1. Appendix - Quasi-Maxwell form of Einstein's equations

1.1. **Stationary regions, space manifold and global time.**

Definition 1.1. *A region U of a spacetime $(Q, \langle , \rangle = g)$ is said to be* stationary *if there exists a timelike Killing vector field T defined in U.*

Recall that T is a *Killing vector field iff* $\pounds_T g = 0$, or, equivalently, *iff*

$$\langle \nabla_X T, Y \rangle + \langle \nabla_Y T, X \rangle = 0$$

for all vector fields X, Y.

Exercise 1.2. *Show that if T is a Killing vector field then*

$$T \cdot \langle T, T \rangle = 0$$

(i.e., the norm of T is constant along its integral lines). Deduce that if T is timelike in some region then T cannot vanish along any of its integral lines leaving that region. Show that if fT is also a Killing vector field for some nonvanishing smooth function f then f is constant along the integral lines of T, and that if T is timelike then f must be a constant function. Conclude that a timelike Killing vector field T is determined by its integral lines up to multiplication by a constant.

We shall assume that U contains a 3-dimensional submanifold Σ such that each integral line of T intersect Σ exactly once (so that Σ coincides with the quotient of U by the integral lines of T). This can always be achieved by restricting U conveniently.

Definition 1.3. *We will call Σ the* space manifold*.*

Notice that the integral lines of T provide a natural projection $\pi : U \to \Sigma$ (corresponding to the quotient map).

We can now define a *global time function* $t : U \to \mathbb{R}$ by setting $t(p)$ equal to the parameter corresponding to $p \in U$ along the integral line of T through p, where we assign $t = 0$ to the intersection of the integral line with Σ (hence Σ is the level hypersurface $t = 0$).

We will have to consider tensor fields defined both in the space manifold Σ or in all the stationary region U. For that reason we shall take Latin indices to run from 1 to 3, and Greek indices from 0 to 3. We shall also use Einstein's summation convention that whenever a repeated index occurs it is understood to be summed over its range.

If $\{x^i\}$ are local coordinates in Σ, we can use the integral lines of T to extend them as functions to the whole of U.

Exercise 1.4. *Show that $\{t, x^i\}$ are local coordinates on U and $T = \frac{\partial}{\partial t}$.*

Exercise 1.5. *Show that in these coordinates one has*

$$\frac{\partial g_{\alpha\beta}}{\partial t} = 0$$

corresponding to the intuitive idea that in a stationary region the metric should not depend on time.

In the coordinates $\{t, x^i\}$ the line element is written

$$\begin{aligned} ds^2 &= g_{00}dt^2 + 2g_{0i}dtdx^i + g_{ij}dx^i dx^j \\ &= g_{00}\left(dt + \frac{g_{0i}}{g_{00}}dx^i\right)^2 - \frac{g_{0i}g_{0j}}{g_{00}}dx^i dx^j + g_{ij}dx^i dx^j \\ &= -e^{2\phi}\left(dt + A_i dx^i\right)^2 + \gamma_{ij}dx^i dx^j \end{aligned}$$

where the definitions of ϕ, A_i and γ_{ij} should be obvious. Here we've used the fact that T is timelike and therefore

$$g_{00} = \langle T, T\rangle < 0.$$

Exercise 1.6. *Use the time independence of the components $g_{\alpha\beta}$ to show that ϕ, $A = A_i dx^i$ and $\gamma = \gamma_{ij}dx^i \otimes dx^j$ satisfy*

$$\begin{aligned} \phi &= \pi^*\left(\phi\,|_\Sigma\right); \\ A &= \pi^*\left(A\,|_\Sigma\right); \\ \gamma &= \pi^*\left(\gamma\,|_\Sigma\right). \end{aligned}$$

Conclude that ϕ, A and γ can be interpreted as tensor fields defined on the space manifold.

We are using the timelike Killing vector field T to identify a special class of observers, namely those whose worldlines are the integral curves of T (to whom we shall refer as *stationary observers*), and a special global time function. From this point of view, the space manifold is just a convenient way to keep track of these stationary observers, and we might as well have picked a different space manifold. Also, there's no reason why we should pick T among all the Killing fields e^cT ($c \in \mathbb{R}$) with the same integral curves. Now the equation of any other space manifold is

$$t = f\left(x^1, x^2, x^3\right)$$

and picking *this* space manifold *and* the Killing vector field e^cT amounts picking a new global time function, i.e., to making the coordinate transformation

$$t' = e^{-c}\left(t - f\right).$$

(obviously one can use the same local coordinates $\{x^i\}$ on the new space manifold). With these new coordinates the line element is written

$$\begin{aligned} ds^2 &= -e^{2\phi}\left[d\left(e^c t' + f\right) + A\right]^2 + \gamma_{ij}dx^i dx^j \\ &= -e^{2(\phi+c)}\left[dt' + e^{-c}\left(A + df\right)\right]^2 + \gamma_{ij}dx^i dx^j \end{aligned}$$

and hence

$$\begin{aligned} \phi' &= \phi + c; \\ A' &= e^{-c}\left(A + df\right); \\ \gamma' &= \gamma. \end{aligned}$$

In particular, we see that the differential forms

$$\begin{aligned} G &= -d\phi \\ H &= -e^{\phi}dA \end{aligned}$$

(which can be thought of as defined on the space manifold) have an invariant meaning associated to the given family of stationary observers (i.e., do not depend on the choice global time function).

Exercise 1.7. *If $u, v \in T_x\Sigma \subseteq T_{(0,x)}Q$, show that*

$$\gamma(u, v) = \langle u^\perp, v^\perp \rangle$$

where $u^\perp$ is the component of u orthogonal to T. Conclude that (Σ, γ) is a Riemannian manifold.

Notice that γ has the physical meaning of being the (local) distance measured by the stationary observers using, say, radar measurements. The fact that T is a Killing vector field means that

$$\frac{\partial \gamma_{ij}}{\partial t} = 0$$

i.e., distances between stationary observers do not change with time. This is just about as close as General Relativity gets to the notion of a "global frame of reference".

1.2. **Connection forms and equations of motion.** Having chosen a global time t (and hence a space manifold $\Sigma = \{t = 0\}$), an orthonormal coframe defined in the stationary region U is

$$\begin{aligned} \omega^0 &= e^\phi (dt + A) \\ \omega^i &= \pi^* \widehat{\omega}^i \end{aligned}$$

where $\{\widehat{\omega}^i\}$ is an orthonormal coframe for (Σ, γ) (we shall, for simplicity, drop π^* out of the equations).

The corresponding orthonormal basis $\{\mathbf{e}_\alpha\}$satisfies

$$\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \rangle = \eta_{\alpha\beta}$$

(where $\eta_{\alpha\beta}$ is -1 if $\alpha = \beta = 0$, 1 if $\alpha = \beta \neq 0$ and 0 if $\alpha \neq \beta$), and consequently it is easy to check that the connection forms satisfy

$$\eta_{\alpha\delta}\omega_\beta^\delta + \eta_{\beta\delta}\omega_\alpha^\delta = 0$$

rather than the more familiar identity for the case of a Riemannian manifold. This plus Cartan's first structure equations

$$d\omega^\alpha = \omega^\beta \wedge \omega_\beta^\alpha$$

completely determine the connection forms. Now we have

$$\begin{aligned} d\omega^0 &= d\phi \wedge \omega^0 + e^\phi dA \\ &= -G \wedge \omega^0 - H \\ &= -G_i \omega^i \wedge \omega^0 - \frac{1}{2} H_{ij} \omega^i \wedge \omega^j \\ &= \omega^i \wedge \left(-G_i \omega^0 - \frac{1}{2} H_{ij} \omega^j \right) \\ &= \omega^i \wedge \omega_i^0 \end{aligned}$$

and

$$\begin{aligned} d\omega^i &= d\widehat{\omega}^i \\ &= \widehat{\omega}^j \wedge \widehat{\omega}_j^i \\ &= \omega^j \wedge \widehat{\omega}_j^i \\ &= \omega^0 \wedge \omega_0^i + \omega^j \wedge \omega_j^i \end{aligned}$$

(where $\widehat{\omega}^i_j$ are the connection forms corresponding to the orthonormal coframe $\{\widehat{\omega}^i\}$ in the space manifold). Consequently, if we make the obvious ansatz

$$\omega^0_i = \omega^i_0 = -G_i\omega^0 - \frac{1}{2}H_{ij}\omega^j$$

we will have

$$\begin{aligned}\omega^j \wedge \omega^i_j &= \omega^j \wedge \widehat{\omega}^i_j - \omega^0 \wedge \left(-G_i\omega^0 - \frac{1}{2}H_{ij}\omega^j\right) \\ &= \omega^j \wedge \widehat{\omega}^i_j + \frac{1}{2}H_{ij}\omega^0 \wedge \omega^j \\ &= \omega^j \wedge \left(\widehat{\omega}^i_j - \frac{1}{2}H_{ij}\omega^0\right)\end{aligned}$$

i.e.,

$$\omega^i_j = \widehat{\omega}^i_j - \frac{1}{2}H_{ij}\omega^0$$

which indeed satisfy the required skew-symmetry properties.

Consider a timelike geodesic representing the motion of a material particle, and let

$$u = u^0\mathbf{e}_0 + u^i\mathbf{e}_i$$

be its unit tangent vector. Clearly u^0 is the energy per unit rest mass that a stationary observer measures for the particle, and

$$\mathbf{u} = u^i\mathbf{e}_i$$

(which can be interpreted as a vector on the space manifold, as $\{\pi_*\mathbf{e}_i\}$ is an orthonormal frame for (Σ, γ) - we shall, for simplicity, stop worrying about the projection and freely identify $\mathbf{e}_i$ and $\pi_*\mathbf{e}_i$) is just $u^0\mathbf{v}$, where $\mathbf{v}$ is the velocity measured by the stationary observer for the particle. We have

$$\langle u, u\rangle = -1 \Leftrightarrow -\left(u^0\right)^2 + \mathbf{u}^2 = -1 \Leftrightarrow \left(u^0\right)^2 = 1 + \mathbf{u}^2$$

where

$$\mathbf{u}^2 = g\left(\mathbf{u}, \mathbf{u}\right) = u^i u^i = \gamma\left(\mathbf{u}, \mathbf{u}\right).$$

Recalling that

$$\nabla_v \mathbf{e}_\alpha = \omega^\beta_\alpha\left(v\right)\mathbf{e}_\beta$$

we can write the geodesic equation as

$$\begin{aligned}\nabla_u u &= 0 \Leftrightarrow \nabla_u\left(u^0\mathbf{e}_0 + u^i\mathbf{e}_i\right) = 0 \\ &\Leftrightarrow \frac{du^0}{d\tau}\mathbf{e}_0 + u^0\omega^i_0\left(u\right)\mathbf{e}_i + \frac{du^i}{d\tau}\mathbf{e}_i + u^i\omega^0_i\left(u\right)\mathbf{e}_0 + u^i\omega^j_i\left(u\right)\mathbf{e}_j = 0\end{aligned}$$

and hence the component along $\mathbf{e}_0$ is

$$\begin{aligned}\frac{du^0}{d\tau} + u^i\omega^0_i\left(u\right) &= 0 \Leftrightarrow \frac{du^0}{d\tau} - u^iG_iu^0 - \frac{1}{2}H_{ij}u^iu^j = 0 \\ &\Leftrightarrow \frac{du^0}{d\tau} = u^0u^iG_i\end{aligned}$$

whereas the component along $\mathbf{e}_i$ is

$$\begin{aligned}\frac{du^i}{d\tau}+u^0\omega_0^i(u)+u^j\omega_j^i(u) &= 0\\ &\Leftrightarrow \frac{du^i}{d\tau}-u^0\left(G_iu^0+\frac{1}{2}H_{ij}u^j\right)+u^j\left(\widehat{\omega}_j^i(u)-\frac{1}{2}H_{ij}u^0\right)=0\\ &\Leftrightarrow \left(\frac{\widehat{D}\mathbf{u}}{d\tau}\right)^i=\left(u^0\right)^2G_i+u^0H_{ij}u^j\end{aligned}$$

(here $\frac{\widehat{D}}{dt}$ refers to the Levi-Civita connection $\widehat{\nabla}$ on (Σ,γ)).

We now use the fact that (Σ,γ) is a three dimensional Riemannian manifold, and that consequently

$$\dim T_x\Sigma=\dim T_x^*\Sigma=\dim\Lambda^2T_x^*\Sigma=3$$

for all $x\in\Sigma$. The Riemannian metric provides a bijection $i_1:T_x\Sigma\to T_x^*\Sigma$ defined through $i_1(\mathbf{v})=\gamma(\mathbf{v},\cdot)$, i.e.,

$$i_1\left(v^1\mathbf{e}_1+v^2\mathbf{e}_2+v^3\mathbf{e}_3\right)=v^1\widehat{\omega}^1+v^2\widehat{\omega}^2+v^3\widehat{\omega}^3.$$

Similarly, one can define a bijection $i_2:T_x\Sigma\to\Lambda^2T_x^*\Sigma$ through

$$i_2\left(v^1\mathbf{e}_1+v^2\mathbf{e}_2+v^3\mathbf{e}_3\right)=v^1\widehat{\omega}^2\wedge\widehat{\omega}^3+v^2\widehat{\omega}^3\wedge\widehat{\omega}^1+v^3\widehat{\omega}^1\wedge\widehat{\omega}^2.$$

Exercise 1.8. *Show that i_2 is well defined, i.e., that it does not depend on the choice of the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$.*

Exercise 1.9. *Show that*

$$i_2(\mathbf{u}\times\mathbf{v})=i_1(\mathbf{u})\wedge i_1(\mathbf{v}).$$

Definition 1.10. *On the space manifold Σ we define the* gravitational vector field $\mathbf{G}$ *and the* gravitomagnetic vector field $\mathbf{H}$ *through*

$$\begin{aligned}G&=i_1(\mathbf{G});\\ H&=i_2(\mathbf{H}).\end{aligned}$$

It should be clear that

$$\begin{aligned}\left(H_{ij}u^j\right) &= \begin{pmatrix}0 & H^3 & -H^2\\ -H^3 & 0 & H^1\\ H^2 & -H^1 & 0\end{pmatrix}\begin{pmatrix}u^1\\ u^2\\ u^3\end{pmatrix}\\ &= \begin{pmatrix}u^2H^3-u^3H^2\\ u^3H^1-u^1H^3\\ u^1H^2-u^2H^1\end{pmatrix}=(\mathbf{u}\times\mathbf{H})^i\end{aligned}$$

and consequently the component of the motion equation along $\mathbf{e}_i$ can be written as

$$\begin{aligned}\frac{\widehat{D}\mathbf{u}}{d\tau} &= \left(u^0\right)^2\mathbf{G}+u^0\mathbf{u}\times\mathbf{H}\\ &= \left(1+\mathbf{u}^2\right)^{\frac{1}{2}}\left(\left(1+\mathbf{u}^2\right)^{\frac{1}{2}}\mathbf{G}+\mathbf{u}\times\mathbf{H}\right).\end{aligned}$$

Thus when all stationary observers compare their local observations they conclude that the particle moves in the space manifold under the influence of a gravitational field $\mathbf{G}$ and a gravitomagnetic field $\mathbf{H}$ in a way that closely resembles electromagnetism. To check how accurate this analogy is we now take a short detour.

Exercise 1.11. *Show that the component of the motion equation along* $\mathbf{e}_0$ *may be written as*

$$\frac{du^0}{d\tau} = u^0 \mathbf{u} \cdot \mathbf{G}$$

and is a simple consequence of the motion equation in the space manifold. Also, show that this equation can still be written as

$$\frac{du^0}{d\tau} = -u^0 \widehat{\nabla}_{\mathbf{u}} \phi$$

and deduce the energy conservation principle

$$\frac{d}{d\tau}\left(u^0 e^{\phi}\right) = 0 \Leftrightarrow \frac{d}{d\tau}\left(\left(1 + \mathbf{u}^2\right)^{\frac{1}{2}} e^{\phi}\right) = 0$$

holds.

Notice that $u^0 e^{\phi}$ is just $\langle T, u \rangle$. For low speeds and weak gravitational fields this conserved quantity is

$$u^0 e^{\phi} = \left(1 - \mathbf{v}^2\right)^{-\frac{1}{2}} e^{\phi} \simeq \left(1 + \frac{1}{2}\mathbf{v}^2\right)(1 + \phi) \simeq 1 + \frac{1}{2}\mathbf{v}^2 + \phi$$

i.e., is just the rest energy plus the Newtonian mechanical energy (per unit rest mass).

Exercise 1.12. *Show that in general stationary observers are accelerated observers, and that their* proper acceleration *is*

$$\frac{D}{d\tau}\left(e^{-\phi} T\right) = -G^i \mathbf{e}_i$$

(this is the acceleration measured by, say, an accelerometer carried by a stationary observer).

1.3. **Stationary Maxwell equations.** Recall that the motion equations for a particle of rest mass m and electric charge e under the influence of an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ are

$$\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

where $\mathbf{p}$ is the particle's (relativistic) momentum and $\mathbf{v}$ is its velocity. If τ is the particle's proper time and $\mathbf{x} = \mathbf{x}(\tau)$ its spatial path, then one has

$$\mathbf{p} = m\frac{d\mathbf{x}}{d\tau} = m\mathbf{u}$$

and

$$-\left(\frac{dt}{d\tau}\right)^2 + \mathbf{u}^2 = -1 \Leftrightarrow \frac{dt}{d\tau} = \left(1 + \mathbf{u}^2\right)^{\frac{1}{2}}.$$

Consequently,

$$\mathbf{u} = \frac{d\mathbf{x}}{dt}\frac{dt}{d\tau} = \left(1 + \mathbf{u}^2\right)^{\frac{1}{2}} \mathbf{v}$$

and the motion equation may be written as

$$\frac{d\mathbf{u}}{d\tau} = \frac{e}{m}\left(\left(1 + \mathbf{u}^2\right)^{\frac{1}{2}} \mathbf{E} + \mathbf{u} \times \mathbf{B}\right).$$

Thus we see that the motion equation for a free falling particle in the space manifold of a stationary region is the curved space generalization of this equation with the ratio $\frac{e}{m}$ replaced by $\left(1+\mathbf{u}^2\right)^{\frac{1}{2}}$. This is reasonable to expect, as $\left(1+\mathbf{u}^2\right)^{\frac{1}{2}}$ is the ratio between the particle's total energy as measured by a stationary observer (which is what one would expect the gravitational field to couple to) and the particle's rest mass.

It is interesting to see how far this analogy goes, and in particular whether Einstein's equation in a stationary region in any way mirrors Maxwell's equations. Recall that in natural units ($c = \varepsilon_0 = 1$) Maxwell's equations for stationary (i.e., time-independent) electric and magnetic fields are written

$$\begin{aligned} \operatorname{div}(\mathbf{E}) &= \rho; \\ \operatorname{div}(\mathbf{B}) &= 0; \\ \operatorname{curl}(\mathbf{E}) &= \mathbf{0}; \\ \operatorname{curl}(\mathbf{B}) &= \mathbf{j} \end{aligned}$$

where ρ is the electric charge density and $\mathbf{j}$ is the electric current density.

Assuming these equations hold in a contractible region of space, the homogeneous equations imply (due to Poincaré's lemma) the existence of an electric potential ϕ and a vector potential $\mathbf{A}$ such that

$$\begin{aligned} \mathbf{E} &= -\operatorname{grad}(\phi)\,; \\ \mathbf{B} &= -\operatorname{curl}(\mathbf{A})\,. \end{aligned}$$

Clearly ϕ is defined up to the addition of a constant function, whereas $\mathbf{A}$ is defined up to the addition of a gradient field.

It is possible to show that Maxwell's equations are fully relativistic, and that electromagnetic fields carry energy and momentum. The energy density, energy density current and stress of the electromagnetic field are given, respectively, by

$$\begin{aligned} \rho_{field} &= \frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right); \\ \mathbf{j}_{field} &= \mathbf{E}\times\mathbf{B}; \\ T_{field} &= \frac{1}{2}\left(\mathbf{E}^2+\mathbf{B}^2\right) I - \mathbf{E}\otimes\mathbf{E} - \mathbf{B}\otimes\mathbf{B}. \end{aligned}$$

1.4. **Curvature forms and Ricci tensor.** We will now try to write Einstein's equation as a set of equations in the space manifold involving the vector fields $\mathbf{G}$ and $\mathbf{H}$. We start by noticing that

$$i_1(\mathbf{G}) = -d\phi \Leftrightarrow \mathbf{G} = -\operatorname{grad}(\phi)$$

in an exact analogue of the corresponding electrostatic formula.

Exercise 1.13. *Show that in $\mathbb{R}^3$ with the usual Euclidean metric one has*

$$i_2(\operatorname{curl}(\mathbf{v})) = d(i_1(\mathbf{v}))\,.$$

Definition 1.14. *If (Σ,γ) is an arbitrary 3-dimensional Riemannian manifold and $\mathbf{v}$ is a vector field defined on Σ we* define $\operatorname{curl}(\mathbf{v})$ *as the unique vector field satisfying*

$$i_2(\operatorname{curl}(\mathbf{v})) = d(i_1(\mathbf{v}))\,.$$

Thus we have

$$\mathbf{H} = -e^{\phi}\operatorname{curl}(\mathbf{A})$$

closely resembling the corresponding magnetostatic expression. Thus the equations defining the gravitational and gravitomagnetic fields $\mathbf{G}$ and $\mathbf{H}$ parallel the homogeneous Maxwell equations.

In order to write Einstein's equations in the orthonormal frame $\{\mathbf{e}_\alpha\}$ we will have to compute the components of the Ricci tensor in this frame. These can be obtained from the curvature forms Ω_α^β, which in turn are given by Cartan's second structure equations

$$\Omega_\alpha^\beta = -d\omega_\alpha^\beta + \omega_\alpha^\delta \wedge \omega_\delta^\beta.$$

Before computing these forms, we notice that since

$$\widehat{\nabla}\widehat{\omega}^i = -\widehat{\omega}^j \otimes \widehat{\omega}_j^i$$

we have

$$\begin{aligned} \widehat{\nabla}G &= \left(\widehat{\nabla}_j G_i\right)\widehat{\omega}^i \otimes \widehat{\omega}^j = \widehat{\nabla}\left(G_i\widehat{\omega}^i\right) \\ &= \widehat{\omega}^i \otimes dG_i - G_i\widehat{\omega}^j \otimes \widehat{\omega}_j^i \\ &= \widehat{\omega}^i \otimes \left(dG_i - G_j\widehat{\omega}_i^j\right) \end{aligned}$$

and

$$\begin{aligned} \widehat{\nabla}H &= \left(\widehat{\nabla}_k H_{ij}\right)\widehat{\omega}^i \otimes \widehat{\omega}^j \otimes \widehat{\omega}^k = \widehat{\nabla}\left(H_{ij}\widehat{\omega}^i \otimes \widehat{\omega}^j\right) \\ &= \widehat{\omega}^i \otimes \widehat{\omega}^j \otimes dH_{ij} - H_{ij}\widehat{\omega}^k \otimes \widehat{\omega}^j \otimes \widehat{\omega}_k^i - H_{ij}\widehat{\omega}^i \otimes \widehat{\omega}^k \otimes \widehat{\omega}_k^j \\ &= \widehat{\omega}^i \otimes \widehat{\omega}^j \otimes \left(dH_{ij} - H_{kj}\widehat{\omega}_i^k - H_{ik}\widehat{\omega}_j^k\right) \end{aligned}$$

(where we've taken the chance to introduce the notation $\widehat{\nabla}_i G_j$ and $\widehat{\nabla}_i H_{jk}$ for the components of the covariant differential of G and H). In other words, one has

$$\left(\widehat{\nabla}_j G_i\right)\widehat{\omega}^j = dG_i - G_j\widehat{\omega}_i^j$$

and

$$\left(\widehat{\nabla}_k H_{ij}\right)\widehat{\omega}^k = dH_{ij} - H_{kj}\widehat{\omega}_i^k - H_{ik}\widehat{\omega}_j^k.$$

Exercise 1.15. *Use the formulae above and the known expressions*

$$\begin{aligned} \omega_i^0 = \omega_0^i &= -G_i\omega^0 - \frac{1}{2}H_{ij}\omega^j; \\ \omega_j^i &= \widehat{\omega}_j^i - \frac{1}{2}H_{ij}\omega^0; \\ d\omega^0 &= -G \wedge \omega^0 - H; \\ d\omega^i &= \widehat{\omega}^j \wedge \widehat{\omega}_j^i \end{aligned}$$

in Cartan's second structure equations

$$\begin{aligned} \Omega_i^0 &= d\omega_i^0 - \omega_i^j \wedge \omega_j^0; \\ \Omega_i^j &= d\omega_i^j - \omega_i^0 \wedge \omega_0^j - \omega_i^k \wedge \omega_k^j \end{aligned}$$

to show that the curvature forms are given by

$$\Omega_i^0 = \Omega_0^i = \left(\widehat{\nabla}_j G_i - G_iG_j + \frac{1}{4}H_{ik}H_{kj}\right)\omega^0 \wedge \omega^j + \frac{1}{2}\left(-\widehat{\nabla}_j H_{ik} + G_iH_{jk}\right)\omega^j \wedge \omega^k;$$

$$\Omega_i^j = -\Omega_j^i = \widehat{\Omega}_i^j + \frac{1}{2}\left(-\widehat{\nabla}_k H_{ij} + G_jH_{ik} - G_iH_{jk} + G_kH_{ij}\right)\omega^0 \wedge \omega^k - \frac{1}{4}\left(H_{ij}H_{kl} + H_{ik}H_{jl}\right)\omega^k \wedge \omega^l.$$

Since

$$\omega^\alpha \wedge \omega^\beta = \omega^\alpha \otimes \omega^\beta - \omega^\beta \otimes \omega^\alpha$$

the independent components of the Riemann tensor in this orthonormal frame are given by

$$\begin{aligned}
R^0_{i0j} &= \widehat{\nabla}_j G_i - G_i G_j + \frac{1}{4} H_{ik} H_{kj}; \\
R^0_{ijk} &= \frac{1}{2} \left(-\widehat{\nabla}_j H_{ik} + \widehat{\nabla}_k H_{ij} + 2 G_i H_{jk} \right); \\
R^j_{i0k} &= \frac{1}{2} \left(-\widehat{\nabla}_k H_{ij} + G_j H_{ik} - G_i H_{jk} + G_k H_{ij} \right); \\
R^j_{ikl} &= \widehat{R}^j_{ikl} - \frac{1}{4} \left(2 H_{ij} H_{kl} + H_{ik} H_{jl} - H_{il} H_{jk} \right),
\end{aligned}$$

where $\widehat{R}^j_{ikl}$ are the components of the Riemann tensor of the space manifold on the corresponding orthonormal basis.

Exercise 1.16. *Show that because of the Riemann tensor symmetries one has*

$$R^0_{ijk} = -R^j_{k0i}.$$

Deduce that G and H must satisfy

$$\widehat{\nabla}_i H_{jk} + \widehat{\nabla}_j H_{ki} + \widehat{\nabla}_k H_{ij} + G_i H_{jk} + G_j H_{ki} + G_k H_{ij} = 0.$$

Rewrite this condition as

$$dH + G \wedge H = 0$$

and show that it follows trivially from

$$H = -e^\phi dA.$$

It is now a simple task to compute the components of the Ricci tensor in our orthonormal frame. For example, one has

$$\begin{aligned}
(Ric)_{00} &= R^i_{0i0} = -\widehat{\nabla}_i G_i + G_i G_i - \frac{1}{4} H_{ik} H_{ki} \\
&= -\operatorname{div}(\mathbf{G}) + \mathbf{G}^2 + \frac{1}{4} H_{ik} H_{ik} \\
&= -\operatorname{div}(\mathbf{G}) + \mathbf{G}^2 + \frac{1}{2} \mathbf{H}^2
\end{aligned}$$

and

$$\begin{aligned}
(Ric)_{0i} &= R^j_{ij0} = \frac{1}{2} \left(\widehat{\nabla}_j H_{ij} - G_j H_{ij} + G_i H_{jj} - G_j H_{ij} \right) \\
&= \frac{1}{2} \left(\widehat{\nabla}_j H_{ij} - 2 H_{ij} G_j \right).
\end{aligned}$$

Since

$$\begin{aligned}
\left(\widehat{\nabla}_j H_{ij} \right) &= \left(-\widehat{\nabla}_j H_{ji} \right) = -\begin{pmatrix} \widehat{\nabla}_1 & \widehat{\nabla}_2 & \widehat{\nabla}_3 \end{pmatrix} \begin{pmatrix} 0 & H^3 & -H^2 \\ -H^3 & 0 & H^1 \\ H^2 & -H^1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \widehat{\nabla}_2 H^3 - \widehat{\nabla}_3 H^2, & \widehat{\nabla}_3 H^1 - \widehat{\nabla}_1 H^3, & \widehat{\nabla}_1 H^2 - \widehat{\nabla}_2 H^1 \end{pmatrix} \\
&= \begin{pmatrix} \left(d \left(i_1 (\mathbf{H}) \right) \right)_{23}, & \left(d \left(i_1 (\mathbf{H}) \right) \right)_{31}, & \left(d \left(i_1 (\mathbf{H}) \right) \right)_{12} \end{pmatrix} \\
&= \begin{pmatrix} \left(\operatorname{curl}(\mathbf{H}) \right)^1, & \left(\operatorname{curl}(\mathbf{H}) \right)^2, & \left(\operatorname{curl}(\mathbf{H}) \right)^3 \end{pmatrix}
\end{aligned}$$

we see that

$$(Ric)_{0i}\,\mathbf{e}_i = \left(\frac{1}{2}\operatorname{curl}(\mathbf{H}) - \mathbf{G}\times\mathbf{H}\right).$$

Finally, we have

$$\begin{aligned}(Ric)_{ij} &= R^0_{i0j} + R^k_{ikj} = \widehat{\nabla}_j G_i - G_i G_j + \frac{1}{4}H_{ik}H_{kj} + \widehat{R}^k_{ikj} - \frac{1}{4}\left(2H_{ik}H_{kj} + H_{ik}H_{kj} - H_{ij}H_{kk}\right)\\ &= \left(\widehat{Ric}\right)_{ij} + \widehat{\nabla}_i G_j - G_i G_j - \frac{1}{2}H_{ik}H_{kj}\end{aligned}$$

where $\left(\widehat{Ric}\right)_{ij}$ are the components of the Ricci tensor of the space manifold on the corresponding orthonormal basis and we've used the fact that $\widehat{\nabla}_i G_j$ is minus the Hessian of ϕ (hence symmetric). As

$$\begin{aligned}(H_{ik}H_{kj}) &= \begin{pmatrix} 0 & H^3 & -H^2\\ -H^3 & 0 & H^1\\ H^2 & -H^1 & 0\end{pmatrix}\begin{pmatrix} 0 & H^3 & -H^2\\ -H^3 & 0 & H^1\\ H^2 & -H^1 & 0\end{pmatrix}\\ &= \begin{pmatrix} -\left(H^2\right)^2 - \left(H^3\right)^2 & H^1H^2 & H^1H^3\\ H^1H^2 & -\left(H^1\right)^2 - \left(H^3\right)^2 & H^2H^3\\ H^1H^3 & H^2H^3 & -\left(H^1\right)^2 - \left(H^2\right)^2\end{pmatrix}\\ &= H^iH^j - \mathbf{H}^2\gamma_{ij}\end{aligned}$$

we can write

$$(Ric)_{ij} = \left(\widehat{Ric}\right)_{ij} + \widehat{\nabla}_i G_j - G_i G_j - \frac{1}{2}H^iH^j + \frac{1}{2}\mathbf{H}^2\gamma_{ij}.$$

1.5. **Quasi-Maxwell equations.**

Definition 1.17. *A* perfect fluid *is defined as a fluid such that the only stresses measured by a comoving observer correspond to an isotropic pressure.*

So if $\{\mathbf{e}_\alpha\}$ is a orthonormal frame associated to a comoving observer, the energy-momentum tensor of a perfect fluid is by definition

$$T = \rho\mathbf{e}_0\otimes\mathbf{e}_0 + p\mathbf{e}_i\otimes\mathbf{e}_i$$

where ρ is the rest energy density of the fluid and p is the rest pressure (note that there are no components in $\mathbf{e}_0\otimes\mathbf{e}_i$ as the observer is at rest with respect to the fluid and therefore must measure zero energy current density). Since the raised indices metric tensor clearly is

$$g = -\mathbf{e}_0\otimes\mathbf{e}_0 + \mathbf{e}_i\otimes\mathbf{e}_i$$

we see that

$$\begin{aligned}T &= \rho\mathbf{e}_0\otimes\mathbf{e}_0 + p\left(g + \mathbf{e}_0\otimes\mathbf{e}_0\right)\\ &= (\rho + p)\,\mathbf{e}_0\otimes\mathbf{e}_0 + pg\end{aligned}$$

or, since $\mathbf{e}_0$ is just the 4-velocity u of the fluid,

$$T = (\rho + p)\,u\otimes u + pg.$$

Exercise 1.18. *Show that Einstein's equation implies the motion equation*

$$(\rho + p)\,\nabla_u u + \operatorname{div}\left((\rho + p)\,u\right)u = -\operatorname{grad}(p)$$

for a perfect fluid (here div *and* grad *refer to the full spacetime metric g).*

Exercise 1.19. *A perfect fluid satisfying $p = -\rho = \frac{\lambda}{8\pi}$ is said to correspond to a* cosmological constant λ *(notice that such fluid does not possess a rest frame). Show that the motion equations imply that λ is indeed constant.*

Recall that

$$Ric = G - \frac{1}{2}C(G)\,g$$

where G is Einstein's tensor. Since Einstein's equation is

$$G = 8\pi T$$

we conclude that

$$Ric = 8\pi\left(T - \frac{1}{2}C(T)\,g\right).$$

Since

$$C(T) = -(\rho + p) + 4p = 3p - \rho$$

we have

$$\begin{aligned} Ric &= 8\pi\left((\rho+p)\,u\otimes u + pg - \frac{1}{2}(3p-\rho)\,g\right) \\ &= 8\pi\left((\rho+p)\,u\otimes u + \frac{1}{2}(\rho-p)\,g\right) \end{aligned}$$

In the stationary orthonormal frame, we have

$$u = u^0\mathbf{e}_0 + \mathbf{u}$$

and consequently

$$Ric = 8\pi\left((\rho+p)\left(\left(u^0\right)^2\mathbf{e}_0\otimes\mathbf{e}_0 + u^0\mathbf{e}_0\otimes\mathbf{u} + u^0\mathbf{u}\otimes\mathbf{e}_0 + \mathbf{u}\otimes\mathbf{u}\right) + \frac{1}{2}(\rho-p)\left(-\mathbf{e}_0\otimes\mathbf{e}_0 + \gamma\right)\right)$$

i.e.,

$$\begin{aligned} (Ric)^{00} &= 8\pi\left((\rho+p)\left(u^0\right)^2 - \frac{1}{2}(\rho-p)\right) = 4\pi\left(\left(2\left(u^0\right)^2 - 1\right)\rho + \left(2\left(u^0\right)^2 + 1\right)p\right); \\ (Ric)^{0i}\,\mathbf{e}_i &= 8\pi(\rho+p)\,u^0\mathbf{u}; \\ (Ric)^{ij} &= 8\pi\left((\rho+p)\,u^iu^j + \frac{1}{2}(\rho-p)\,\gamma^{ij}\right). \end{aligned}$$

Since we are using an orthonormal frame, it is simple to equate these components to those obtained from the expression of the line element:

$$\begin{aligned} -\operatorname{div}(\mathbf{G}) + \mathbf{G}^2 + \frac{1}{2}\mathbf{H}^2 &= 4\pi\left(\left(2\left(u^0\right)^2 - 1\right)\rho + \left(2\left(u^0\right)^2 + 1\right)p\right); \\ \frac{1}{2}\operatorname{curl}(\mathbf{H}) - \mathbf{G}\times\mathbf{H} &= -8\pi(\rho+p)\,u^0\mathbf{u}; \\ \left(\widehat{Ric}\right)_{ij} + \widehat{\nabla}_iG_j - G_iG_j - \frac{1}{2}H^iH^j + \frac{1}{2}\mathbf{H}^2\gamma_{ij} &= 8\pi\left((\rho+p)\,u^iu^j + \frac{1}{2}(\rho-p)\,\gamma^{ij}\right). \end{aligned}$$

Rearranging these equations slightly, and remembering we are using an orthonormal frame, we can finally write

$$
\begin{aligned}
(1) \qquad \operatorname{div}(\mathbf{G}) &= \mathbf{G}^2 + \frac{1}{2}\mathbf{H}^2 - 4\pi\left(\left(2\left(u^0\right)^2 - 1\right)\rho + \left(2\left(u^0\right)^2 + 1\right)p\right);\\
(2) \qquad \operatorname{curl}(\mathbf{H}) &= 2\mathbf{G}\times\mathbf{H} - 16\pi\,(\rho + p)\,u^0\mathbf{u};\\
(3) \qquad \left(\widehat{R}\right)_{ij} + \widehat{\nabla}_i G_j &= G_iG_j + \frac{1}{2}H_iH_j - \frac{1}{2}\mathbf{H}^2\gamma_{ij} + 8\pi\left((\rho+p)\,u_iu_j + \frac{1}{2}(\rho - p)\,\gamma_{ij}\right).
\end{aligned}
$$

These equations are now either tensor equations or the components of tensor equations on the space manifold, and therefore hold in any frame.

Definition 1.20. *Equations (1), (2) and (3) are called the* quasi-Maxwell equations *corresponding to the given family of stationary observers.*

Notice that on contraction equation (3) yields

$$\widehat{S} + \operatorname{div}(\mathbf{G}) = \mathbf{G}^2 + \frac{1}{2}\mathbf{H}^2 - \frac{3}{2}\mathbf{H}^2 + 8\pi\left((\rho+p)\,\mathbf{u}^2 + \frac{3}{2}(\rho - p)\right)$$

where $\widehat{S}$ is the scalar curvature of the space manifold; using (1), one gets

$$
\begin{aligned}
\widehat{S} &= -\frac{3}{2}\mathbf{H}^2 + 8\pi\left((\rho+p)\,\mathbf{u}^2 + \frac{3}{2}(\rho-p)\right) + 4\pi\left(\left(2\left(u^0\right)^2 - 1\right)\rho + \left(2\left(u^0\right)^2 + 1\right)p\right)\\
&= -\frac{3}{2}\mathbf{H}^2 + 4\pi\left(2\mathbf{u}^2 + 3 + 2\left(u^0\right)^2 - 1\right)\rho + 4\pi\left(2\mathbf{u}^2 - 3 + 2\left(u^0\right)^2 + 1\right)p\\
&= -\frac{3}{2}\mathbf{H}^2 + 16\pi\left(u^0\right)^2\rho + 16\pi\left(\left(u^0\right)^2 - 1\right)p\\
&= -\frac{3}{2}\mathbf{H}^2 + 16\pi T_{00}
\end{aligned}
$$

where T_{00} is the fluid's energy density as measured by the stationary observers.

Equations (1) and (2) are analogues of the non-homogeneous Maxwell equations for stationary fields. They basically state that the source of the gravitational field $\mathbf{G}$ is proportional to

$$\rho_{matter} = \left(2\left(u^0\right)^2 - 1\right)\rho + \left(2\left(u^0\right)^2 + 1\right)p$$

whereas the source of the gravitomagnetic field $\mathbf{H}$ is proportional to

$$\mathbf{j}_{matter} = (\rho + p)\,u^0\mathbf{u}.$$

For low speeds one usually has $p << \rho$ in our units; therefore, to first order in $\mathbf{v}$ one has $\rho_{matter} = \rho$ and $\mathbf{j}_{matter} = \rho\mathbf{v}$. In other words, the gravitational field is basically generated by the fluid's mass, whereas the gravitomagnetic field is basically generated by the fluid's mass *current* with respect to the stationary observers. This completely parallels the situation in electrostatics and magnetostatics.

More interestingly, nonlinear terms occur in equations (1) and (2) (reflecting the fact that the Einstein equation is highly nonlinear), in such a way that $\mathbf{G}$ and $\mathbf{H}$ act as a source of themselves. These terms are

$$\rho_{field} = \mathbf{G}^2 + \frac{1}{2}\mathbf{H}^2$$

and

$$\mathbf{j}_{field} = 2\mathbf{G}\times\mathbf{H},$$

strikingly similar to the expressions for the energy density and energy current density of the electromagnetic field. With these definitions, equations (1) and (2) are written

$$\begin{aligned} \operatorname{div}(\mathbf{G}) &= \rho_{field} - 4\pi\rho_{matter}; \\ \operatorname{curl}(\mathbf{H}) &= \mathbf{j}_{field} - 16\pi\mathbf{j}_{matter}, \end{aligned}$$

clearly bringing out their resemblance to the non-homogeneous Maxwell equations. Notice that the source terms corresponding to the fields occur with an opposite sign to the source terms corresponding to the fluid; this is in line with the general idea that the gravitational field is always attractive and hence should have negative energy. (Actually, the energy of the gravitational field in General Relativity is much more subtle: it is a nonlocal concept, as any observer can eliminate his local gravitational field by being in free fall).

The analogy between the quasi-Maxwell form of the Einstein equation and Maxwell's equations for stationary fields is remarkable, but there are also important differences, the most obvious of which is the existence of equation (3), with no electromagnetic analogue. Notice that this equation, which is a kind of Einstein equation for the space manifold, has 6 independent components (as many as 2 vector equations), and can be written as

$$\widehat{Ric} + \widehat{\nabla} G = 8\pi \widehat{T}_{matter} - \widehat{T}_{field}$$

with

$$\widehat{T}_{matter} = (\rho + p)\,\mathbf{u} \otimes \mathbf{u} + \frac{1}{2}(\rho - p)\,\gamma$$

and

$$\widehat{T}_{field} = \frac{1}{2}\mathbf{H}^2\gamma - \mathbf{G} \otimes \mathbf{G} - \frac{1}{2}\mathbf{H} \otimes \mathbf{H}$$

(notice again the similarity with the stress tensor of the electromagnetic field).

In a way, it is hardly surprising that the analogy breaks down at some point. Electromagnetism and gravity are fundamentally different interactions (for example, they correspond to fields of different spins). What is surprising is that the analogy is so good in the first place. It is also essential to the existence of the analogy that we are dealing with stationary fields: gravitational waves essentially correspond to time-varying space metrics.

The quasi-Maxwell formalism allows one to get an immediate grasp of the physical meaning of a stationary metric from the point of view of the family of stationary observers (although if more than one such family exists the picture may change quite considerably, as we shall see). Also, it provides an alternative way of solving Einstein's equation: one postulates a metric for the space manifolds (eventually depending on one or more unknown functions) and tries to solve for the fields (eventually imposing some sort of relation between the fields' directions and the space manifold geometry). We shall see this at work presently.

1.6. **Examples.** We will now analyze a number of examples of application of the quasi-Maxwell equations.

The simplest example is clearly Minkowski spacetime. In the usual $\{t, x, y, z\}$ coordinates the line element is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

and thus $\frac{\partial}{\partial t}$ is a timelike Killing vector field. For the global time coordinate t we have $\phi = 0$, $A = 0$ and the space manifold is just Euclidean 3-space, with line element

$$dl^2 = dx^2 + dy^2 + dz^2.$$

Thus for this family of stationary observers $\mathbf{G} = \mathbf{H} = \mathbf{0}$.

Interestingly, however, Minkowski spacetime has many different Killing vector fields.

Exercise 1.21. *Show that the Killing equation $\pounds_k g = 0$ in Minkowski space reduces to*

$$\frac{\partial k_\beta}{\partial x^\alpha} + \frac{\partial k_\alpha}{\partial x^\beta} = 0.$$

Show that this equation implies that k_α is an affine function of the coordinates x^β, and then solve it. Prove that the space of all Killing vector fields is 10-dimensional, and that a basis for it is

$$\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right\}.$$

Notice that the one-parameter families of isometries generated by these Killing fields are (respectively) translations along each of the axes, boosts along each of the spatial axes and rotations about each of the spatial axes.

Exercise 1.22. *Show that making the coordinate transformation*

$$t = a \sinh u$$
$$x = a \cosh u$$

in the $t < |x|$ region of Minkowski spacetime one gets the so-called Rindler spacetime *line element*

$$ds^2 = -a^2 du^2 + da^2 + dy^2 + dz^2.$$

Show that the timelike Killing vector field $\frac{\partial}{\partial u}$ is just

$$\frac{\partial}{\partial u} = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial t}$$

and corresponds to a family of stationary observers who measure and Euclidean space manifold and $\mathbf{G} = -\frac{1}{a}\frac{\partial}{\partial a}$, $\mathbf{H} = 0$. Check that the quasi-Maxwell equations hold in this example.

Thus we see that stationary observers may measure gravitational fields in a flat spacetime. This happens when the orbits of the timelike Killing vector field corresponds to accelerated motions, which in General Relativity are locally indistinguishable from observers accelerating to oppose gravity in order to remain stationary. The stationary observers of Rindler spacetime are the relativistic analogue of an uniformly accelerated frame. Notice that while the distances between them remain constant, each observer measures a different acceleration.

Another simple kind of accelerated motion is uniform circular motion.

Exercise 1.23. *Take the Minkowski line element in cylindrical coordinates,*

$$ds^2 = -dt^2 + dr^2 + r^2 d\varphi^2 + dz^2,$$

and make the coordinate transformation $\theta = \varphi - \omega t$. *Check that in these new coordinates* $\frac{\partial}{\partial t}$ *is a timelike Killing vector field for* $r < \frac{1}{\omega}$, *corresponding to a family of uniformly rotating observers with angular velocity* ω, *and that in fact it is just*

$$\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \varphi} = \frac{\partial}{\partial t} + \omega \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$

in the old coordinates. Check that for this family of stationary observers the space manifold line element is

$$dl^2 = dr^2 + \frac{r^2}{1 - \omega^2 r^2} d\theta^2 + dz^2,$$

that

$$\mathbf{G} = \frac{\omega^2 r}{1 - \omega^2 r^2} \frac{\partial}{\partial r},$$
$$\mathbf{H} = \frac{2\omega}{1 - \omega^2 r^2} \frac{\partial}{\partial z}$$

and that the quasi-Maxwell equations hold.

Thus again accelerated stationary observers in flat spacetime measure nonzero fields. From the equations of motion one easily sees the gravitational field corresponds to the centrifugal acceleration, whereas the gravitomagnetic field is responsible for the Coriolis forces.

Notice that

$$\widehat{S} = -\frac{3}{2} \mathbf{H}^2 = -\frac{6\omega^2}{(1 - \omega^2 r^2)^2}$$

and hence the space manifold is curved, although the full spacetime is flat (Einstein used this example, which he analyzed in terms of length contraction of rulers in the tangential direction, to start thinking of curved geometries in connection with gravity). We now investigate whether the reverse is also possible:

Exercise 1.24. *Show that if the space manifold is Euclidean 3-space and no fluid is present then* $\mathbf{H} = \mathbf{0}$ *and hence the quasi-Maxwell equations reduce to*

$$\frac{\partial^2 \phi}{\partial x^i \partial x^j} = -\frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j}.$$

Show that the appropriate initial data for these equations is the value of ϕ *and its first partial derivatives at a* point, *and argue that it suffices to solve the equation for the particular case when this point is the origin and all partial derivatives but one vanish. Solve such equation and prove that all stationary vacuum spacetimes with Euclidean space manifolds are either Minkowski or Rindler spacetime.*

Thus to get curved spacetimes with Euclidean space manifolds we must introduce matter.

Exercise 1.25. *Assume that the space manifold is flat but there is a fluid present. Make the ansatz*

$$\begin{aligned} \mathbf{G} &= G \frac{\partial}{\partial x}; \\ \mathbf{H} &= H \frac{\partial}{\partial y}; \\ \mathbf{u} &= u \frac{\partial}{\partial z} \end{aligned}$$

where G, H and u are constants. Show that for each $\rho \geq 0$ the quasi-Maxwell equations have a unique solution of this form given by

$$\begin{aligned} G &= 4\sqrt{\pi\rho}; \\ H &= 4\sqrt{2\pi\rho}; \\ u &= 1; \\ p &= \rho, \end{aligned}$$

and show that the corresponding spacetime metric is

$$\begin{aligned} ds^2 &= -e^{-8\sqrt{\pi\rho}x}\left(dt + \sqrt{2}e^{4\sqrt{\pi\rho}x}dz\right)^2 + dx^2 + dy^2 + dz^2 \\ &= -\left(dz + \sqrt{2}e^{-4\sqrt{\pi\rho}x}dt\right)^2 + e^{-8\sqrt{\pi\rho}x}dt^2 + dx^2 + dy^2. \end{aligned}$$

Conclude that $\frac{\partial}{\partial z}$ is also *a timelike Killing vector field and that for the corresponding family of observers*

$$\begin{aligned} \mathbf{G} &= \mathbf{0}; \\ \mathbf{H} &= 4\sqrt{2\pi\rho}\frac{\partial}{\partial y}; \\ \mathbf{u} &= 0. \end{aligned}$$

Thus these observers are comoving with the fluid. Show that the 2-dimensional line element

$$e^{-8\sqrt{\pi\rho}x}dt^2 + dx^2$$

is that of a hyperbolic plane (and thus the comoving observers' space manifold is just a hyperbolic plane times $\mathbb{R}$).

This is the line element for the so-called *Godel universe*, which was discovered by Kurt Godel in 1949. It describes a fluid which is rotating about each of the comoving observers. This solution caused considerable unrest among physicists at the time, as it was shown by Godel to contain closed timelike curves (see [H]).

This feature could already be found in an exact solution discovered by Von Stockum in 1936:

Exercise 1.26. *Show that setting $\mathbf{G} = \mathbf{u} = \mathbf{0}$ and $p = 0$ in the quasi-Maxwell equations turns them into*

$$\begin{aligned} \mathbf{H} &= \operatorname{grad}(\psi); \\ \rho &= \frac{1}{8\pi}\mathbf{H}^2; \\ \left(\widehat{Ric}\right)_{ij} &= \frac{1}{2}H_iH_j. \end{aligned}$$

Take as line element for the space manifold

$$dl^2 = F(r)\left(dr^2 + dz^2\right) + r^2 d\varphi^2$$

where F is an arbitrary function satisfying $F(0) = 1$, and set

$$\psi = 2az$$

so that

$$\widehat{Ric} = 2a^2 dz \otimes dz.$$

Show that the quasi-Maxwell equations have the unique solution

$$F = e^{-a^2 r^2}$$

and that consequently one has the rest density

$$\rho = \frac{a^2}{2\pi} e^{a^2 r^2}$$

and the line element

$$ds^2 = -\left(dt - ar^2 d\varphi\right)^2 + e^{-a^2 r^2}\left(dr^2 + dz^2\right) + r^2 d\varphi^2.$$

This solution describes a rigidly rotating cylinder such that the gravitational attraction is exactly balanced by the centrifugal acceleration. Notice that $\frac{\partial}{\partial\varphi}$ becomes timelike for $r > \frac{1}{a}$, thus leading to closed timelike curves. In 1974, Tipler matched Von Stockum's solution to an exterior vacuum solution at $r = R < \frac{1}{a}$, thus obtaining the field *outside* a rigidly rotating cylinder of finite radius, and also got closed timelike curves there (see [T]). Using these he was able to prove that any two events outside the cylinder could be joined by a timelike curve.

The quasi-Maxwell formalism can be successfully employed to get other kinds of stationary solutions of Einstein's equation:

Exercise 1.27. *Consider a spherically symmetric space manifold,*

$$dl^2 = C^2(r)\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

and radial gravitational and gravitomagnetic fields,

$$\mathbf{G} = G(r)\,\frac{\partial}{\partial r};$$
$$\mathbf{H} = H(r)\,\frac{\partial}{\partial r}.$$

Show that there exists a two-parameter family of asymptotically flat solutions of the quasi-Maxwell vacuum equations given by

$$e^{2\phi} = 1 - \frac{2}{r^2}\left(q^2 + M\left(r^2 - q^2\right)^{\frac{1}{2}}\right);$$
$$C^2 = \left(1 - \frac{q^2}{r^2}\right)^{-1} e^{-2\phi};$$
$$H = -e^{\phi}\frac{2q}{Cr^2},$$

yielding the line element

$$ds^2 = -e^{2\phi}\left(dt - 2q\cos\theta d\varphi\right)^2 + \left(1 - \frac{q^2}{r^2}\right)^{-1} e^{-2\phi} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

This is the so-called Newman-Unti-Tamburino (NUT) solution (see [H]), and represents a gravitational monopole. Notice that for $q = 0$ it reduces to the Schwarzschild solution.

Exercise 1.28. *On a five-dimensional spacetime let* $\frac{\partial}{\partial x^4}$ *be a spacelike Killing vector field with constant norm and write the line element as*

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \widehat{g}_{44}\left(dx^4 + A_\mu dx^\mu\right)^2.$$

Use a similar method to that used to obtain the quasi-Maxwell equations to show that the 5-dimensional Einstein tensor has the components

$$\begin{aligned}\widehat{G}_{\mu 4} &= \frac{1}{2}\widehat{g}_{44}\nabla^{\alpha}F_{\alpha\mu};\\ \widehat{G}_{\mu\nu} &= G_{\mu\nu} - \frac{1}{2}\widehat{g}_{44}\left(F_{\mu\alpha}F^{\alpha}{}_{\nu} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu}\right),\end{aligned}$$

where $F = dA$.

Setting $\widehat{g}_{44} = 16\pi$ we see that the vanishing of these components is equivalent to *simultaneously* satisfying the coupled Einstein and Maxwell equations (F being interpreted as the Faraday electromagnetic tensor). This observation is the starting point of Kaluza-Klein theory unifying gravity and electromagnetism in a geometric framework.

References

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[T] Tipler, F. J., *Rotating Cylinders and the Possibility of Causality Violation*, Physical Review D 9 (15 April), 2203-2206 (1974).