$, $\alpha, \beta = 2, \ldots, n$, be a positive-definite matrix, whose elements are smooth functions of the sole variable x_0; further, let $(a_{\alpha\beta})$ be its inverse matrix and $a = \mathrm{Det}(a_{\alpha\beta})$. Then

$$L[u] \equiv 2\frac{\partial^2 u}{\partial x_0 \partial x_1} + \frac{1}{\sqrt{a}} \cdot \frac{\partial \sqrt{a}}{\partial x_0}\frac{\partial u}{\partial x_1} - \sum_{\alpha,\beta=2}^{n} a^{\alpha\beta}\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} \tag{17}$$

is always a Huygens operator, which is trivial only for certain special matrices $(a^{\alpha\beta})$. The associated pseudo-Riemannian metric reads as follows

$$ds^2 = 2dx_0dx_1 - \sum_{\alpha,\beta=2}^{n} a_{\alpha\beta}(x_0)dx_\alpha dx_\beta; \tag{18}$$

metrics of this kind are known as plane-wave metrics [8].

In 1969 R. G. McLenaghan studied the general operator (15) with $m = 4$ under the assumption that the Ricci tensor R_{ij} of the associated metric vanishes. (In general relativity such a metric defines an empty space-time.) He proved by use of the moments of order at most 4 that a Huygens operator of this kind is equivalent under (a) and (b) to an operator of the form (17) [16].

In 1981 Sigurdur Helgason discovered a new class of non-trivial Huygens operators with $m \geqslant 6$ independent variables

$$L[u] \equiv \frac{\partial^2 u}{\partial x_0^2} - \left(\Delta - \frac{R}{6}\right)u.$$

Here Δ denotes the Laplace-Beltrami operator of certain symmetric Riemannian manifolds (M,g) of odd dimension $n \geqslant 5$, $m = n + 1$, with scalar curvature R. For instance, $M = G$ is a compact simple Lie group and g the bi-invariant metric arising from the negative of the associated Killing form. Taking $G = SO(\mathbf{R},3)$, one obtains a trivial Huygens operator, but $G = SO(\mathbf{R},6)$, $n = 15$, leads to a non-trivial one [12].

In 1986/88 J. Carminati and R. G. McLenaghan treated again Hadamard's problem for the operator (15) with $A^i = 0$ and $m = 4$. Their starting point was the Petrov classification for the Weyl curvature tensor of the associated metric g. They settled the problem in the cases of Petrov type N, D and (partially) III; in these investigations they found no other Huygens operators than the above-mentioned ones. The same result was proved by R. G. McLenaghan and T. F. Walton even in the case $A^i \neq 0$ for a Petrov type N metric [3], [17].

Concluding Remarks

Several other papers contain further contributions to Hadamard's problem. We mention those of V. Wünsch, R. Schimming, R. G. McLenaghan, N. X. Ibragimov, and R. Illge. Their papers also deal with operators that act on sections of vector bundles instead of scalar functions u; they are quoted in the bibliography of [9].

In general relativity the propagation of electromagnetic waves is described by Maxwell's equations taken for a 4-dimensional pseudo-Riemannian metric of signature $(+,-,-,-)$. The following result in some sense solves Hadamard's problem within the framework of Einstein's theory.

Let (M,g) be an empty space-time. Then Maxwell's equations obey Huygens' principle if and only if g is a plane-wave metric (18). For the "if" part see H. P. Künzle [14], R. Schimming [18]; for the "only if" part see P. Günther and V. Wünsch [10].

On the other hand, the non-validity of Huygens' principle leads to an aftereffect; it is an effect of second order at least for an empty space-time, and therefore not so remarkable as the classical ones (deviation of light rays, red shift).

Mathematics is not the art of computation but the art of minimal computation.

Anon.

References

1. Leifur Asgeirsson, Some hints on Huygens' principle and Hadamard's conjecture, *Comm. Pure Appl. Math.* 9 (1956), 307–326.
2. Luigi Bianchi, *Vorlesungen uber Differentialgeometrie*, Leipzig: B. G. Teubner-Verlag (1899).
3. J. Carminati and R. G. McLenaghan, An explicit determination of the space-times on which the conformally invariant scalar wave equation satisfies Huygens' principle, *Ann. Inst. Henri Poincaré, Phys. théor.* 44 (1986), 115–153; Part II, 47 (1987), 337–354; Part III, 48 (1988), 77–96.
4. Richard Courant and David Hilbert, *Methoden der Mathematischen Physik*, Band 2, Berlin: Springer-Verlag (1937).
5. F. G. Friedlander, *The wave equation on a curved space-time*, Cambridge: Cambridge Univ. Press (1975).
6. Paul Günther, Zur Gültigkeit des Huygens'schen Prinzips bei partiellen Differentialgleichungen vom normalen hyperbolischen Typ, *Ber. Verh. Sächs. Akad. d. Wiss. Leipzig, Math.-nat. Klasse* 100, Heft 2 (1952).
7. Paul Günther, Uber einige spezielle Probleme aus der Theorie der linearen partiellen Differentialgleichungen zweiter Ordnung, *Ber. Verh. Sächs. Akad. d. Wiss. Leipzig, Math.-nat. Klasse* 102, Heft 1 (1957).
8. Paul Günther, Ein Beispiel einer nichttrivialen Huygens'schen Differentialgleichung mit vier unabhangigen Veränderlichen, *Archive Rat. Mech. and Analysis* 18 (1965), 103–106.
9. Paul Günther, *Huygens' Principle and Hyperbolic Equations*, Boston: Academic Press (1988).
10. Paul Günther and Volkmar Wünsch, Maxwell'sche Gleichungen und Huygens'sches Prinzip, *Math. Nachr.* 63 (1974), 97–121.
11. Jacques Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, New Haven: Yale University Press (1923).
12. Sigurdur Helgason, *Wave Equations on Homogeneous Spaces*, Preprint 1981; Lectures Notes in Mathematics 1077, Springer-Verlag (1984).
13. Ernst Holder, Poissonsche Wellenformel in nichteuklidischen Räumen. *Ber. Verh. Sächs. Akad. d. Wiss. Leipzig*, 99 (1938), 55–66.
14. H. P. Künzle, Maxwell fields satisfying Huygens' principle, *Proc. Cambr. Phil. Soc.* 64 (1968), 779–785.
15. Myron Mathisson, Le problème de M. Hadamard relatif à la diffusion des ondes, *Acta Math.* 71 (1939), 249–282.
16. R. G. McLenaghan, An explicit determination of the empty space-times on which the wave equation satisfies Huygens' principle, *Proc. Cambr. Phil. Soc.* 65 (1969), 139–155.
17. R. G. McLenaghan and T. F. Walton, An explicit determination of the non-self-adjoint wave equations on curved space-time that satisfy Huygens' principle. Part I: Petrov type N background space-times, *Ann. Inst. Henri Poincaré, Phys. Théor.* 48(3) (1988), 267–280.
18. Rainer Schimming, Zur Gültigkeit des Huygens'schen Prinzips bei einer speziellen Metrik, *ZAMM* 51 (1971), 201–208.
19. Karl L. Stellmacher, Eine Klasse huygensscher Differentialgleichungen und ihre Integration, *Math. Ann.* 130 (1955), 219–233.
20. Jules Verne, *Le tour du monde en 80 jours.*

Sektion Mathematik, Karl-Marx-Universität
Karl-Marx-Platz
O-7010 Leipzig
Federal Republic of Germany