TÉCNICO LISBOA

# Min-Oo's Conjecture: Positive Mass Theorem in the de Sitter Universe

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# Resumo

Depois das demonstrações de Schoen, Yau e Witten do Teorema da Massa Positiva para variedades Riemannianas assimptoticamente planas, têm sido feitos esforços no sentido de se encontrar um resultado análogo para esferas, ao qual a Conjectura de Min-Oo (o foco desta tese) inicialmente parecia uma candidata natural. Esta conjectura é, no entanto, falsa, apesar de permanecer válida sob certas formulações mais fracas, algumas das quais discutidas aqui. De qualquer forma, se alguma proposição similar sobre esferas for verdadeira, esta proibirá a existência de um buraco negro único no Universo de de Sitter, algo à partida sugerido pela solução estacionária de Schwarzschild-de Sitter. Também expomos sucintamente a ideia da refutação da conjectura, descrevemos uma possível maneira de procurar contra-exemplos explícitos e propomos uma nova conjectura para a substituir.

**Palavras-chave:** Conjectura de Min-Oo, Teorema da Massa Positiva, rigidez, métrica de Schwarzschild-de Sitter.

# Abstract

After Schoen, Yau and Witten's proofs of the Positive Mass Theorem for asymptotically flat Riemannian manifolds, efforts have been made towards finding an analogous result for spheres, to which Min-Oo's Conjecture (the focus of this thesis) initially seemed a rather natural candidate. This conjecture is, nevertheless, false, but still remains true under weaker formulations, a few of which we discuss here. In any case, if some similar statement about spheres happens to be true, it shall prohibit the existence of a unique black hole in the de Sitter Universe, something already sugested by the stationary Schwarzschild-de Sitter metric. We also briefly delineate how the conjecture was disproved, describe a possible way of searching for explicit counterexamples and propose a new conjecture to replace it.

**Keywords:** Min-Oo's Conjecture, Positive Mass Theorem, rigidity, Schwarzschild-de Sitter metric.

# Contents

# Chapter 1

# Introduction

In the past decades several theorems regarding Riemannian manifolds of non-negative scalar curvature have appeared, the most relevant of which being the Positive Mass Theorem, first proven by Richard Schoen and Shing-Tung Yau for manifolds of dimension $n \leq 7$ [15] [16] and later by Edward Witten for spin manifolds of any dimension [19]. This result states that if a complete Riemannian manifold $(M, g)$ is asymptotically flat and if its scalar curvature is non-negative, then its ADM mass [1] [2],

$$m_{ADM} = \frac{1}{2\,(n-1)\,\omega_{n-1}} \lim_{r \to +\infty} \int_{\mathbb{S}_r^{n-1}} \sum_{i,j} \left(g_{ij,i} - g_{ii,j}\right) \nu_j \; d\Omega_{\mathbb{S}_r^{n-1}} \;, \qquad (1.0.1)$$

[a scalar quantity conveniently defined so as to coincide with the mass of the Schwarzschild metric[1], with $\omega_{n-1}$ being the volume of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$, $\mathbb{S}_r^{n-1}$ the coordinate $(n-1)$-sphere of radius $r$, $d\Omega_{\mathbb{S}_r^{n-1}}$ its volume element and $\nu$ its outward-pointing unit normal] should be non-negative as well, being zero if and only if the metric is flat. This is of special and immediate importance to General Relativity as it asserts that if an energy condition assuming the non-negativity of the energy density is postulated then the total mass cannot be negative.

[1] Setting $G = c = 1$ and using isotropic coordinates:
$$ds^2 \;=\; -\left(\frac{1-\frac{M}{2r^{n-2}}}{1+\frac{M}{2r^{n-2}}}\right)^2 dt^2 \;+\; \left(1+\frac{M}{2r^{n-2}}\right)^{\frac{4}{n-2}} \left(dx_1^2 + \ldots + dx_n^2\right) \;.$$

## 1.1 Motivation

Evidently, one might be tempted to find similar propositions applicable to other kinds of differentiable manifolds, namely spheres. Of particular interest in this matter is Min-Oo's Conjecture.

Let us start by considering the $n$-dimensional closed ball of radius $r$ centered at the origin of $\mathbb{R}^n$, $\bar{\mathbb{B}}^n_r$, and equiped with a metric $g$ that satisfies the boundary conditions

$$g_{|\partial\mathbb{B}^n_r} \;=\; r^2 \, d\Omega^2_{\mathbb{S}^{n-1}} \;, \tag{1.1.1}$$

$$A \;=\; \frac{1}{r}\, g_{|\partial\mathbb{B}^n_r} \;, \tag{1.1.2}$$

(where $d\Omega^2_{\mathbb{S}^n}$ is the round metric of the unit $n$-sphere and $A$ is the extrinsic curvature[2] ) and such that the scalar curvature is $S \geq 0$. These boundary conditions allow us to glue $(\bar{\mathbb{B}}^n_r, g)$ and $(\mathbb{R}^n \backslash \bar{\mathbb{B}}^n_r, \delta)$ (where $\delta$ is the Euclidean metric) along $\mathbb{S}^{n-1}_r$ since the boundaries of both manifolds are isometric (to each other and to the round $\mathbb{S}^{n-1}_r$) and their extrinsic curvatures coincide [3], thus forming an asymptotically flat complete Riemannian manifold with non-negative scalar curvature and zero ADM mass, which is guaranteed to be $(\mathbb{R}^n, \delta)$ by the Positive Mass Theorem. This implies that $g = \delta$.

This result could have an analogue in the case of the round sphere $\left(\mathbb{S}^n, dr^2 + \sin^2 r \; d\Omega^2_{\mathbb{S}^{n-1}}\right)$, with scalar curvature $S = n(n-1)$. In this situation we have:

[2] We use the definition $A := \frac{1}{2}\, \mathcal{L}_\nu \, g$, where $\mathcal{L}$ represents the Lie derivative and $\nu$ is the outward-pointing unit normal.
It is worth mentioning the relation between $A$ and the second fundamental form $B$. It is well-known that $B(X,Y) \;=\; -\left\langle \nabla_X \nu, Y \right\rangle \nu \;=\; -\left\langle \nabla_Y \nu, X\right\rangle \nu$. On the other hand, recalling the metric compatibility of the Levi-Civita connection,

$$\begin{aligned} A(X,Y) \;&=\; \frac{1}{2}\mathcal{L}_\nu \, g(X,Y) \;=\; \frac{1}{2}\left[\, \nu \cdot g(X,Y) - g(\mathcal{L}_\nu X, Y) - g(X, \mathcal{L}_\nu Y)\,\right] \;= \\ &=\; \frac{1}{2}\left[\, \nu \cdot g(X,Y) - g(\nabla_\nu X - \nabla_X \nu, Y) - g(X, \nabla_\nu Y - \nabla_Y \nu)\,\right] \;= \\ &=\; \frac{1}{2}\left(\, \langle \nabla_X \nu, Y\rangle + \langle \nabla_Y \nu, X\rangle \,\right) \;=\; \langle \nabla_X \nu, Y \rangle \;. \end{aligned} \tag{1.1.3}$$

We therefore have $A_{ij} = \left\langle \nabla_{X_i} \nu, X_j \right\rangle = \nabla\nu\,(X_i, X_j) = \nabla_i \, \nu_j$.

$$g_{|\partial B_r(0)} \;=\; \sin^2 r \, d\Omega^2_{\mathbb{S}^{n-1}} \;, \tag{1.1.4}$$

$$A \;=\; \operatorname{cotan} r \, g_{|\partial B_r(0)} \;, \tag{1.1.5}$$

where $B_r(0)$ denotes the geodesic ball of radius $r$ centered at the north pole ($r = 0$). Though the analogy definitely fails when $r > \pi/2$ (this is intuitively clear if one thinks of how a spherical cap smaller than the hemisphere can be completed by a manifold with boundary that "curves faster" than a sphere and, as such, has greater scalar curvature), the highly non-trivial situation where $r = \pi/2$ appears reasonable, and that is precisely why it seems natural to establish the following conjecture, proposed by Min-Oo in 1995 [14], and regard it as a good candidate for a Positive Mass Theorem analogue for spheres:

**Conjecture 1.1.1. (Min-Oo's Conjecture)** *Let $(M, g)$ be an $n$-dimensional compact Riemannian manifold with boundary. Suppose $(\partial M, i^* g)$ [3] is isometric to $\left(\mathbb{S}^{n-1}, d\Omega^2_{\mathbb{S}^{n-1}}\right)$ and totally geodesic[4] in $(M, g)$. If $(M, g)$ has scalar curvature $S \geq n(n-1)$, then it is isometric to $\left(\mathbb{S}^n_+, d\Omega^2_{\mathbb{S}^n_+}\right)$.*

However, despite holding true for a number of quite important particular subcases – some of which shall be discussed in the next chapter – and somewhat unexpectedly, it was proved to be false by Simon Brendle, Fernando Codá Marques and André Neves [5] [13], as we will later see. But not before stressing the relevance of this study to Cosmology.

## 1.2 Black holes in the de Sitter universe

The $n$-sphere represents the spatial component of the de Sitter universe, which itself is the maximally symmetric vacuum solution of the Einstein Field Equations with a positive cos-

[3] The notation $i^* g$ represents the pullback of $g$ by the inclusion map $i : \partial M \hookrightarrow M$.

[4] That is, that the geodesics of $\partial M$ are geodesics of $M$. This is equivalent to having $A \equiv 0$.

mological constant ($\Lambda > 0$). Specifically, the case where $n = 3$ has obvious interest to Physics. The argument presented in this section is built on the hypothesis that the effects of a gravitational field on a spherical space should in some way resemble the effects of an electric field. Also, from now on, and unless otherwise indicated, the Einstein summation convention is assumed.

Let us start from Laplace's equation and work out the fields created by a point charge in the three-dimensional flat and spherical spaces equiped with their respective standard metrics.

In $\left(\mathbb{R}^3, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$, and assuming $\Phi = \Phi(r)$,

$$\Delta\Phi \;=\; 0 \;\Leftrightarrow \tag{1.2.1}$$

$$\Leftrightarrow \frac{1}{\sqrt{\det g}}\,\partial_i\left(\sqrt{\det g}\,\partial^i\Phi\right) \;=\; 0 \;\Leftrightarrow$$

$$\Leftrightarrow \partial_r\left(r^2\;\sin\theta\,\partial_r\Phi\right) \;=\; 0 \;\Leftrightarrow$$

$$\Leftrightarrow \partial_r\left(r^2\,\partial_r\Phi\right) \;=\; 0 \;\Leftrightarrow$$

$$\Leftrightarrow \Phi \;=\; \frac{a}{r} + b\,, \qquad a, b \in \mathbb{R}\,. \tag{1.2.2}$$

Similarly, in $\left(\mathbb{S}^3, dr^2 + \sin^2 r\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right)$, and again assuming $\Phi = \Phi(r)$,

$$\Delta\Phi \;=\; 0 \;\Leftrightarrow$$

$$\Leftrightarrow \frac{1}{\sqrt{\det g}}\,\partial_i\left(\sqrt{\det g}\,\partial^i\Phi\right) \;=\; 0 \;\Leftrightarrow$$

$$\Leftrightarrow \partial_r\left(\sin^2 r\;\sin\theta\,\partial_r\Phi\right) \;=\; 0 \;\Leftrightarrow$$

$$\Leftrightarrow \partial_r\left(\sin^2 r\,\partial_r\,\Phi\right) \;=\; 0 \;\Leftrightarrow$$

$$\Leftrightarrow \Phi \;=\; a\,\mathrm{cotan}\,r + b\,, \qquad a, b \in \mathbb{R}\,. \tag{1.2.3}$$

In the flat space the potential possesses only one singularity at $r = 0$, but in the spherical setting there are two (and of opposite signs, indicating the presence of point charges of opposite signs): one at $r = 0$ and another at $r = \pi$, that is, one at each pole. Which makes sense because according to Gauss's Law we have

$$\int_{\mathbb{S}^3} \mathrm{div}\vec{E} \;=\; \int_{\mathbb{S}^3} \rho \,, \tag{1.2.4}$$

and since the boundary of a sphere is empty the total charge must be zero.

With gravity, a similar phenomenon occurs. Let us recall the Einstein Field Equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} \;=\; (n-1)\,\omega_{n-1}\,T_{\mu\nu} \,, \tag{1.2.5}$$

where $G = \mathrm{Ric} - \frac{1}{2}\,S\,g$ is the Einstein tensor and $T$ the energy-momentum tensor. The energy density is $\rho = T_{00}$ and we can make a choice of units that sets $\Lambda = n(n-1)/2$ in such a way that the simplest solution, the de Sitter metric, takes the form

$$ds^2 = -dt^2 + \cosh^2(t)\,d\Omega^2_{\mathbb{S}^n} \,. \tag{1.2.6}$$

If we take the time slice $\{t = 0\}$ then the spatial component becomes isometric to the round $\mathbb{S}^n$ and also totally geodesic[5]. In addition, it can be shown [18] that

$$G_{00} = \frac{1}{2}\left[S + (A^i_i)^2 - A_{ij}A^{ij}\right] \,, \tag{1.2.7}$$

where $S$ is the scalar curvature of the spacetime's spatial component and $A$ is its extrinsic curvature. In the end, combining (1.2.5) and (1.2.7) we get

$$S \;=\; n(n-1) \;+\; 2(n-1)\,\omega_{n-1}\,\rho \,. \tag{1.2.8}$$

Much like in the remarkable and previously mentioned consequence of the Positive Mass Theorem, if we posit that $\rho \geq 0$, then $S \geq n(n-1)$. In other words, by putting matter in a neighbourhood of a point (which can be taken as the north pole without loss of generality) in the de Sitter universe, we increase the scalar curvature locally and, if the amount of matter is small enough, by increasing the density sufficiently, a black hole will eventually form.

[5] Because $ds^2|_{\{t=0\}} = d\Omega^2_{\mathbb{S}^n}$ and $A = \frac{1}{2}\mathcal{L}_{\frac{\partial}{\partial t}|_{t=0}} ds^2 = \sinh(0)\cosh(0)\,d\Omega^2_{\mathbb{S}^n} = 0$.

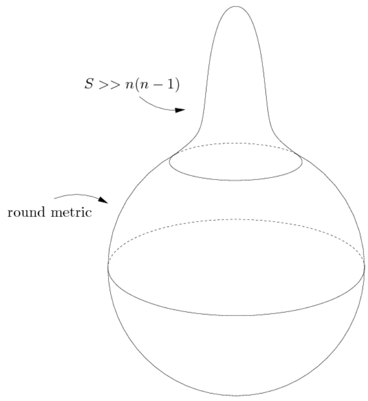


Figure 1.1: Sketch of the scalar curvature in the presence of matter leading to the formation of a single black hole.

Now the Schwarzschild-de Sitter solution[6] with $0 < M < \frac{1}{n}\left[\frac{(n-1)(n-2)}{2\Lambda}\right]^{\frac{n-2}{2}}$, describing stationary black holes in the spherical de Sitter universe, predicts the existence of two horizons, each of them surrounding one of the poles, hence, presumably, the initial conditions leading to a single black hole, that is, a single local deformation that increases the scalar curvature, should be forbidden. In fact, forcing the southern hemisphere, boundary included, to have the round metric (which is physically equivalent to establishing the initial condition of eliminating all traces of matter and gravitational waves in that region), Min-Oo's Conjecture, when true, prohibits the existence of a single black hole.

6 $ds^2 = -Vdt^2 + Vdr^2 + r^2 d\Omega^2_{\mathbb{S}^{n-1}}$, where $V = 1 - \frac{2\Lambda}{n(n-1)}r^2 - \frac{2M}{r^{n-2}}$ and $\Lambda > 0$.

# Chapter 2

# Some partial results

In what follows we present four particular cases where Min-Oo's Conjecture holds: the case where the manifold is the graph of a function (with a proof by Lan-Hsuan Huang and Damin Wu [11]) and the cases where the metric is spherically symmetric or can be obtained by a conformal transformation (original proof by Fengbo Hang and Xiaodong Wang [9]) or a one-parameter variation of the round metric (for all of which we offer a new proof).

## 2.1 Graph of a function

**Theorem 2.1.1.** *Consider a function $f \in C^2(\mathbb{B}^n) \cap C^0(\bar{\mathbb{B}}^n)$, with $n \geq 2$ and where $\mathbb{B}^n$ is the $n$-dimensional unit ball centered at the origin of $\mathbb{R}^n$, and let $M$ be the graph of $f$ in $\mathbb{R}^{n+1}$. If $(M, \delta + df^2)$ has induced scalar curvature $S \geq n(n-1)$, then it is isometric to $\left(\mathbb{S}^n_+, d\Omega^2_{\mathbb{S}^n_+}\right)$.*

*Proof.* Let $M$ be the graph of $f$, that is,

$$M = \left\{ \left(x^1, \ldots, x^n, x^{n+1}\right) \in \bar{\mathbb{B}}^n \times \mathbb{R} : \ x^{n+1} = f\left(x^1, \ldots, x^n\right) \right\} . \qquad (2.1.1)$$

It is easy to notice that the upward-pointing unit normal is $\nu = \frac{(-\nabla f,\, 1)}{\sqrt{1+\|\nabla f\|^2}}$ and, using (1.1.3), we find the sum of the principal curvatures (the eigenvalues of $\langle B, \nu \rangle = -A$, which

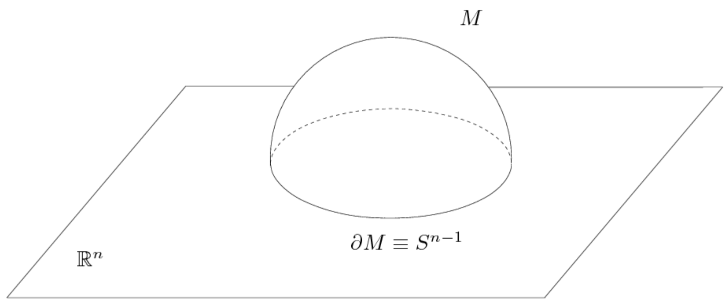


Figure 2.1: $M$ is the graph of $f : \bar{\mathbb{B}}^n \to \mathbb{R}$.

we denote by $k_i$ , $i = 1, \ldots, n$) to be

$$H_0 \;:=\; \sum_{i=1}^{n} k_i \;=\; -\mathrm{tr}\, A \;=\; -\mathrm{div}\, \nu \;=\; -\sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left( \frac{\partial f / \partial x^i}{\sqrt{1 + \|\nabla f\|^2}} \right) . \tag{2.1.2}$$

On the other hand, we can take an orthonormal field of frames tangent to $M$, $\{E_i\}_{i=1}^n$, and let $\Pi_{ij}$ be the two-dimensional subspace of the tangent space generated by $\{E_i, E_j\}$, $i \neq j$; the sectional curvature of $\Pi_{ij}$ is

$$K(\Pi_{ij}) \;=\; R_{ijij} \;=\; A_{ii}A_{jj} - A_{ij}^2 \;=\; k_i k_j \tag{2.1.3}$$

(where the Einstein summation convention is not being used). The scalar curvature then becomes

$$S \;=\; \sum_{i \neq j} k_i k_j \;=\; 2 \sum_{i<j} k_i k_j \;, \tag{2.1.4}$$

and so we can write

$$H_0^2 \;=\; S + \mathrm{tr}\left(A^2\right) \;. \tag{2.1.5}$$

We also define $\bar{A}$ to be the traceless part of the extrinsic curvature; explicitly,

$$\left(\bar{A}\right)_{ij} = (A)_{ij} + \frac{H_0}{n}\delta_{ij} \;. \tag{2.1.6}$$

Inserting (2.1.6) into (2.1.5) yields

$$\left(\frac{H_0}{n}\right)^2 = \frac{S + \operatorname{tr}\left(\bar{A}^2\right)}{n(n-1)} \ . \tag{2.1.7}$$

Since $S \geq n(n-1)$, we have $H_0^2 \geq n^2$, which by continuity implies either $H_0 \geq n$ everywhere or $H_0 \leq -n$ everywhere.

However, using (2.1.2) and the divergence theorem, we have

$$\begin{aligned} \left|\int_{\mathbb{B}^n_a} H_0 \, d^n x\right| \ &= \ \left|\int_{\partial\mathbb{B}^n_a} \frac{\nabla f \cdot y/a}{\sqrt{1+\|\nabla f\|^2}} \, d\Omega_{\mathbb{S}^{n-1}_a}(y)\right| \ \leq \\ &\leq \ \int_{\partial\mathbb{B}^n_a} \frac{\|\nabla f\|}{\sqrt{1+\|\nabla f\|^2}} \, d\Omega_{\mathbb{S}^{n-1}_a}(y) \ \leq \\ &\leq \ \operatorname{vol}(\partial\mathbb{B}^n_a) \ = \ \frac{n}{a} \operatorname{vol}(\mathbb{B}^n_a) \ . \end{aligned} \tag{2.1.8}$$

Taking the limit when $a \to 1$,

$$\left|\int_{\mathbb{B}^n} H_0 \, d^n x\right| \ \leq \ n \operatorname{vol}(\mathbb{B}^n) \ , \tag{2.1.9}$$

and we conclude that either $H_0 = n$ everywhere or $H_0 = -n$ everywhere; we can assume the former without loss of generality, as that depends solely on whether $\nu$ is outward or inward-pointing (and simply makes $M$ lie above or below $\mathbb{R}^n \times \{0\}$, respectively, having no practical importance whatsoever). Thus, from (2.1.7), $\bar{A} \equiv 0$ and, as such, $M$ is totally umbillic with all the principal curvatures identically equal to 1, therefore being isometric to $\mathbb{S}^n_+$.

□

*Remark* 2.1.2. It should be noted that theorem 2.1.1 does not require the boundary to be totally geodesic and therefore, in this sense, it is even stronger than Min-Oo's Conjecture.

## 2.2 Spherically symmetric Riemannian manifold

Before we proceed to prove the theorem, we start with the definition of the kind of manifold we refer to.

**Definition 2.2.1.** A Riemannian manifold $\left(M, ds^2\right)$ is said to be spherically symmetric if $ds^2 = dr^2 + R^2(r)\ d\Omega^2_{\mathbb{S}^{n-1}}$, for some non-negative differentiable function $R$.

**Theorem 2.2.2.** *Let $\left(M, ds^2\right)$ be an $n$-dimensional (with $n \geq 2$), compact, spherically symmetric Riemannian manifold whose boundary $\left(\partial M, i^* ds^2\right)$ is isometric to $\left(\mathbb{S}^{n-1}, d\Omega^2_{\mathbb{S}^{n-1}}\right)$ and totally geodesic in $\left(M, ds^2\right)$. If $\left(M, ds^2\right)$ has scalar curvature $S \geq n(n-1)$, then it is isometric to $\left(\mathbb{S}^n_+, d\Omega^2_{\mathbb{S}^n_+}\right)$.*

*Proof.* Since $M$ is a compact, smooth and spherically symmetric manifold, its metric is of the form $ds^2 = dr^2 + R^2(r)\ d\Omega^2_{\mathbb{S}^{n-1}}$, where $R$ is a non-negative differentiable function whose domain is $[0, r_0]$ for some $r_0 > 0$ and such that $R(0) = 1$, $R(r_0) = 0$ and $R > 0$ everywhere else.

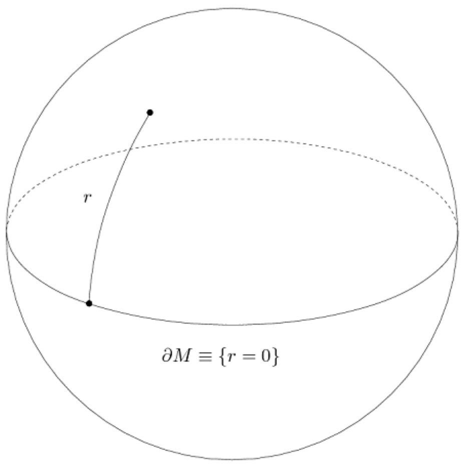


Figure 2.2: A spherically symmetric manifold.

Also, the volume of a geodesic ball approaches that of a Euclidean ball when the radius goes to 0, meaning that

$$\lim_{h \to 0} \frac{R^n(r_0 - h)}{h^n} \;=\; 1 \;\;\Leftrightarrow$$

$$\Leftrightarrow\;\; \lim_{h \to 0} \frac{R(r_0 - h)}{h} \;=\; 1 \;\;\Leftrightarrow$$

$$\Leftrightarrow\;\; R'(r_0) \;=\; -1\,. \tag{2.2.1}$$

By hypothesis, the extrinsic curvature of the boundary, which is given by

$$A \;=\; \frac{1}{2}\mathcal{L}_{\frac{\partial}{\partial r}} ds^2 \;=\; R(0)R'(0)\, d\Omega^2_{S^{n-1}}\,, \tag{2.2.2}$$

must be zero, and so $R'(0) = 0$ as well.

Next, by straightforward computation it is easy to see that the scalar curvature is

$$S \;=\; -2(n-1)\frac{R''}{R} \;+\; (n-1)(n-2)\frac{1-(R')^2}{R^2}\,. \tag{2.2.3}$$

Since $S \geq n(n-1)$, and defining $\rho = 1 - (R')^2$ which therefore gives $\rho' = -2R'R''$, we have

$$-\,2(n-1)\frac{R''}{R} \;+\; (n-1)(n-2)\frac{1-(R')^2}{R^2} \;\geq\; n(n-1) \;\;\Leftrightarrow \tag{2.2.4}$$

$$\Leftrightarrow\;\; (n-2)(n-1)\rho \;-\; 2(n-1)RR'' \;\geq\; n(n-1)R^2 \;\;\Leftrightarrow$$

$$\Leftrightarrow\;\; (n-2)\rho \;-\; 2RR'' \;\geq\; nR^2\,. \tag{2.2.5}$$

Replacing the values of $R(0)$ and $R'(0)$ in (2.2.4), we find that $R''(0) \leq -1$. As such, $R'$ is decreasing when $r = 0$ and negative in $]0, \epsilon[$ for some $\epsilon > 0$.

Then, in $[0, \epsilon]$, we have

$$(n-2)\rho R' \; - \; 2RR'R'' \; \leq \; nR'R^2 \;\; \Leftrightarrow \tag{2.2.6}$$

$$\Leftrightarrow \; (n-2)\rho R' \; + \; R\rho' \; \leq \; nR'R^2 \;\; \Leftrightarrow$$

$$\Leftrightarrow \; (n-2)R^{n-3}R'\rho \; + \; R^{n-2}\rho' \; \leq \; nR^{n-1}R' \;\; \Leftrightarrow$$

$$\Leftrightarrow \; \left(R^{n-2}\rho\right)' \; \leq \; (R^n)' \; . \tag{2.2.7}$$

Integrating over $[0, r] \subseteq [0, \epsilon]$,

$$R^{n-2}\rho - 1 \; \leq \; R^n - 1 \;\; \Leftrightarrow \tag{2.2.8}$$

$$\Leftrightarrow \; R^{n-2}\rho \; \leq \; R^n \;\; \Leftrightarrow$$

$$\Leftrightarrow \; R^{n-2}\left[1 - (R')^2\right] \; \leq \; R^n \;\; \Leftrightarrow$$

$$\Leftrightarrow \; 1 - (R')^2 \; \leq \; R^2 \;\; \Leftrightarrow$$

$$\Leftrightarrow \; R^2 + (R')^2 \; \geq \; 1 \; . \tag{2.2.9}$$

Therefore, $R' \leq -\sqrt{1 - R^2}$ in $[0, \epsilon]$, which means $R$ must decrease until it reaches $0$, that is, until $r = r_0$. This way, expression (2.2.9) is valid in $[0, r_0]$ and it now suffices to show it actually becomes an equality.

Let us assume that at some point $r' \in \,]0, r_0[$ the strict inequality $R^2 + (R')^2 > 1$ is observed. Integrating expression (2.2.7) between $r'$ and $r \in [r', r_0]$, yields

$$R^{n-2}(r)\rho(r) - R^{n-2}(r')\,\rho(r') \; \leq \; R^n(r) - R^n(r') \;\; \Leftrightarrow$$

$$\Leftrightarrow \; R^{n-2}(r')\left[R^2(r') - \rho(r')\right] \; \leq \; R^{n-2}(r)\left[R^2(r) - \rho(r)\right] \; . \tag{2.2.10}$$

As it stands, the left-hand side of (2.2.10) is a positive constant and the left-hand side goes to 0 as $r \to r_0$, which is a contradiction. Consequently,

$$\begin{aligned} R^2 + (R')^2 &= 1 \;\Leftrightarrow \\ \Leftrightarrow\; -\frac{R'}{\sqrt{1-R^2}} &= 1 \;\Leftrightarrow \\ \Leftrightarrow\; R(r) &= \cos r\,, \end{aligned} \tag{2.2.11}$$

rendering $ds^2$ equal to the standard metric of the sphere, thus concluding the proof.

□

## 2.3 Riemannian manifold conformal to the round hemisphere

**Theorem 2.3.1.** *Let $(M, g) = \left(\mathbb{S}^n_+, g\right)$ be an $n$-dimensional (with $n \geq 2$) compact Riemannian manifold conformal*[7] *to $\left(\mathbb{S}^n_+, d\Omega^2_{\mathbb{S}^n_+}\right)$ and whose boundary $(\partial M, i^* g)$ is isometric to $\left(\mathbb{S}^{n-1}, d\Omega^2_{\mathbb{S}^{n-1}}\right)$ and totally geodesic in $(M, g)$. If $(M, g)$ has scalar curvature $S \geq n(n-1)$, then it is isometric to $\left(\mathbb{S}^n_+, d\Omega^2_{\mathbb{S}^n_+}\right)$.*

*Proof.* Let us consider $X, Y \in \mathfrak{X}(M)$ and the outward-pointing unit normal $\nu = x$. Denoting by $\nabla$ the Levi-Civita connection on $M$, we have

[7] Which is to say that $g$ and $d\Omega^2_{\mathbb{S}^n_+}$ are conformally equivalent or, in other words, that there exists a positive differentiable function $p$ such that $g = p\, d\Omega^2_{\mathbb{S}^n_+}$.

$$\begin{aligned}
\nabla\left(dx^{n+1}\right)(X,Y) \;&=\; \left(\nabla_X dx^{n+1}\right)(Y) \;= \\
&=\; X\cdot\left[dx^{n+1}(Y)\right] \;-\; dx^{n+1}\left(\nabla_X Y\right) \;= \\
&=\; X\cdot Y^{n+1} \;-\; dx^{n+1}\left(\, X\cdot Y \;-\; \langle X\cdot Y,\nu\rangle\,\nu\,\right) \;= \\
&=\; X\cdot Y^{n+1} \;-\; X\cdot Y^{n+1} \;+\; \langle X\cdot Y,\nu\rangle\;\nu^{n+1} \;= \\
&=\; \left(\, X\cdot\langle Y,\nu\rangle \;-\; \langle X\cdot\nu, Y\rangle\,\right)\nu^{n+1} \;= \\
&=\; -\langle X\cdot\nu, Y\rangle\;\nu^{n+1} \;=\; -\left\langle X^i\partial_i\, x^j\partial_j, Y\right\rangle\; x^{n+1} \;= \\
&=\; -\langle X,Y\rangle\; x^{n+1}\,. \qquad (2.3.1)
\end{aligned}$$

Put differently, $\nabla dx^{n+1} = -x^{n+1}g$ . Taking the trace,

$$(\Delta+n)\,x^{n+1} \;=\; 0\,. \qquad (2.3.2)$$

It is also useful to note the Laplacian's self-adjointness:

$$\begin{aligned}
\int_{\mathbb{S}^n} w\;\Delta x^{n+1} \;&=\; \int_{\mathbb{S}^n} w \operatorname{div}\operatorname{grad} x^{n+1} \;= \\
&=\; \int_{\mathbb{S}^n}\left[\,\operatorname{div}\left(w\operatorname{grad} x^{n+1}\right) - \left\langle \operatorname{grad} w, \operatorname{grad} x^{n+1}\right\rangle\,\right] \;= \\
&=\; -\int_{\mathbb{S}^n}\left\langle \operatorname{grad} w, \operatorname{grad} x^{n+1}\right\rangle \;= \\
&=\; \int_{\mathbb{S}^n}\left[\,\operatorname{div}\left(x^{n+1}\operatorname{grad} w\right) - \left\langle \operatorname{grad} x^{n+1}, \operatorname{grad} w\right\rangle\,\right] \;= \\
&=\; \int_{\mathbb{S}^n} x^{n+1}\operatorname{div}\operatorname{grad} w \;=\; \int_{\mathbb{S}^n} x^{n+1}\;\Delta w\,. \qquad (2.3.3)
\end{aligned}$$

Let us first consider the case where $n = 2$, and equip $\mathbb{S}^2$ with the metric $g = e^{2w} d\Omega^2_{\mathbb{S}^2}$ where $w$ is a smooth function in $\mathbb{S}^2$ such that $w = 0$ in $\mathbb{S}^2_-$ (this is possible because the metric is continuous and the extrinsic curvature of the equator is well-defined, or equivalently, both hemispheres can be matched into a Riemannian manifold). Computing the Gauss curvature $K$ of $\left(\mathbb{S}^n_+, g\right)$, we derive the following formula:

$$\Delta w \;+\; K e^{2w} \;=\; 1\,. \tag{2.3.4}$$

Thus, combining (2.3.2) and (2.3.3) and recalling that $K = S/2 \geq 1$, we have

$$\begin{aligned}
0 \;&=\; \int_{\mathbb{S}^2} w\,(\Delta + 2)\,x^3 \;= \\
&=\; \int_{\mathbb{S}^2} x^3\,(\Delta + 2)\,w \;= \\
&=\; \int_{\mathbb{S}^2_-} x^3\,(\Delta + 2)\,w \;+\; \int_{\mathbb{S}^2_+} x^3\,(\Delta + 2)\,w \;= \\
&=\; \int_{\mathbb{S}^2_+} \left[\,2\,x^3\,w + x^3\left(1 - K e^{2w}\right)\,\right] \;= \\
&=\; \int_{\mathbb{S}^2_+} \left(1 + 2w - K e^{2w}\right) x^3 \;\leq \\
&\leq\; \int_{\mathbb{S}^2_+} \left(1 + 2w - e^{2w}\right) x^3 \;\leq\; 0 \;\;\Rightarrow \qquad (2.3.5) \\
&\Rightarrow\;\; w \;=\; 0 \;\;\Leftrightarrow \\
&\Leftrightarrow\;\; g \;=\; d\Omega^2_{\mathbb{S}^2}\,. \qquad (2.3.6)
\end{aligned}$$

*Remark* 2.3.2. This proves that if a Riemannian metric on a 2-sphere has $K \geq 1$ and if the length of one of its geodesics is $2\pi$, then that sphere is isometric to the round 2-sphere. This statement, known as Toponogov's theorem, is essentially equivalent to Min-Oo's Conjecture for $n = 2$.

Let us now look at the case where $n > 2$. This time we set $g = u^{\frac{4}{n-2}}\ d\Omega^2_{\mathbb{S}^n}$, where $u$ is a positive smooth function such that $u = 1$ on $\mathbb{S}^n_-$ (again, this is possible since the metric is continous and the extrinsic curvature of the equator is well-defined). Computing the scalar curvature $S \geq n(n-1)$, one obtains

$$\frac{4(n-1)}{n-2}\,\Delta u \;+\; S\,u^{\frac{n+2}{n-2}} \;=\; n(n-1)\,u\,. \tag{2.3.7}$$

So, again from (2.3.2) and (2.3.3),

$$\begin{aligned}
0 \;&=\; \int_{\mathbb{S}^n} u\,(\Delta + n)\,x^{n+1} \;= \\
&=\; \int_{\mathbb{S}^n} x^{n+1}\,(\Delta + n)\,u \;= \\
&=\; \int_{\mathbb{S}^n} \left\{ n\,x^{n+1}\,u \;+\; x^{n+1}\,\frac{n-2}{4(n-1)} \left[ n(n-1)\,u \;-\; S\,u^{\frac{n+2}{n-2}} \right] \right\} \;= \\
&=\; \int_{\mathbb{S}^n} \left[ n\,x^{n+1}\,u \;+\; n\,x^{n+1}\,\frac{n-2}{4}u \;-\; \frac{S(n-2)}{4(n-1)}\,x^{n+1}\,u^{\frac{n+2}{n-2}} \right] \;= \\
&=\; \int_{\mathbb{S}^n} \left[ \left(1 + \frac{n-2}{4}\right) n\,x^{n+1}\,u \;-\; \frac{S(n-2)}{4(n-1)}\,x^{n+1}\,u^{\frac{n+2}{n-2}} \right] \;= \\
&=\; \int_{\mathbb{S}^n} \left[ \frac{n+2}{4}\,n\,x^{n+1}\,u \;-\; \frac{S(n-2)}{4(n-1)}\,x^{n+1}\,u^{\frac{n+2}{n-2}} \right] \;= \\
&=\; \int_{\mathbb{S}^n_-} n\,x^{n+1} \;+\; \int_{\mathbb{S}^n_+} \left[ \frac{n+2}{4}\,n\,x^{n+1}\,u \;-\; \frac{S(n-2)}{4(n-1)}\,x^{n+1}\,u^{\frac{n+2}{n-2}} \right] \;\leq \\
&\leq\; \int_{\mathbb{S}^n_-} n\,x^{n+1} \;+\; \int_{\mathbb{S}^n_+} \left[ \frac{n+2}{4}\,n\,x^{n+1}\,u \;-\; \frac{n(n-2)}{4}\,x^{n+1}\,u^{\frac{n+2}{n-2}} \right] \;= \\
&=\; \int_{\mathbb{S}^n_+} \left( \frac{n+2}{4}u \;-\; \frac{n-2}{4}\,u^{\frac{n+2}{n-2}} \;-\; 1 \right) n\,x^{n+1}\,.
\end{aligned} \tag{2.3.8}$$

Now we look at the integrand and define function $h$ for $x \geq 0$, calculate its first and second order derivatives and evaluate them at $x = 1$:

$$h(x) := \frac{n+2}{4}x - \frac{n-2}{4}\, x^{\frac{n+2}{n-2}} - 1\,,$$
$$h(1) = 0\,; \tag{2.3.9}$$

$$h'(x) = \frac{n+2}{4} - \frac{n+2}{4}\, x^{\frac{4}{n-2}}\,,$$
$$h'(1) = 0\,; \tag{2.3.10}$$

$$h''(x) = -\frac{n+2}{n-2}\, x^{\frac{6-n}{n-2}}\,,$$
$$h''(1) = -\frac{n+2}{n-2} \;<\; 0\,. \tag{2.3.11}$$

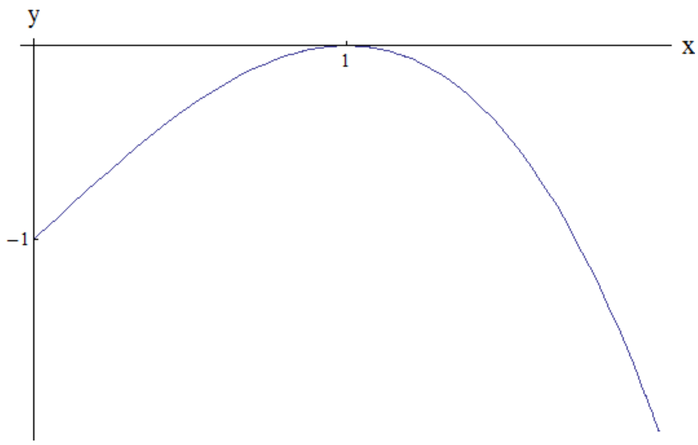


Figure 2.3: The graph of $h$.

This being the case, expression (2.3.8) can be rewritten as

$$0 \;\leq\; \int_{\mathbb{S}^n_+} h(u)\, n\, x^{n+1}\,, \tag{2.3.12}$$

where $h(x) \leq 0 \;\; \forall x \geq 0$ and with equality holding if and only if $x = 1$, ultimately implying that $u = 1$ and, of course, $g = d\Omega^2_{\mathbb{S}^n}$ , as was to be shown.

□

## 2.4 One-parameter variations of the round metric

This section deals with one-parameter deformations on the hemisphere and it can be understood as a linearized version of the conjecture, as it asserts that the first order term in the scalar curvature's series expansion has to be zero.

**Theorem 2.4.1.** *Let $g(\lambda)$ be a one-parameter family of metrics on the $n$-sphere such that $g(0) = d\Omega^2_{\mathbb{S}^n}$ and define $\delta \equiv \frac{d}{d\lambda}_{|\lambda=0}$. Suppose $\delta S \geq 0$ . If $\delta S = 0$ in $\mathbb{S}^n_-$ , then $\delta S = 0$ in $\mathbb{S}^n_+$.*

*Proof.* In this situation, Wald [18] gives the following expression for the variation in the scalar curvature at $\lambda = 0$:

$$\delta S \;=\; \mathrm{Ric}_{ab}\, \delta\left(g^{ab}\right) \;-\; \nabla_c \nabla^c\, \delta\left(g^d_d\right) \;+\; \nabla^c \nabla^d\, \delta\left(g_{cd}\right)\;. \tag{2.4.1}$$

We define $\bar{g}$ as the traceless part of the metric and, since $\mathrm{Ric}_{ab} = (n-1)\, g_{ab}$, we have

$$\begin{aligned}
\delta S \;&=\; (n-1)\, g_{ab}\, \delta\left(g^{ab}\right) \;-\; \nabla_c \nabla^c\, \delta\left(g^d_d\right) \;+\; \nabla^c \nabla^d\, \delta\left(g_{cd}\right) \;= \\
&=\; -(n-1)\, \delta\left(g^d_d\right) \;-\; \nabla_c \nabla^c\, \delta\left(g^d_d\right) \;+\; \nabla^c \nabla^d\, \delta\left(g_{cd}\right) \;= \\
&=\; -(n-1)\, \delta\left(g^d_d\right) \;-\; \nabla_c \nabla^c\, \delta\left(g^d_d\right) \;+\; \nabla^c \nabla^d \left[\delta\left(\bar{g}_{cd}\right) + \frac{1}{n}\delta\left(g^d_d\right) g_{cd}\right] \;= \\
&=\; -(n-1)\, \delta\left(g^d_d\right) \;-\; \left(1 - \frac{1}{n}\right) \nabla_c \nabla^c\, \delta\left(g^d_d\right) \;+\; \nabla^c \nabla^d\, \delta\left(\bar{g}_{cd}\right) \;= \\
&=\; -\frac{n-1}{n}\left(\Delta + n\right) \delta\left(g^d_d\right) \;+\; \nabla^c \nabla^d\, \delta\left(\bar{g}_{cd}\right)\;.
\end{aligned} \tag{2.4.2}$$

Recalling (2.3.1), (2.3.2) and (2.3.3) we can write

$$
\begin{aligned}
\int_{\mathbb{S}^n} x^{n+1}\,\delta S \;&=\; \int_{\mathbb{S}^n} x^{n+1}\,\left[-\frac{n-1}{n}\,(\Delta+n)\,\delta\left(g^d_d\right) \;+\; \nabla^c\nabla^d\,\delta\left(\bar{g}_{cd}\right)\right] \;= \\
&=\; \int_{\mathbb{S}^n} x^{n+1}\,\nabla^c\nabla^d\,\delta\left(\bar{g}_{cd}\right) \;= \\
&=\; \int_{\mathbb{S}^n} \left\{\nabla^c\left[x^{n+1}\,\nabla^d\,\delta\left(\bar{g}_{cd}\right)\right] \;-\; \left(\nabla^c\,x^{n+1}\right)\left[\nabla^d\,\delta\left(\bar{g}_{cd}\right)\right]\right\} \;= \\
&=\; \int_{\mathbb{S}^n} -\;\left(\nabla^c\,x^{n+1}\right)\left[\nabla^d\,\delta\left(\bar{g}_{cd}\right)\right] \;= \\
&=\; \int_{\mathbb{S}^n} \left\{-\,\nabla^d\left[\left(\nabla^c\,x^{n+1}\right)\delta\left(\bar{g}_{cd}\right)\right] \;+\; \delta\left(\bar{g}_{cd}\right)\,\nabla^d\nabla^c\,x^{n+1}\right\} \;= \\
&=\; \int_{\mathbb{S}^n} \delta\left(\bar{g}_{cd}\right)\,\nabla^c\nabla^d\,x^{n+1} \;= \\
&=\; \int_{\mathbb{S}^n} -\,\delta\left(\bar{g}_{cd}\right)\,g^{cd}\,x^{n+1} \;= \\
&=\; 0\,. \qquad (2.4.3)
\end{aligned}
$$

Thus, we cannot have $\delta S = 0$ in just one hemisphere (which is also true if we assume $\delta S \leq 0$ instead of $\delta S \geq 0$).

□

*Remark* 2.4.2. This is useful to have in mind when searching for counterexamples as, at the very most, $S$ will vary with the second power of $\lambda$ when $\lambda \to 0$.

# Chapter 3

# Falseness of the conjecture

As mentioned earlier, and surprisingly, Min-Oo's Conjecture is false. In this chapter we discuss the general idea behind the proof of this fact, describe a possible way of searching for counterexamples and propose a new conjecture that might support our observation regarding Schwarzschild-de Sitter black holes.

## 3.1 Existence of counterexamples

Brendle, Marques and Neves [5] showed that it is possible to find a metric that violates Min-Oo's Conjecture, and here we will briefly outline their reasoning. They start by proving the following result using a perturbation argument:

**Theorem 3.1.1.** *Given any integer $n \geq 3$, there exists a differentiable metric $g$ on $\mathbb{S}^n_+$ that satisfies:*

1. $S_g > n(n-1)$;
2. $g = d\Omega^2_{\mathbb{S}^n_+}$ *in* $\partial\mathbb{S}^n_+$;
3. *the sum of the principal curvatures of* $\partial\mathbb{S}^n_+$ *with respect to* $g$, $H_g$, *is strictly positive.*

Next they prove the following gluing theorem:

**Theorem 3.1.2.** *Let M be an n-dimensional compact manifold with boundary and let $g$ and $\tilde{g}$ be two differentiable Riemannian metrics on $M$ such that $g = \tilde{g}$ in $\partial M$. Assume that $H_g > H_{\tilde{g}}$ in $\partial M$. Then, given any $\epsilon > 0$ and any neighbourhood $\Omega$ of $\partial M$, there exists a differentiable metric $\hat{g}$ on $M$ that satisfies:*

1. $S_{\hat{g}}(x) \geq \min\{S_g(x), S_{\tilde{g}}(x)\} - \epsilon$ , $\forall x \in M$*;*

2. $\hat{g} = g$ *outside of* $\Omega$*;*

3. $\hat{g} = \tilde{g}$ *in a neighbourhood of* $\partial M$*.*

Now, let $g$ be the metric defined in theorem 3.1.1. One can construct a spherically symmetric metric $\tilde{g}$ on $\mathbb{S}^n_+$ that verifies:

- $S_{\tilde{g}} > n(n-1)$ in a neighbourhood of $\partial\mathbb{S}^n_+$;

- $\tilde{g} = d\Omega^2_{\mathbb{S}^n_+}$ in $\partial\mathbb{S}^n_+$;

- $\partial\mathbb{S}^n_+$ is totally geodesic with respect to $\tilde{g}$.

Applying theorem 3.1.2 one then obtains a metric $\hat{g}$ that directly counters Min-Oo's Conjecture:

**Corollary 3.1.3.** *Given any integer $n \geq 3$ there exists a differentiable metric $\hat{g}$ on $\mathbb{S}^n_+$ such that:*

1. $S_{\hat{g}} > n(n-1)$*;*

2. $\hat{g} = d\Omega^2_{\mathbb{S}^n_+}$ *in* $\partial\mathbb{S}^n_+$*;*

3. $\partial\mathbb{S}^n_+$ *is totally geodesic with respect to* $\hat{g}$*.*

## 3.2 Attempt at finding an explicit counterexample

In order to try to find a counterexample, we considered a 4-sphere split in squashed 3-spheres with the metric

$$ds^2 = dr^2 + \frac{1}{4}\left(e^{2a}\,\sigma_1^2 + e^{2b}\,\sigma_2^2 + e^{2c}\,\sigma_3^2\right)\ , \tag{3.2.1}$$

where $a$, $b$ and $c$ are differentiable functions that depend only on $r$ and

$$\begin{cases} \sigma_1 = \cos\psi\; d\theta + \sin\theta\sin\psi\; d\phi \\ \sigma_2 = \sin\psi\; d\theta + \sin\theta\cos\psi\; d\phi \\ \sigma_3 = d\psi + \cos\theta\; d\phi \end{cases} \tag{3.2.2}$$

for $\theta \in \,]0,\pi[\,$, $\phi \in \,]0,2\pi[\,$ and $\psi \in \,]0,2\pi[\,$ are a basis of left-invariant 1-forms.

Next, defining $A = e^a$, $B = e^b$ and $C = e^c$ and using Cartan's structure equations, we find the scalar curvature to be

$$\begin{aligned} S \;=\; 4\left(\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}\right) \;-\; 2\bigg(\frac{A''}{A}+\frac{B''}{B}+\frac{C''}{C}\;+ \\ +\frac{A'B'}{AB}+\frac{A'C'}{AC}+\frac{B'C'}{BC}+\frac{A^2}{B^2C^2}+\frac{B^2}{A^2C^2}+\frac{C^2}{A^2B^2}\bigg)\ . \end{aligned} \tag{3.2.3}$$

As we have seen in section 2.2, $A$, $B$ and $C$ must obey the restrictions

$$\begin{aligned} &A(0) = B(0) = C(0) = 1\ , \\ &A'(0) = B'(0) = C'(0) = 0\ , \\ &A(r_0) = B(r_0) = C(r_0) = 0\ , \\ &A'(r_0) = B'(r_0) = C'(r_0) = -1\ , \end{aligned} \tag{3.2.4}$$

for some $r_0 > 0$.

The goal is to find functions (and assign them to $A$, $B$ and $C$) such that the scalar curvature is at least 12 everywhere and strictly greater than 12 somewhere. Obviously, the three functions cannot be all the same, as that would fall under the case of a spherically symmetric manifold, for which we already know Min-Oo's Conjecture holds, but two of them can be equal, or they can all be different from each other.

For example, choosing $r_0 = 1$, some of our tries were:

$$\begin{aligned} f_1(x) &= x^3 - 2x^2 + 1 \ , \\ f_2(x) &= e^x \left[ \left( 1 - \frac{1}{e} \right) \left( x^3 - x^2 \right) - x + 1 \right] \ , \\ f_3(x) &= \frac{1}{e-3} \left( e - 2 + x + x^2 - e^x \right) \ . \end{aligned} \tag{3.2.5}$$

For $r_0 = \pi/2$:

$$\begin{aligned} g_1(x) &= \cos x \ , \\ g_2(x) &= \left( \frac{16}{\pi^3} - \frac{4}{\pi^2} \right) x^3 + \left( \frac{2}{\pi} - \frac{12}{\pi^2} \right) x^2 + 1 \ , \\ g_3(x) &= \cos x \left[ 1 + \lambda \left( x^3 - \frac{\pi}{2} x^2 \right) \right] \ , \quad \lambda \in \mathbb{R} \ . \end{aligned} \tag{3.2.6}$$

For arbitrary $r_0$ we may consider Fourier series, which offer plenty of possibilities:

$$h(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left[ a_n \cos\left( \frac{\pi n x}{2 r_0} \right) + b_n \sin\left( \frac{\pi n x}{2 r_0} \right) \right] \ , \tag{3.2.7}$$

where the usual restrictions lead to the following conditions:

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n = 1 \ ,$$

$$\sum_{n=1}^{+\infty} n\, b_n = 0 \ ,$$

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_{4n} - a_{4n-2} + b_{4n-3} - b_{4n-1}) = 0 \ , \tag{3.2.8}$$

$$\sum_{n=1}^{+\infty} [\, 4n\, b_{4n} - (4n-2)\, b_{4n-2} - (4n-3)\, a_{4n-3} - (4n-1)\, a_{4n-1}\,] = -\frac{2r_0}{\pi} \ .$$

We then select the terms of (3.2.7) we want to keep (crossing out the others) and work out their respective coefficients using (3.2.8); in the process we obtain families of functions with the required properties. For instance, one such family is

$$\begin{aligned} \tilde{h}(x) \ = \ & (1 - a_1 - a_2 - a_3 - a_4) \ + \ a_1 \cos\left(\frac{\pi}{2r_0}x\right) \ + \ a_2 \cos\left(\frac{\pi}{r_0}x\right) \ + \ a_3 \cos\left(\frac{3\pi}{2r_0}x\right) \ + \\ & + \ a_4 \cos\left(\frac{2\pi}{r_0}x\right) \ + \ \left(-\frac{3}{4} - \frac{r_0}{2\pi} \ + \ a_1 + \frac{3}{2}a_2 - 2b_4\right) \sin\left(\frac{\pi}{2r_0}x\right) \ + \\ & + \ \left(\frac{r_0}{\pi} - \frac{a_1}{2} + \frac{3}{2}a_3 + 2b_4\right) \sin\left(\frac{\pi}{r_0}x\right) \ + \\ & + \ \left(\frac{1}{4} - \frac{r_0}{2\pi} - \frac{a_2}{2} - a_3 - 2b_4\right) \sin\left(\frac{3\pi}{2r_0}x\right) \ + \\ & + \ b_4 \sin\left(\frac{2\pi}{r_0}x\right) \ , \qquad a_1, a_2, a_3, a_4, b_4 \in \mathbb{R} \ . \end{aligned} \tag{3.2.9}$$

Unfortunately, none of our attempts up till now managed to provide an explicit counterexample to Min-Oo's Conjecture.

## 3.3 New conjecture and future prospects

Brendle and Marques were able to prove the following weaker version of Min-Oo's Conjecture [4] [6]:

**Theorem 3.3.1.** *Let $g$ be a metric on $\mathbb{S}^n_+$ that agrees with $d\Omega^2_{\mathbb{S}^n_+}$ in $\left\{x \in \mathbb{S}^n_+ : 0 \le x^{n+1} \le \frac{2}{\sqrt{n+3}}\right\}$ and such that $S \ge n(n-1)$ . There exists $\alpha > 0$ such that if $\left\| g - d\Omega^2_{\mathbb{S}^n_+} \right\|_{C^2} < \alpha$ then $g$ is isometric to $d\Omega^2_{\mathbb{S}^n_+}$ .*

Due to the argument presented in section 1.2, some modified version of the conjecture should, in principle, hold, and one possibility is obtained by dropping from theorem 3.3.1 the requirement that $g - d\Omega^2_{\mathbb{S}^n_+}$ be sufficiently small in the $C^2$-norm. Future work should explore this idea because, as we have seen, the point raised in section 1.1 fails for $r \ge \pi/2$ and this new conjecture corresponds to $r = \arccos\left(\frac{2}{\sqrt{n+3}}\right) < \pi/2$.

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