From Relativistic Elasticity to Cosmic Censorship

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Abstract

We start by developing a framework for relativistic elastic strings whose energy density only depends on how stretched they are. We introduce the cosmic censorship conjecture while exploring its current state of the art and aftewards, we study whether it is possible to spin up a Kerr black hole past extremality using elastic strings and hence violate the conjecture. We find that if the null energy condition is satisfied on the event horizon then the black hole cannot be destroyed and therefore Cosmic Censorship is preserved. Finally, using facts from black hole thermodynamics and some lorentzian geometry, we find out that arbitrary test fields satisfying the null energy condition on the horizon can't destroy extremal black holes. We argue that the case of strings and many other results with test particles present in the literature are just particular cases of this latter result.

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1 Relativistic Elastic Strings

Along the whole text, we follow the conventions of [46, 47]; in particular, we use a system of units for which $c = G = 1$.

The most popular relativistic matter model is certainly the perfect fluid, not only due to its simplicity but also because most astrophysical objects can be described as bodies composed of perfect fluid. Yet, if we want to talk about solids of our everyday life, elasticity is an important property that must be taken into account. Therefore, although most times we don't have to take into account relativistic effects for these objects, it's important to have a working theory which is able to describe them in accordance with the principles of relativity. That is where Relativistic Elasticity emerges.

Relativistic elasticity in its modern form was originally formulated by Carter and Quintana [9]. It has been used to compute the speed of sound in relativistic solids [7] and to study elastic equilibrium states [1, 2] or dynamical situations [8].

Here we consider one-dimensional elastic bodies that we shall call strings. The motion of current carrying Nambu-Goto/cosmic string loops in black hole backgrounds was studied in [11].

1.1 Lagrangian density

We model a string moving on a $(n+1)$-dimensional spacetime (M, g) by an embedding $X : \mathbb{R} \times I \to M$, where $I \subset \mathbb{R}$ is an interval labeling the points of the string. Thus, the curve $\tau \mapsto X(\tau, \lambda)$ is the worldline of the point of the string labeled by $\lambda \in I$. We assume that the parameter $\lambda \in I$ is the arclength in the string's relaxed configuration.

The embedding X induces a metric

$$h_{AB} = g_{\mu\nu}(X)\partial_A X^\mu \partial_B X^\nu \tag{1}$$

on $\mathbb{R} \times I$, which we identify with its image $\Sigma = X(\mathbb{R} \times I)$, sometimes called the string's *worldsheet*. If we choose a local orthonormal frame $\{E_0, E_1\}$ tangent to Σ such that E_0 is the 4-velocity of the string's particles, we have

$$\begin{cases} \dfrac{\partial X}{\partial \tau} = \alpha E_0 \\ \\ \dfrac{\partial X}{\partial \lambda} = \beta E_0 + s E_1 \end{cases}, \tag{2}$$

for some local smooth functions α, β, s. Note that $|s|$ is the factor by which the string is stretched according to an observer comoving with it. The components of the induced metric are then

$$(h_{AB}) = \begin{pmatrix} -\alpha^2 & -\alpha\beta \\ -\alpha\beta & -\beta^2 + s^2 \end{pmatrix}, \tag{3}$$

and so

$$h \equiv \det(h_{AB}) = -\alpha^2 s^2 = h_{00} s^2. \tag{4}$$

Defining the *number density* $n = \frac{1}{|s|}$, we then have

$$n^2 = \frac{h_{00}}{h}. \tag{5}$$

To obtain the string's equation of motion we must choose an action

$$S = \int_{\mathbb{R} \times I} \mathcal{L}(X, \partial X)\, d\tau d\lambda. \tag{6}$$

For an elastic string whose internal energy density ρ depends only on its stretching,

$\rho = F(n^2)$, the Lagrangian density is (see [13])

$$\mathcal{L} = F(n^2)\sqrt{-h}, \tag{7}$$

where $h \equiv \det(h_{AB})$ and n^2 are given as functions of $(X, \partial X)$ from equations (1) and (5).

1.2 Equations of motion

To find the equations of motion we must compute the variation $\delta\mathcal{L}$ of the Lagrangian density resulting from a variation δX of the embedding. From (5) we have

$$\delta n^2 = \frac{\delta h_{00}}{h} - \frac{h_{00}}{h^2}\delta h. \tag{8}$$

Using the well-known formula for the variation of the determinant of the metric,

$$\delta h = h h^{AB} \delta h_{AB} \tag{9}$$

(see e.g. [47]), we obtain

$$\delta n^2 = \frac{\delta_0^A \delta_0^B - h_{00} h^{AB}}{h} \delta h_{AB}. \tag{10}$$

Since

$$\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}\, h^{AB} \delta h_{AB}, \tag{11}$$

we can write

$$\delta\mathcal{L} = \left(F'(n^2) \frac{h_{00} h^{AB} - \delta_0^A \delta_0^B}{\sqrt{-h}} + \frac{1}{2} F(n^2)\sqrt{-h}\, h^{AB} \right) \delta h_{AB}. \tag{12}$$

By analogy with the energy-momentum tensor in general relativity, we define the string's energy-momentum tensor density $\mathcal{T}^{AB}$ by the relation

$$\delta\mathcal{L} = -\frac{1}{2}\mathcal{T}^{AB} \delta h_{AB} \tag{13}$$

(see [47]). We will provide a better justification for this choice after obtaining the equation of motion. The string's energy-momentum tensor T^{AB} is then defined as

$$T^{AB} = \frac{1}{\sqrt{-h}} \mathcal{T}^{AB}, \tag{14}$$

that is,

$$
\begin{aligned}
T^{AB} &= 2F'(n^2)\frac{h_{00}h^{AB} - \delta_0^A\delta_0^B}{h} - F(n^2)h^{AB} \\
&= 2F'(n^2)\frac{h_{00}U^AU^B + h_{00}h^{AB}}{h} - F(n^2)h^{AB} \\
&= 2n^2F'(n^2)U^AU^B + \left[2n^2F'(n^2) - F(n^2)\right] h^{AB},
\end{aligned}
\tag{15}
$$

where U^A is the four-velocity of the string's particles. Therefore, the string's energy density ρ and the string's pressure p are given by[2]

$$
\rho = F(n^2), \qquad p = 2n^2F'(n^2) - F(n^2). \tag{16}
$$

Note that the string's energy density is indeed correct. We shall assume that the pressure is zero when the string is not stretched nor compressed:

$$
2F'(1) - F(1) = 0. \tag{17}
$$

To obtain the equation of motion, we note that

$$
\delta h_{AB} = \partial_\alpha g_{\mu\nu}\delta X^\alpha \partial_A X^\mu \partial_B X^\nu + 2g_{\mu\nu}\partial_A X^\mu \partial_B \delta X^\nu, \tag{18}
$$

and so

$$
\begin{aligned}
-2\delta\mathcal{L} &= \left(\mathcal{T}^{AB}\partial_\alpha g_{\mu\nu}\partial_A X^\mu \partial_B X^\nu\right)\delta X^\alpha + 2\mathcal{T}^{AB}g_{\mu\alpha}\partial_A X^\mu \partial_B \delta X^\alpha \\
&= \left[\mathcal{T}^{AB}\partial_\alpha g_{\mu\nu}\partial_A X^\mu \partial_B X^\nu - \partial_B\left(2\mathcal{T}^{AB}g_{\mu\alpha}\partial_A X^\mu\right)\right]\delta X^\alpha \\
&+ \partial_B\left(2\mathcal{T}^{AB}g_{\mu\alpha}\partial_A X^\mu \delta X^\alpha\right).
\end{aligned}
\tag{19}
$$

Discarding the total divergence in Hamilton's principle

$$
\delta S = 0 \Leftrightarrow \int_{\mathbb{R}\times I} \delta\mathcal{L}(X, \partial X)\, d\tau d\lambda = 0, \tag{20}
$$

[2]Since the worldsheet is two-dimensional, this pressure is actually a force (tension or compression of the string).

we are led to

$$\begin{aligned}
&\partial_B \left(\mathcal{T}^{AB} g_{\mu\alpha}\partial_A X^\mu\right) - \frac{1}{2}\mathcal{T}^{AB}\partial_\alpha g_{\mu\nu}\partial_A X^\mu \partial_B X^\nu = 0 \Leftrightarrow \\
&g_{\mu\alpha}\partial_B \left(\mathcal{T}^{AB}\partial_A X^\mu\right) + \mathcal{T}^{AB}\partial_\nu g_{\mu\alpha}\partial_B X^\nu \partial_A X^\mu \\
&\qquad\qquad\qquad - \frac{1}{2}\mathcal{T}^{AB}\partial_\alpha g_{\mu\nu}\partial_A X^\mu \partial_B X^\nu = 0 \Leftrightarrow \\
&g_{\mu\alpha}\partial_B \left(\mathcal{T}^{AB}\partial_A X^\mu\right) + \frac{1}{2}\mathcal{T}^{AB}\left(2\partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)\partial_A X^\mu \partial_B X^\nu = 0.
\end{aligned} \tag{21}$$

Since $\mathcal{T}^{AB}$ is symmetric, the equation of motion can be put in the form

$$\partial_B \left(\mathcal{T}^{AB}\partial_A X^\alpha\right) + \mathcal{T}^{AB}\Gamma^\alpha_{\mu\nu}\partial_A X^\mu \partial_B X^\nu = 0, \tag{22}$$

or, equivalently,

$$\frac{1}{\sqrt{-h}}\partial_B \left(\sqrt{-h}T^{AB}\partial_A X^\alpha\right) + T^{AB}\Gamma^\alpha_{\mu\nu}\partial_A X^\mu \partial_B X^\nu = 0. \tag{23}$$

If T^{AB} is proportional to h^{AB}, that is, if $F(n^2)$ is a nonvanishing constant, then we obtain the Nambu-Goto equation of motion,

$$\frac{1}{\sqrt{-h}}\partial_B \left(\sqrt{-h}h^{AB}\partial_A X^\alpha\right) + h^{AB}\Gamma^\alpha_{\mu\nu}\partial_A X^\mu \partial_B X^\nu = 0. \tag{24}$$

This equation is closely related to the harmonic map/wave map/nonlinear sigma model equation (see [12]).

1.3 Adapted coordinates

To better understand the equation of motion, we extend the local coordinates $(x^A) = (\tau, \lambda)$ on the worldsheet Σ to a local coordinate system (x^A, x^i) defined on a neighborhood of Σ by choosing an orthonormal frame $\{E_2, \ldots, E_n\}$ for the normal bundle of Σ and letting (x^A, x^i) parameterize the point $\exp_p(x^i E_i)$, where $\exp_p$ is the geodesic exponential map and $p \in \Sigma$ is the point with coordinates (x^A). The worldsheet is given in these coordinates by $x^i = 0$, and the spacetime metric by

$$g = g_{AB}dx^A dx^B + 2g_{Ai}dx^A dx^i + g_{ij}dx^i dx^j. \tag{25}$$

Note that on Σ we have

$$g = h_{AB}dx^A dx^B + \delta_{ij}dx^i dx^j. \tag{26}$$

The tensor

$$K^i_{AB} = \frac{1}{2}\partial_i g_{AB}, \tag{27}$$

defined on Σ, is called the *second fundamental form* of Σ in the direction of E_i. It easily seen that on Σ

$$\Gamma^i_{AB} = -K^i_{AB} \tag{28}$$

and

$$\Gamma^C_{AB} = \overline{\Gamma}^C_{AB}, \tag{29}$$

where $\overline{\Gamma}^C_{AB}$ are the Christoffel symbols for the Levi-Civita connection $\overline{\nabla}$ of h_{AB}. In this coordinate system, the embedding is simply given by $(X^A, X^i) = (x^A, 0)$, and so we can write the first two components of equation (83) as

$$\frac{1}{\sqrt{-h}}\partial_B\left(\sqrt{-h}T^{BC}\right) + T^{AB}\overline{\Gamma}^C_{AB} = 0, \tag{30}$$

and the last $n-1$ as

$$T^{AB}K^i_{AB} = 0. \tag{31}$$

Using the well-known formula (see [47])

$$\partial_B \log\sqrt{-h} = \overline{\Gamma}^A_{BA}, \tag{32}$$

equation (30) is easily seen to be equivalent to

$$\overline{\nabla}_B T^{BC} = 0. \tag{33}$$

This justifies the choice of T^{AB} as the string's energy-momentum tensor.

The equations of motion of the string can then be understood as constraints of the geometry of the worldsheet, given by (31), plus conservation of energy-momentum, given by (33) (see [10] for similar equations in the context of blackfolds). For the Nambu-Goto strings, for instance, where T_{AB} is proportional to h_{AB}, the constraints are the condition that the worldsheet is a minimal surface, and the conservation equation is automatically satisfied.

Interestingly, the component of the conservation equations along U^A is always trivial, even in the general case. Indeed, from

$$n = \frac{\sqrt{-h_{00}}}{\sqrt{-h}} \tag{34}$$

and

$$U^A = (-h_{00})^{-\frac{1}{2}}\delta_0^A, \tag{35}$$

it is clear that

$$\overline{\nabla}_A(n\,U^A) = \frac{1}{\sqrt{-h}}\partial_A(\sqrt{-h}\,n\,U^A) = 0, \tag{36}$$

that is,

$$\overline{\nabla}_U n + n\overline{\nabla}_A U^A = 0. \tag{37}$$

Consequently,

$$\overline{\nabla}_U \rho + (\rho + p)\overline{\nabla}_A U^A = 2nF'(n^2)\overline{\nabla}_U n + 2n^2F'(n^2)\overline{\nabla}_A U^A = 0, \tag{38}$$

which is precisely the component along U of

$$\overline{\nabla}_A T^{AB} = 0 \Leftrightarrow \overline{\nabla}_A\left((\rho+p)U^AU^B + p\,h^{AB}\right) = 0. \tag{39}$$

1.4 Conserved quantities

If (M, g) admits a Killing vector field ξ,

$$\nabla_{(\mu}\xi_{\nu)} = 0, \tag{40}$$

then in the coordinates above we have

$$\partial_{(A}\xi_{B)} + \overline{\Gamma}^C_{AB}\xi_C + \Gamma^i_{AB}\xi_i = 0 \Leftrightarrow \overline{\nabla}_{(A}\xi_{B)} = K^i_{AB}\xi_i, \tag{41}$$

that is, the projection of ξ on $T\Sigma$ is not, in general, a Killing vector field of h_{AB}. Nevertheless,

$$\overline{\nabla}_A(T^{AB}\xi_B) = T^{AB}K^i_{AB}\xi_i = 0 \tag{42}$$

in view of (31), that is, the vector field

$$j^A = T^{AB}\xi_B \tag{43}$$

is divergenceless on Σ. As a consequence, the quantity

$$E^\xi = \int_{\{\tau=\text{constant}\}} j^A\nu_A\sqrt{h_{11}}\,d\lambda \tag{44}$$

is conserved, where

$$\nu_A = \frac{\delta^0_A}{\sqrt{-h^{00}}} = \frac{\sqrt{-h}}{\sqrt{h_{11}}}\,\delta^0_A \tag{45}$$

is the past-pointing normal to the spacelike curve $\{\tau = \text{constant}\}$. In other words, we have the conserved quantity

$$E^\xi = \int_{\{\tau=\text{constant}\}} j^0 \sqrt{-h}\, d\lambda. \tag{46}$$

2 Cosmic Censorship Conjecture

2.1 Singularity Theorems

One of the biggest surprises that General Relativity has given us is that under certain circumstances the theory predicts its own limitations. There are two physical situations where we expect that General Relativity breaks down. The first is the gravitational collapse of certain massive stars when their nuclear fuel is spent. The second one is the Big Bang, or in other words the point where spacetime itself was born. In both cases we expect that the geometry of spacetime will show some pathological behaviour. Usually, we say that spacetime will exhibit a singularity. But what do we mean by that? Usually one thinks of singularities as those points of spacetime where the curvature blows up, however, rigorously speaking, singularities are points that don't even belong to the manifold representing spacetime. With this idea in mind, finding a singularity is equivalent to finding a point where the manifold ceases to exist. Hence, in order to recognize singularities we just search for incomplete geodesics, that is geodesics which can't be extended to infinite values of the affine parameter. Intuitively, geodesic incompleteness describes that there is an obstruction to free falling observers to continue travelling through spacetime. In some sense, they have reached the edge of spacetime in a finite amount of time. From this discussion, the definition follows.

Definition 2.1. *A spacetime is singular if it is timelike or null geodesically incomplete, but can't be embedded in a larger spacetime.*

Note that it is important to check if a geodesically incomplete spacetime can be embedded in a larger one, because, if it can, then maybe this larger manifold is complete and therefore the former incompleteness is probably not physically relevant.

Between 1965 and 1970, Hawking and Penrose proved a series of singularity theorems that predicted the inevitability of singularities under certain conditions. The general structure of these theorems establish that if on a spacetime $(\mathcal{M}, g_{\mu\nu})$

- The matter content satisfies an energy condition.
- Gravity is strong enough in some region.
- A global causal condition is met.

Then $(\mathcal{M}, g_{\mu\nu})$ must be geodesically incomplete.

It's obvious that this is not a proper statement of the theorem. Still, it's important to know what each of the above conditions could be. Firstly, if we want to talk about physically reasonable matter, we usually need to assume that it obeys some kind of energy condition which can be turned into a mathematical statement about the energy-momentum tensor. As a simple example, we can take the *weak energy condition* which states that the energy density of any matter distribution, as measured by any observer in spacetime must be nonnegative. Since an observer with four velocity u^α measures the energy density to be $T_{\mu\nu}u^\mu u^\nu$, the condition is just

$$T_{\mu\nu}u^\mu u^\nu \geq 0\,. \tag{47}$$

What does it mean for gravity being strong enough? Formally speaking, it means that there exists a closed trapped surface $\mathcal{T}$, that is a closed 2-surface such that both the ingoing and outgoing null geodesics orthogonal to it are converging. The intuitive idea is simple: the gravitational field is becoming so strong in some region that light rays (and so all other forms of matter) are trapped inside a succession of 2-surfaces of smaller and smaller area. Finally, a global causal condition is just a statement about how different regions of spacetime affect each other: for example, one such statement may be 'There are no closed timelike curves', roughly meaning that if a massive particle starts out in some point p, it cannot travel and come back to that point, that is it can't go to the future and then come back to its past.

I believe it's now clear that the kind of assumptions which appear in the singularity theorems are reasonable from the physical point of view and probably hold in our Universe. Hence, we need to take their consequences seriously. As we said in the beginning, the theorems predict singularities in two situations: one is in the future of gravitational collapse of stars and other massive bodies and the other is in the beginning of the present expansion of the universe. This latter singularity is particularly important because it supports the idea of the Big Bang, that is the existence of a point where spacetime begins.

2.2 Cosmic Censorship

Now we know that probably our Universe has singularities. What does that mean for us as observers? Well, it means that we cannot predict what is going to happen to us or anything and therefore classical General Relativity is not a complete theory. This happens because the singular points have to be taken out of the spacetime manifold and hence one can't predict what is going to come out of them. As far as the singularity in the past is concerned, one can do no more than consider that it will eventually be resolved in a theory of quantum gravity. Nevertheless, the singularities which are predicted in the future may have a resolution within the classical theory. In the wake of the proofs of the singularity theorems in general relativity [14–16], Penrose formulated the weak cosmic censorship conjecture [17, 18], according to which, generically, the singularities resulting from gravitational collapse are hidden from the observers at infinity by a black hole event horizon. Penrose's expectation was that, independently of what might happen inside black holes, the evolution of the outside universe would proceed undisturbed. We must keep in mind that this is just an intuitive way of saying what cosmic censorship is. Rigorously speaking, the cosmic censorship conjecture is a statement about the evolution of Einstein's equations, which is a system of nonlinear PDEs.

Conjecture 2.2. *For generic asymptotically flat initial data for 'reasonable' Einstein-matter systems, the maximal Cauchy development has a complete null infinity.*

Nowadays, this is the most used formulation [38] of cosmic censorship conjecture by the Mathematical Relativity community. Yet, it is important to note that because there are physical considerations to take into account, it is very hard to come up with a very strict formulation. For example, in this case, one may wonder about what is reasonable matter. Anyway, for our purposes, we'll take the following claim

Claim 2.3. *Singularities arising from gravitational collapse always appear inside an event horizon.*

Although this notion captures the idea of the conjecture if we 'find' some singularity that can be seen from infinity, we certainly need to go back to the real formulation of it in order to see if what we found is really a violation. Hence, violating this claim is not a disproof of cosmic censorship conjecture, yet it would lead to an inconsistency that should be further explored.

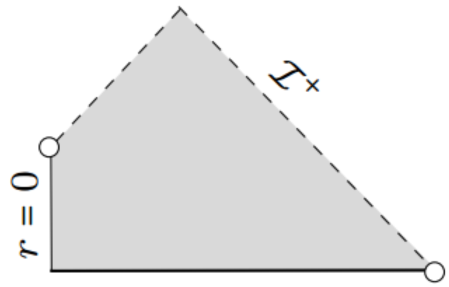

Figure 1: An evolution of the initial data that leads to a naked singularity. Roughly speaking, the conjecture tells us that observers at null infinity cannot observe for infinite proper time, because when they see the singularity for the first time we lose predictability about their future. So, in some sense, if we can follow the observers at infinity for infinite proper time it possibly means that the singularity has always been hidden.

2.3 Gedanken Experiments

In order to test claim (2.3), Wald [19] devised a thought experiment to destroy extremal Kerr-Newman black holes, already on the verge of becoming naked singularities, by dropping charged and/or spinning test particles into the event horizon. Both him and subsequent authors [20, 21] found that if the parameters of the infalling particle (energy, angular momentum, charge and/or spin) were suited to overspin/overcharge the black hole then the particle would not go in, in agreement with the cosmic censorship conjecture. Similar conclusions were reached by analyzing scalar and electromagnetic test fields propagating in extremal Kerr-Newman black hole backgrounds [22–25]. In this case, the fluxes of energy, angular momentum and charge across the event horizon were found to be always insufficient to overspin/overcharge the black hole. Some of these results have been extended to higher dimensions [26] and also to the case when there is a negative cosmological constant [27, 28].

More recently, it was noticed that Wald's thought experiment may produce naked singularities when applied to nearly extremal black holes [29–32]. However, in this case the perturbation cannot be assumed to be infinitesimal, and so backreaction effects have to be taken into account; when this is done, the validity of the cosmic censorship conjecture appears to be restored [33–37]. It can also be argued that the third law of black hole thermodynamics [39], for which there is some evidence [40–42], forbids subextremal black holes from ever becoming extremal, and so, presumably, from being destroyed. Nonetheless, this cannot be taken as a definitive argument, since, for instance, extremal Reissner-Nordström black holes can be formed by collapsing charged thin shells [43].

We may now wonder what would happen if, instead of trying to spin up black holes with point particles, we use the elastic strings we developed in the first section. In the next section, we study that possibility, which is going to take us to a more general result in section 4.

3 Destroying a black hole with rotating elastic strings

3.1 Embedding in Kerr

We want to spin up a Kerr black hole past extremality through the absorption of rotating elastic strings described by an arbitrary law, therefore we need to setup the embedding of the string in this geometry. We will consider the extremal Kerr solution, with mass $M > 0$ and angular momentum M^2. The Kerr metric in Painlevé-Gullstrand coordinates [48] for the equatorial plane $\theta = \frac{\pi}{2}$ is the following

$$ds^2 = -dt^2 + \frac{r^2}{\Sigma}(dr - vdt)^2 + \Sigma(d\phi - \Omega dt)^2\,, \tag{48}$$

where

$$\Sigma = r^2 + a^2 + \frac{2Ma^2}{r} \tag{49}$$

$$\Omega = \frac{2Ma}{r\Sigma} \tag{50}$$

$$v = -\frac{\sqrt{2Mr(r^2+a^2)}}{r^2}\,. \tag{51}$$

Note that v and Ω are just the familiar expressions for the angular velocity and radial proper velocity of a zero angular momentum observer dropped from infinity.

We have the embedding

$$\begin{cases} t(\tau,\lambda) = \tau \\ r(\tau,\lambda) = R(\tau) \\ \varphi(\tau,\lambda) = \Phi(\tau) + k\lambda \end{cases}\,, \tag{52}$$

where k and λ are defined as before. On the other hand, for now, $R(\tau)$ and $\omega(\tau)$ are just undetermined functions of τ. Replacing (52) in (48) yields

$$(h_{AB}) = \begin{pmatrix} -1 + \frac{R^2}{\Sigma}\nu_R^2 + \Sigma\nu_\omega^2 & k\Sigma\nu_\Phi \\ k\Sigma\nu_\Phi & \Sigma k^2 \end{pmatrix}\,. \tag{53}$$

where

$$\begin{cases} \nu_R = \dot{R} - v \\ \nu_\Phi = \dot{\Phi} - \Omega \end{cases}, \tag{54}$$

for the sake of notation simplicity. Moreover, we have

$$h = -k^2(\Sigma - R^2\nu_R^2) \tag{55}$$

and

$$(h^{AB}) = \begin{pmatrix} -1 + \frac{\rho^2}{\Sigma}\nu_R^2 + \Sigma\nu_\Phi^2 & k\Sigma\nu_\Phi \\ k\Sigma\nu_\Phi & \Sigma k^2 \end{pmatrix}. \tag{56}$$

The Kerr metric admits two Killing vector fields – $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ – whose associated covector fields are given by

$$\begin{cases} \xi^t = g\left(\frac{\partial}{\partial t}, .\right) = \left(-1 - \frac{R^2}{\Sigma}v\nu_R - \Sigma\Omega\nu_\Phi\right) d\tau - (\Sigma\Omega k)\, d\lambda \\ \xi^\phi = g\left(\frac{\partial}{\partial \phi}, .\right) = (\Sigma\nu_\Phi)d\tau + (\Sigma k)d\lambda \end{cases}, \tag{57}$$

written in the embedding coordinates. We are working in a spacetime which admits two conserved quantities - the energy E and angular momentum L of the elastic string - these quantities are going to be relevant and can now be computed using (46) and (57). We obtain

$$\begin{cases} E = -\frac{2\pi\sqrt{-h}}{k}\left(\left((\Sigma + vR^2\nu_R)\left(\frac{1}{\Sigma h_{00}} - \frac{k^2}{h}\right) + \frac{\Sigma\nu_\Phi}{h_{00}}\Omega\right)p + \left(\frac{1}{h_{00}} + \frac{vR^2\nu_R}{\Sigma h_{00}} + \frac{\Sigma\nu_\Phi}{h_{00}}\Omega\right)\rho\right) \\ L = -\frac{2\pi\sqrt{-h}\Sigma\nu_\Phi}{kh_{00}}(p+\rho) = -\frac{2\pi\Sigma^2\nu_\Phi\sqrt{\Sigma - R^2\nu_R^2}}{R^2\nu_R^2 + \Sigma\left(\Sigma\nu_\Phi^2 - 1\right)}(p+\rho) \end{cases}. \tag{58}$$

3.2 Elastic strings can't destroy BHs

Now, we assume that an elastic string of infinitesimal energy E and angular momentum L is going to enter inside an extremal black hole. Ignoring the backreaction of the string on spacetime, we can easily find a relationship between E and L that must be satisfied in order to destroy the black hole

$$\frac{a_f}{M_f} = \frac{Ma + L}{(M+E)^2} \approx \frac{a}{M} + \frac{E}{M}\left(\frac{L}{ME} - \frac{2a}{M}\right) \tag{59}$$

At extremality $a = M$ and the black hole is destroyed if $a_f > M_f$, where a_f and M_f are just the parameters of the black hole after absorbing the string. Hence

$$\frac{L}{EM} > 2 \tag{60}$$

must be satisfied for a dangerous string. From (58), we can write the energy of the string in the following way

$$E = -\frac{2\pi\sqrt{-h}}{k} \cdot \frac{(R^2 v\beta + \Sigma)\left(2\dot{F}h + (Fh - 2\dot{F}h_{00})k^2\Sigma\right)}{h^2\Sigma} + L\Omega\,, \tag{61}$$

In order to simplify our expression, we considered an extremal black hole of unit mass. Note that for such a black hole (60) reduces to

$$\frac{L}{2} > E \tag{62}$$

Since we we want to restrict our attention to the strings capable of spinning up the BH, that is the ones which enter the horizon, we evaluate the energy at the horizon, which then takes the following form

$$E = \dot{F}\left(\frac{2 + 8(-1+\dot{\Phi})\dot{\Phi}}{k^2\sqrt{-\dot{R}}(4+\dot{R})^{\frac{3}{2}}}\right) + F\frac{\sqrt{-\dot{R}}}{\sqrt{4+\dot{R}}} + \frac{L}{2}\,. \tag{63}$$

Note that all the functions appearing in the energy expression are implicitly being evaluated at $R = M = 1$. By (58), we can further consider that $\dot{\Phi}$ is given by

$$\dot{\Phi} = \Omega + \frac{\sqrt{-h}kL}{4\dot{F}\pi\Sigma} \tag{64}$$

and if we replace it in the energy function evaluated at the horizon, we obtain

$$E = -\frac{\left(\frac{L^2k^2}{8\pi\dot{F}} + 4F\pi\right)}{\sqrt{-\dot{R}(4+\dot{R})}}\dot{R} + \frac{L}{2} \tag{65}$$

Now, by assuming that the motion of the string must be timelike, we can get a nice restriction for $\dot{R}$. Remember that $\frac{\partial}{\partial\tau}$ is the vector field in the the string manifold that represents the *4-velocity* . Therefore, it is natural to use its pushforward by the embedding as the 4-velocity in spacetime. With this in mind, we have the following inequality

$$g\left(\frac{\partial}{\partial t}+\dot{R}\frac{\partial}{\partial r}+\dot{\Phi}\frac{\partial}{\partial\phi},\frac{\partial}{\partial t}+\dot{R}\frac{\partial}{\partial r}+\dot{\Phi}\frac{\partial}{\partial\phi}\right)<0 \tag{66}$$

It follows that

$$-1+\frac{R^2}{\Sigma}(\dot{R}-v)^2+\Sigma(\dot{\Phi}-\Omega)^2<0 \tag{67}$$

and consequently

$$\frac{R^2}{\Sigma}(\dot{R}-v)^2<1\,. \tag{68}$$

At the horizon $R=1$, this expression reduces to

$$(\dot{R}+2)^2<4\,. \tag{69}$$

Hence, it's easy to conclude that the maximum interval for $\dot{R}$ is $-4<\dot{R}<0$ in accordance to what we need for (65) to be well defined. Since $\dot{R}<0$ at the horizon for timelike motions, if the black hole is not destroyed, we should have

$$\frac{L^2k^2}{8\pi\dot{F}}+4F\pi\geq 0\,, \tag{70}$$

since only this way (60) won't be satisfied. Indeed, having $\dot{F}>0$ and $F\geq 0$ is enough to satisfy the above inequality and this is exactly assuming that the string obeys the weak energy condition in the worldsheet. Therefore

$$T_{AB}u^Au^B\geq 0\,. \tag{71}$$

for every causal vector u in the worldsheet. If we extend the coordinate system to the whole spacetime, we can see that the null energy condition on spacetime is equivalent to the weak energy condition on the string manifold. The null energy condition states that

$$T_{\mu\nu}k^\mu k^\nu\geq 0\,. \tag{72}$$

for every null vector k. But we can decompose any null vector into a part tangent to the worldsheet and an orthogonal one, from where we can write (following the notation of the first section)

$$T_{AB}k^Ak^B+T_{ij}k^ik^j=T_{AB}k^Ak^B\geq 0\,, \tag{73}$$

since the energy-momentum tensor vanishes for the orthogonal part. Furthermore, as k is null

$$k_\mu k^\mu=k_Ak^A+k_ik^i=0\,. \tag{74}$$

Since $k_i k^i \geq 0$, because it only includes spatial components, we conclude that $k_A k^A \leq 0$, that is the tangent part of k is causal. It follows that (71) and (72) are equivalent statements for the elastic string. Hence, if the elastic string satisfies the null energy condition (which is necesssary for reasonable classical matter) at the horizon, we won't be able to undress the singularity.

4 Test Fields cannot destroy extremal black holes

Considering the result we have just derived and many others from the literature, one easily concludes that it is really hard to destroy an extremal black hole. The most natural question to ask is: Is there any general reason that can explain this fact?

In this section [61], we consider arbitrary (possibly charged) test fields propagating in extremal Kerr-Newman or Kerr-Newman-anti de Sitter (AdS) black hole backgrounds. Apart from ignoring their gravitational and electromagnetic backreaction, we make no further hypotheses on these fields: they could be any combination of scalar, vector or tensor fields, charged fluids, sigma models, elastic media, or other types of matter. This also includes test particles and elastic strings, since they can be seen as singular limits of continuous media [44, 45]. We give a general proof that if the test fields satisfy the null energy condition at the event horizon then they cannot overspin/overcharge the black hole. This is done by first establishing, in Section 4.2, a test field version of the second law of black hole thermodynamics for extremal Kerr-Newman or Kerr-Newman-AdS black holes (which does not assume cosmic censorship). We use this result in Section 4.3, together with the Smarr formula and the first law, to conclude the proof. This last step requires the black hole to be extremal, and cannot be extended to near-extremal black holes. In Section 4.4 we discuss generalizations of our result to other extremal black holes, including higher dimensions and alternative theories of gravity.

4.1 Divergence theorem on a Lorentzian manifold

In this section we recall the divergence theorem on a Lorentzian manifold, which is slightly more involved that the familiar divergence theorem on a Riemannian manifold. The statement of the theorem is the same: if (M, g) is an n-dimensional Lorentzian manifold, $U \subset M$ is an open region whose boundary ∂U is a closed $(n-1)$-manifold (possibly with corners), and X is a smooth vector field on M, then

$$\int_U (\nabla_\mu X^\mu) dV_n = \int_{\partial U} (X^\mu N_\mu) dV_{n-1}, \tag{75}$$

where ∇ is the Levi-Civita connection of g and N is a unit normal vector field wherever ∂U is not null. At points where ∂U is timelike N is the (spacelike) outward-pointing unit normal, and at points where ∂U is spacelike N is the (timelike) inward-pointing unit normal. At points where ∂U is null N is simply any null normal whose time orientation is compatible with the time orientation of the adjacent timelike unit normal, as suggested in Figure 2; it is therefore only determined up to a positive function. The volume element dV_{n-1} to be used on the null portions of ∂U depends on the choice of N, but for our purposes it suffices to know that the volume of a null open subset of ∂U is always positive.

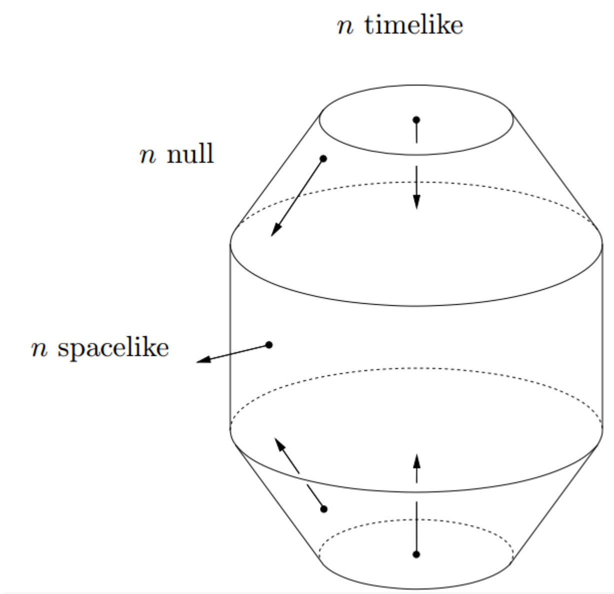

Figure 2: Normal vector for the divergence theorem on a Lorentzian manifold.

4.2 Second law for test fields

In this section we prove that a version of the second law of black hole thermodynamics holds in the case of (possibly charged) test fields propagating on a background Kerr-Newman or Kerr-Newman-AdS black hole (either subextremal or extremal). This calculation is similar to the one in [49], but we do not assume cosmic censorship, i.e. we do not assume that the black hole is not destroyed by interacting with the test field.

We start by recalling the Kerr-Newman-AdS metric, given in Boyer-Lindquist coordinates by

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\varphi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(a\,dt - \frac{r^2+a^2}{\Xi}d\varphi\right)^2, \tag{76}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta; \tag{77}$$

$$\Xi = 1 - \frac{a^2}{l^2}; \tag{78}$$

$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2mr + q^2; \tag{79}$$

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta \tag{80}$$

(see for instance [50]). Here m, a and q denote the mass, rotation and electric charge parameters, respectively. These parameters are related to the physical mass M, angular momentum J and electric charge Q by

$$M = \frac{m}{\Xi^2}, \qquad J = \frac{ma}{\Xi^2}, \qquad Q = \frac{q}{\Xi}. \tag{81}$$

The cosmological constant is $\Lambda = -\frac{3}{l^2}$, and so the Kerr-Newman metric can be obtained by taking the limit $l \to +\infty$. To avoid repeating ourselves, we will present all calculations below for the Kerr-Newman-AdS metric only; the corresponding formulae for the Kerr-Newman metric can be easily retrieved by making $l \to +\infty$.

The Kerr-Newman-AdS metric, together with the electromagnetic 4-potential

$$A = -\frac{qr}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\varphi\right), \tag{82}$$

is a solution of the Einstein-Maxwell equations with cosmological constant Λ. It admits a two-dimensional group of isometries, generated by the Killing vector fields $X = \frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial\varphi}$.

We consider arbitrary (possibly charged) test fields propagating in this background. Apart from ignoring their gravitational and electromagnetic backreaction, we make no further hypotheses on the fields: they could be any combination of scalar, vector or tensor fields, charged fluids, sigma models, elastic media, or other types of matter. Since the fields may be charged, their energy-momentum tensor T satisfies the generalized Lorentz law[3]

$$\nabla_\mu T^{\mu\nu} = F^{\nu\alpha} j_\alpha, \tag{83}$$

where $F = dA$ is the Faraday tensor of the background electromagnetic field and j is the charge current density 4-vector associated to the test fields. Using the symmetry of T and the Killing equation,

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0, \tag{84}$$

we have

$$\nabla_\mu (T^{\mu\nu} X_\nu) = F^{\nu\alpha} j_\alpha X_\nu. \tag{85}$$

On the other hand, using the charge conservation equation,

$$\nabla_\mu j^\mu = 0, \tag{86}$$

we obtain

$$\begin{aligned}
\nabla_\mu (j^\mu A^\nu X_\nu) &= j^\mu (\nabla_\mu A^\nu) X_\nu + j^\mu A^\nu \nabla_\mu X_\nu \\
&= j^\mu ({F_\mu}^\nu + \nabla^\nu A_\mu) X_\nu - j^\mu A^\nu \nabla_\nu X_\mu \\
&= F^{\mu\nu} j_\mu X_\nu + j_\mu (X^\nu \nabla_\nu A^\mu - A^\nu \nabla_\nu X^\mu).
\end{aligned} \tag{87}$$

Since A is invariant under time translations, we have

$$\mathcal{L}_X A = 0 \Leftrightarrow [X, A] = 0 \Leftrightarrow X^\nu \nabla_\nu A^\mu - A^\nu \nabla_\nu X^\mu = 0, \tag{88}$$

and so from (85) and (87) we obtain

$$\nabla_\mu (T^{\mu\nu} X_\nu + j^\mu A^\nu X_\nu) = 0. \tag{89}$$

This conservation law suggests that the total field energy on a given spacelike hypersurface S extending from the black hole event horizon $\mathscr{H}^+$ to infinity (Figure 3) should be

$$E' = \int_S (T^{\mu\nu} + j^\mu A^\nu) X_\nu N_\mu dV_3, \tag{90}$$

where N is the future-pointing unit normal to S. However, in the Kerr-Newman-AdS case the non-rotating observers at infinity are rotating with respect to the Killing vector

[3]See the Appendix for a complete explanation of the origin and meaning of this equation.

field X with angular velocity

$$\Omega_\infty = -\frac{a}{l^2}, \tag{91}$$

and so, as shown in [51], the physical energy should be computed with respect to the non-rotating Killing vector field

$$K = X + \Omega_\infty Y = X - \frac{a}{l^2} Y, \tag{92}$$

that is, the physical energy is actually

$$E = \int_S (T^{\mu\nu} + j^\mu A^\nu) K_\nu N_\mu dV_3. \tag{93}$$

This correction was implemented for test particles in [27]. The need for the corresponding correction in the calculation of the physical black hole mass has been stressed in [52, 53]. Note that in the Kerr-Newman case $\Omega_\infty = 0$, and no correction is needed.

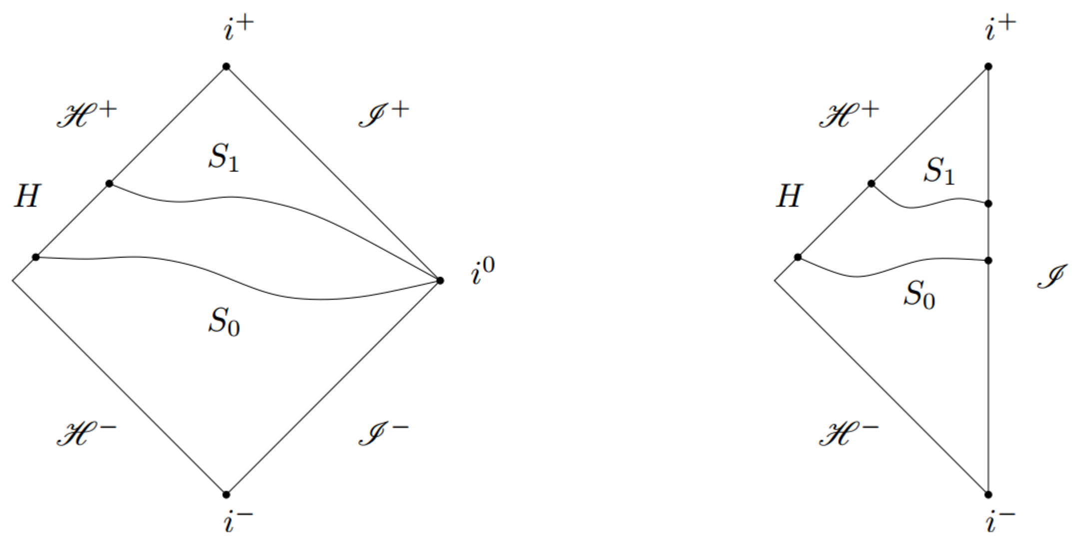

Figure 3: Penrose diagrams for the region of outer communication of the Kerr-Newman (left) and Kerr-Newman-AdS (right) spacetimes.

Analogously, but now without ambiguity, the total field angular momentum on a spacelike hypersurface S extending from the event horizon to infinity is

$$L = -\int_S (T^{\mu\nu} + j^\mu A^\nu) Y_\nu N_\mu dV_3, \tag{94}$$

where the minus sign accounts for the timelike unit normal.

Consider now two such spacelike hypersurfaces, S_0 and S_1, with S_1 to the future of S_0 (Figure 3). We assume reflecting boundary conditions in the Kerr-Newman-AdS case, so that all fluxes vanish at infinity. The energy absorbed by the black hole across the subset H of $\mathscr{H}^+$ between S_0 and S_1 is then

$$\Delta M = \int_{S_0} (T^{\mu\nu} + j^\mu A^\nu) K_\nu N_\mu dV_3 - \int_{S_1} (T^{\mu\nu} + j^\mu A^\nu) K_\nu N_\mu dV_3, \tag{95}$$

whereas the angular momentum absorbed by the black hole across H is

$$\Delta J = -\int_{S_0} (T^{\mu\nu} + j^\mu A^\nu) Y_\nu N_\mu dV_3 + \int_{S_1} (T^{\mu\nu} + j^\mu A^\nu) Y_\nu N_\mu dV_3. \tag{96}$$

Recall that the angular velocity of the black hole horizon is

$$\Omega_H = \frac{a\Xi}{r_+^2 + a^2}, \tag{97}$$

where r_+ is the largest root of $\Delta_r = 0$. This means that the (future-pointing) Killing generator of $\mathscr{H}^+$ is

$$Z = X + \Omega_H Y = K + \Omega Y, \tag{98}$$

where

$$\Omega = \Omega_H - \Omega_\infty \tag{99}$$

is precisely the thermodynamic angular velocity, that is, the angular velocity that occurs in the first law for Kerr-Newman-AdS black holes [52]. Therefore, we have

$$\Delta M - \Omega \Delta J = \int_{S_0} (T^{\mu\nu} + j^\mu A^\nu) Z_\nu N_\mu dV_3 - \int_{S_1} (T^{\mu\nu} + j^\mu A^\nu) Z_\nu N_\mu dV_3. \tag{100}$$

Because Z is also a Killing vector field,

$$\nabla_\mu (T^{\mu\nu} Z_\nu + j^\mu A^\nu Z_\nu) = 0, \tag{101}$$

and so the divergence theorem, applied to the region bounded by S_0, S_1 and H, yields

$$\Delta M - \Omega \Delta J = \int_H (T^{\mu\nu} + j^\mu A^\nu) Z_\nu Z_\mu dV_3 \tag{102}$$

(we use $-Z$ as the null normal[4] on H). Since on $\mathscr{H}^+$

$$A^\mu Z_\mu = -\frac{er_+}{r_+^2 + a^2} = -\Phi, \tag{103}$$

where Φ is the horizon's electric potential, we have

$$\int_H j^\mu A^\nu Z_\nu Z_\mu dV_3 = -\Phi \int_H j^\mu Z_\mu dV_3. \tag{104}$$

Using again the divergence theorem, this time together with the charge conservation equation (86), we obtain

$$\int_H j^\mu A^\nu Z_\nu Z_\mu dV_3 = -\Phi \int_{S_0} j^\mu N_\mu dV_3 + \Phi \int_{S_1} j^\mu N_\mu dV_3. \tag{105}$$

Now the total charge on a spacelike hypersurface S extending from the event horizon to infinity is

$$-\int_S j^\mu N_\mu dV_3, \tag{106}$$

where the minus sign accounts for the timelike unit normal. Therefore, denoting by ΔQ the electric charge absorbed by the black hole across H, we have

$$\int_H j^\mu A^\nu Z_\nu Z_\mu dV_3 = \Phi \Delta Q, \tag{107}$$

and so equation (102) can then be written as

$$\Delta M - \Omega \Delta J - \Phi \Delta Q = \int_H (T^{\mu\nu} Z_\mu Z_\nu) dV_3. \tag{108}$$

Since Z is null on H, we have the following test field version of the second law of black hole thermodynamics.

Theorem 4.1. *If the energy-momentum tensor T corresponding to any collection of test fields propagating on a Kerr-Newman or Kerr-Newman-AdS black hole background satisfies the null energy condition at the event horizon and appropriate boundary conditions at infinity then the energy ΔM, angular momentum ΔJ and electric charge ΔQ*

[4]Recall that the divergence theorem on a Lorentzian manifold requires that the unit normal is outward-pointing when spacelike and inward-pointing when timelike. When the normal is null it is non-unique, and the volume element depends on the choice of normal; it should be past-pointing in the future null subset of the boundary, and future-pointing in the past null subset of the boundary.

absorbed by the black hole satisfy

$$\Delta M \geq \Omega \Delta J + \Phi \Delta Q. \tag{109}$$

It should be stressed that (109) is valid for extremal black holes, and it does not assume cosmic censorship, i.e. it does not assume that the Kerr-Newman-AdS metric with physical mass $M+\Delta M$, angular momentum $J+\Delta J$ and electric charge is $Q+\Delta Q$ represents a black hole rather than a naked singularity. Note that this scenario where the test fields interact with the geometry and change the values of the black hole charges is not in contradiction with the test field approximation, since the change is supposed to be infinitesimal.

4.3 Proof of the result

We can now prove our main result.

Theorem 4.2. *Test fields satisfying the null energy condition at the event horizon and appropriate boundary conditions at infinity cannot destroy extremal Kerr-Newman or Kerr-Newman-AdS black holes. More precisely, if an extremal black hole is characterized by the physical quantities* (M, J, Q)*, and absorbs energy, angular momentum and electric charge* $(\Delta M, \Delta J, \Delta Q)$ *by interacting with the test fields, then the metric corresponding to the physical quantities* $(M + \Delta M, J + \Delta J, Q + \Delta Q)$ *represents either a subextremal or an extremal black hole.*

Proof. The physical mass of a Kerr-Newman or Kerr-Newman-AdS black hole, given in (81), is completely determined by the black hole's event horizon area A, angular momentum J and electric charge Q through a Smarr formula

$$M = M(A, J, Q). \tag{110}$$

From the first law of black hole thermodynamics, we know that this function satisfies

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ, \tag{111}$$

where κ is the surface gravity of the event horizon [39, 50, 52]. The condition for the black hole to be extremal is

$$\kappa = 0 \Leftrightarrow \frac{\partial M}{\partial A}(A, J, Q) = 0, \tag{112}$$

which can be solved to yield the area of an extremal black hole as a function of its angular momentum and charge,

$$A = A_{\text{ext}}(J, Q). \tag{113}$$

The mass of an extremal black hole with angular momentum J and electric charge Q is then

$$M_{\text{ext}}(J, Q) = M(A_{\text{ext}}(J, Q), J, Q). \tag{114}$$

A Kerr-Newman-AdS metric characterized by M, J and Q will represent a black hole if $M \geq M_{\text{ext}}(J, Q)$, and a naked singularity if $M < M_{\text{ext}}(J, Q)$. We have

$$\begin{aligned} dM_{\text{ext}} &= \left(\frac{\partial M}{\partial A} \frac{\partial A_{\text{ext}}}{\partial J} + \frac{\partial M}{\partial J} \right) dJ + \left(\frac{\partial M}{\partial A} \frac{\partial A_{\text{ext}}}{\partial Q} + \frac{\partial M}{\partial Q} \right) dQ \\ &= \left(\frac{\kappa}{8\pi} \frac{\partial A_{\text{ext}}}{\partial J} + \Omega \right) dJ + \left(\frac{\kappa}{8\pi} \frac{\partial A_{\text{ext}}}{\partial Q} + \Phi \right) dQ \\ &= \Omega dJ + \Phi dQ, \end{aligned} \tag{115}$$

where all quantities are evaluated at the extremal black hole.

Consider now an extremal black hole with angular momentum J, electric charge Q and mass $M = M_{\text{ext}}(J, Q)$. After interacting with the test fields, its angular momentum is $J + \Delta J$, its electric charge is $Q + \Delta Q$ and its mass is, using (109) and (115),

$$\begin{aligned} M + \Delta M &\geq M + \Omega \Delta J + \Phi \Delta Q \\ &= M_{\text{ext}}(J, Q) + \Delta M_{\text{ext}} \\ &= M_{\text{ext}}(J + \Delta J, Q + \Delta Q). \end{aligned} \tag{116}$$

In other words, the final mass is above the mass of an extremal black hole with the same angular momentum and electric charge, meaning that the final metric does not represent a naked singularity, that is, the black hole has not been destroyed. □

4.4 Discussion

In this paper we proved that extremal Kerr-Newman or Kerr-Newman-AdS black holes cannot be destroyed by interacting with (possibly charged) test fields satisfying the null energy condition at the event horizon and appropriate boundary conditions at infinity. This includes as particular cases all previous results of this kind obtained for scalar and electromagnetic test fields [22–25]. The corresponding results for test particles [19–21] can also be considered particular cases, since particles can be seen as singular limits of continuous media [44, 45]. It is interesting to note that if the null energy condition is not satisfied then the weak cosmic censorship conjecture may indeed be violated, as shown in [54, 55] for Dirac fields. Moreover, this result is also important for the consistency of the generalized second law of thermodynamics [59].

Our proof depends only on certain generic features of the Kerr-Newman or Kerr-Newman-AdS metric, and can therefore be adapted to other black holes. In fact, Theorem 4.2 can be generalized as follows.

Theorem 4.3. *Consider a family of charged and spinning black holes in some metric theory of gravity, with suitable asymptotic regions, and test fields propagating in these backgrounds, such that:*

1. *There exists an asymptotically timelike Killing vector field K, determining the black hole's physical mass, and angular Killing vector fields Y_i, yielding the black hole's angular momenta, such that event horizon's Killing generator is*

$$Z = K + \sum_i \Omega_i Y_i, \tag{117}$$

where Ω_i are the thermodynamic angular velocities (that is, the angular velocities that occur in the first law).

2. *There exists a Smarr formula relating the black hole's physical mass M, its entropy S, its angular momenta J_i and its electric charge Q,*

$$M = M(S, J_i, Q), \tag{118}$$

yielding the first law of black hole thermodynamics,

$$dM = TdS + \sum_i \Omega_i dJ_i + \Phi dQ, \tag{119}$$

where T is the black hole temperature and Φ is the event horizon's electric potential.

3. *Extremal black holes (that is, black holes with $T = 0$) are characterized by $M = M_{ext}(J_i, Q)$, and subextremal black holes by $M > M_{ext}(J_i, Q)$.*

4. *The test fields satisfy the null energy condition at the event horizon and appropriate boundary conditions at infinity.*

Then the test fields cannot destroy extremal black holes. More precisely, if an extremal black hole is characterized by the physical quantities (M, J_i, Q)*, and absorbs energy, angular momenta and electric charge* $(\Delta M, \Delta J_i, \Delta Q)$ *by interacting with the test fields, then the metric corresponding to the physical quantities* $(M + \Delta M, J_i + \Delta J_i, Q + \Delta Q)$ *represents either a subextremal or an extremal black hole.*

It is easy to check that this result applies to black holes in higher dimensions [56], including the case of a negative[5] cosmological constant [52]. It can also be used for other black holes, like accelerated black holes with conical singularities [57] or black holes in alternative theories of gravity [58]. There is, however, no *a priori* reason why it should apply to arbitrary parametrized deformations of the Kerr metric [60].

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Appendix

To obtain equation (83), we observe that the charged test fields generate an extra electromagnetic field f satisfying the Maxwell equations $df = 0$ and

$$\nabla_\mu f^{\mu\nu} = -j^\nu. \tag{120}$$

The total electromagnetic energy-momentum tensor is then

$$T^{EM}_{\mu\nu} = (F_{\mu\alpha} + f_{\mu\alpha})(F_\nu{}^\alpha + f_\nu{}^\alpha) - \frac{1}{4} g_{\mu\nu}(F_{\alpha\beta} + f_{\alpha\beta})(F^{\alpha\beta} + f^{\alpha\beta}). \tag{121}$$

[5]Theorem 4.3 does not apply to the case of a positive cosmological constant, because the first hypothesis is not satisfied.

Besides the stationary part, due solely to F, one has to consider, in the test field approximation, the cross terms

$$t_{\mu\nu} = f_{\mu\alpha}F_\nu{}^\alpha + F_{\mu\alpha}f_\nu{}^\alpha - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}f^{\alpha\beta}. \tag{122}$$

We have

$$\begin{aligned} \nabla^\mu t_{\mu\nu} &= -j_\alpha F_\nu{}^\alpha + f_{\mu\alpha}\nabla^\mu F_\nu{}^\alpha + F_{\mu\alpha}\nabla^\mu f_\nu{}^\alpha - \frac{1}{2}(\nabla_\nu F_{\alpha\beta})f^{\alpha\beta} - \frac{1}{2}F_{\alpha\beta}\nabla_\nu f^{\alpha\beta} \\ &= -F_{\nu\alpha}j^\alpha, \end{aligned} \tag{123}$$

where we used (120), the Maxwell equation $\nabla^\mu F_{\mu\alpha} = 0$, the fact that

$$f_{\mu\alpha}\nabla^\mu F^{\nu\alpha} - \frac{1}{2}(\nabla^\nu F_{\alpha\beta})f^{\alpha\beta} = \frac{1}{2}f_{\alpha\beta}\left(\nabla^\alpha F^{\nu\beta} + \nabla^\beta F^{\alpha\nu} - \nabla^\nu F^{\alpha\beta}\right) = 0 \tag{124}$$

(because of the Maxwell equation $dF = 0$), and the fact that

$$F_{\mu\alpha}\nabla^\mu f^{\nu\alpha} - \frac{1}{2}F_{\alpha\beta}\nabla^\nu f^{\alpha\beta} = \frac{1}{2}F_{\alpha\beta}\left(\nabla^\alpha f^{\nu\beta} + \nabla^\beta f^{\alpha\nu} - \nabla^\nu f^{\alpha\beta}\right) = 0 \tag{125}$$

(because of the Maxwell equation $df = 0$). Therefore, in the test field approximation, we have

$$\nabla^\mu\left(T_{\mu\nu} + T^{EM}_{\mu\nu}\right) = 0 \Leftrightarrow \nabla^\mu\left(T_{\mu\nu} + t_{\mu\nu}\right) = 0 \Leftrightarrow \nabla^\mu T_{\mu\nu} = F_{\nu\alpha}j^\alpha, \tag{126}$$

which is equation (83).

One may wonder why not use the conserved current

$$\nabla_\mu(T^{\mu\nu}K_\nu + t^{\mu\nu}K_\nu) = 0 \tag{127}$$

to define the energy of the test field as

$$E'' = \int_S (T^{\mu\nu} + t^{\mu\nu})K_\nu N_\mu dV_3. \tag{128}$$

The reason is that this expression accounts for the energy of the interaction between the charged field and the background electromagnetic field through the electromagnetic cross terms (122), whereas (93) localizes it on the charges. As is well known, the physical mass of a charged black hole includes the energy of its background electromagnetic field; when charge enters the black hole, the interaction energy should be transferred from the energy of the electromagnetic field to the black hole's mass. This accounting is accomplished by (93), but not[6] by (128).

One might also worry that the presence of the extra energy-momentum tensor t with nonzero divergence (123) could invalidate our previous conclusions. That is not the case, however, because t does not contribute to the flux across the horizon. In fact, using (88) and the Killing equation (84), we have

$$\begin{aligned}
&\int_H t_{\mu\nu} Z^\mu Z^\nu = \int_H 2 f_\mu{}^\alpha F_{\nu\alpha} Z^\mu Z^\nu = \int_H 2 f_\mu{}^\alpha (\nabla_\nu A_\alpha - \nabla_\alpha A_\nu) Z^\mu Z^\nu \\
&= \int_H 2 f_\mu{}^\alpha (A^\nu \nabla_\nu Z_\alpha - Z^\nu \nabla_\alpha A_\nu) Z^\mu = \int_H 2 f_\mu{}^\alpha (-A^\nu \nabla_\alpha Z_\nu - Z^\nu \nabla_\alpha A_\nu) Z^\mu \\
&= -\int_H 2 Z^\mu f_\mu{}^\alpha \nabla_\alpha (A^\nu Z_\nu) = 0,
\end{aligned} \tag{129}$$

since the vector field $Z^\mu f_\mu{}^\alpha$ is tangent to the event horizon,

$$Z^\mu f_\mu{}^\alpha Z_\alpha = 0, \tag{130}$$

and $A^\nu Z_\nu = \Phi$ is constant along the event horizon.

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