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The Yamabe Problem

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Para a minha Mãe.

Declaration

I declare that this document is an original work of my own authorship and that it fulfills all the requirements of the Code of Conduct and Good Practices of the Universidade de Lisboa.

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Resumo

Nesta dissertação apresenta-se, de forma detalhada e auto-contida, a solução do problema de Yamabe. Historicamente, problema de Yamabe surgiu quando Hidehiko Yamabe queria resolver a conjectura de Poincaré. Matematicamente, o problema de Yamabe consiste em responder à seguinte pergunta: dada uma variedade Riemanniana compacta e conexa, será que existe uma métrica na variedade conforme à original que tenha curvatura escalar constante? Analiticamente, o problema de Yamabe é equivalente a resolver uma equação com derivadas parciais elíptica não linear numa variedade Riemanniana. Ao longo da dissertação seguimos de perto, adicionando todos os detalhes, o artigo acerca do problema de Yamabe de John M. Lee e Thomas H. Parker [13].

Palavras-chave: Análise Geométrica, Equações com Derivadas Parciais Elípticas, Geometria Riemanniana, Geometria Conforme, Funções de Green.

Abstract

This dissertation aims to present, in detail and in a self-contained form, the solution to the Yamabe problem. From a historical point of view, the Yamabe problem came to be when, in 1960 Hidehiko Yamabe was interested in solving the Poincaré conjecture. Mathematically speaking, the Yamabe problem consists in answering the following: given a compact, connected Riemannian manifold, is there a metric conformal to the original one on the manifold that has constant scalar curvature? From an analytical point of view, the Yamabe problem is equivalent to solving a nonlinear elliptic partial differential equation on the Riemannian manifold. Throughout this dissertation, we followed very closely John M. Lee's and Thomas H. Parker's paper on the Yamabe problem [13] where we have added all the details.

Keywords: Geometric Analysis, Elliptic Partial Differential Equations, Riemannian Geometry, Conformal Geometry, Green Functions.

Contents

Nomenclature

(g^{ij}) Components of the inverse matrix of $g = (g_{ij})$.

(M, g) Riemannian manifold.

$2^* = \frac{2n}{n-2}$ Critical exponent of the Sobolev inequality in $\mathbb{R}^n$

Δ Euclidean Laplacian, given by $\Delta f = -\partial_{ii} f$

Δ_g Laplace-Beltrami operator of (M, g), given in local coordinates by $\Delta_g f = -\frac{1}{\sqrt{\det(g_{ij})}} \partial_i \left(g^{ij} \sqrt{\det(g_{ij})} \partial_j f \right)$.

Γ Green function of the conformal Laplacian.

Γ_{ij}^k Christoffel symbols of the Levi-Civita connection.

$\langle \cdot, \cdot \rangle$ Inner product in a Riemannian Manifold (M, g).

$\mathcal{L}_g$ Conformal Laplacian of the metric g, defined as $\mathcal{L}_g = a\Delta_g + S$, where $a = 4\frac{n-1}{n-2}$.

∇ Levi-Civita connection.

ω_{n-1} Volume of the Euclidean unit sphere in $\mathbb{R}^n$.

R Curvature operator of Levi-Civita connection.

Ric Ricci tensor.

Rm Riemann curvature tensor.

$\mathbb{R}^n$ Standard Euclidean space equipped with the standard Euclidean metric.

$\overset{\circ}{\mathrm{Ric}}$ Traceless Ricci tensor.

W Weyl tensor.

$ds^2 = dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n$ Euclidean metric in $\mathbb{R}^n$.

dV_g Riemannian volume form of (M, g).

S Scalar Curvature of (M, g).

Chapter 1

Introduction

Hidehiko Yamabe, wanting to solve the Poincaré Conjecture (see [4]), thought that a good first step would be to, given a compact, connected smooth Riemannian manifold, equip that manifold with a Riemannian metric of constant scalar curvature. In this setting, Yamabe considered the simplest change of metric, which is a conformal transformation, and in 1960 he sought to prove in [19] that every compact connected Riemannian manifold (M, g) of dimension $\dim M \geq 3$ admits a metric conformal to g that has constant scalar curvature. From an analytical point of view, this problem is equivalent to solving a nonlinear elliptic partial differential equation on a Riemannian manifold. Indeed, if (M, g) is a compact, connected Riemannian manifold of dimension n, every metric conformal to g can uniquely be written as $\widetilde{g} = \varphi^{2^*-2}g$ for some (unique) positive smooth function φ in M, where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent. So, the transformation law (A.0.19) yields that the scalar curvature of the metric $\widetilde{g}$, $\widetilde{S}$, is given by

$$\widetilde{S} = \varphi^{1-2^*} \left(\frac{4(n-1)}{n-2} \Delta_g \varphi + S\varphi \right), \tag{1.0.1}$$

where S denotes the scalar curvature of g and Δ_g denotes the Laplace-Beltrami operator, that, in local coordinates $\{x^i\}$, takes the form $\Delta_g \varphi = -\det(g_{ij})^{-1/2} \partial_i \left(g^{ij} \det(g_{ij})^{1/2} \partial_j \varphi \right)$. Thus, finding a conformal metric with constant scalar curvature is equivalent to finding a smooth positive function on M for which the right-hand side of (1.0.1) is constant. In the very same paper where Yamabe conjectured the existence of such a metric, he attempted to solve this problem. Yamabe's approach was to formulate a variational problem that, if solved, would imply his statement. Indeed, Yamabe considered the functional

$$\mathcal{Q}(\widetilde{g}) := \frac{\displaystyle\int_M \widetilde{S} dV_{\widetilde{g}}}{\left(\displaystyle\int_M dV_{\widetilde{g}} \right)^{2/2^*}}, \tag{1.0.2}$$

where $\widetilde{g}$ is a metric conformal to g. Due to the one-to-one correspondence between conformal metrics and smooth positive functions on M, by setting $\mathcal{Q}_g(\varphi) := \mathcal{Q}(\varphi^{2^*-2}g)$, equation (1.0.1) implies that

$$\mathcal{Q}_g(\varphi) = \frac{\displaystyle\int_M a\varphi\Delta_g\varphi + S\varphi^2 dV_g}{\left(\displaystyle\int_M \varphi^{2^*} dV_g\right)^{2/2^*}} = \frac{\displaystyle\int_M a|\nabla\varphi|^2 + S\varphi^2 dV_g}{\left(\displaystyle\int_M \varphi^{2^*} dV_g\right)^{2/2^*}},$$

where $a := 4\frac{n-1}{n-2}$. Yamabe noted that the Euler-Lagrange equation for this functional is precisely equation (1.0.1). Indeed, if φ is a positive, smooth critical point of $\mathcal{Q}_g$, then φ is a solution of (1.0.1) with $\widetilde{S} = \mathcal{Q}_g(\varphi)$, as shown in (2.0.7).

Yamabe then defined the now-called Yamabe invariant:

$$\lambda(M) = \inf\{\mathcal{Q}_g(\varphi) : \varphi \in \mathcal{C}^\infty(M) \text{ positive}\}.$$

As we will see in Chapter 2, in the definition of $\lambda(M)$ we can extend the space of functions to $H := \{u \in H^1(M) : u \neq 0\}$ (which is a Hilbert space). This gives rise to the problem of minimizing $\mathcal{Q}_g$ over H. Associated with this problem is the Yamabe equation

$$a\Delta_g u + Su = \lambda(M)u^{2^*-2}, \text{ on } M. \tag{1.0.3}$$

The operator $a\Delta_g + S$ is usually called the conformal Laplacian, and from now onwards we will denote it by $\mathcal{L}_g$. This nomenclature comes from the fact that when $\widetilde{g} = \varphi^{2^*-2}g$ is a conformal metric to g, then

$$\mathcal{L}_{\widetilde{g}}(\varphi^{-1}u) = \varphi^{1-2^*}\mathcal{L}_g u.$$

Yamabe presented in [19] a solution to this variational problem and (1.0.3) using methods from Calculus of Variations and Elliptic Partial Differential Equations.

Later on, in the same year, Yamabe died from a stroke (see [9]), just months after accepting a full professorship at Northwestern University, believing his solution of the Yamabe problem was correct. However, our story does not end here, for in 1968 Neil Trudinger found a mistake in Yamabe's proof (see [18]), and was able to adapt Yamabe's proof with an extra assumption imposed on the manifold. More precisely, he proved that if $\lambda(M)$ is below a certain (non-explicit) positive threshold, Yamabe's proof would work, and so Yamabe's conjecture was proved correct when $\lambda(M) \leq 0$. However, the general question remained open: does every compact, connected Riemannian manifold of dimension greater or equal to 3 admit a conformal metric of constant scalar curvature?

A few years later, in 1976, Thierry Aubin extended Trudinger's results in [3]. Indeed, Aubin showed that the Yamabe invariant on any compact, connected Riemannian manifold (M, g) satisfies $\lambda(M) \leq \lambda(\mathbb{S}^n)$. In the same paper, Aubin also showed that the threshold obtained by Trudinger could be taken to be $\lambda(\mathbb{S}^n)$. The combined efforts of Yamabe, Trudinger and Aubin put together led to the following:

Theorem 1.0.1 (Yamabe, Trudinger, Aubin)**.** *For any compact, connected Riemannian manifold of di-*

mension $n \geq 3$, we have $\lambda(M) \leq \lambda(\mathbb{S}^n)$. Furthermore, if $\lambda(M) < \lambda(\mathbb{S}^n)$, then there is a conformal metric to g with constant scalar curvature equal to $\lambda(M)$, where $\mathbb{S}^n$ is the standard sphere.

We will present a proof of this theorem, which is the one from [13]. To prove this theorem, the idea is to perturb the Yamabe equation (1.0.3) with a parameter such that, as the parameter goes to 2^*, we get closer and closer to the Yamabe equation (1.0.3). This was Yamabe's original plan (see [19]). Indeed, he considered the family of problems

$$a\Delta_g u + Su = \lambda_s(M)u^{s-1} \text{ on } M, \tag{1.0.4}$$

where $s \in [2, 2^*[$, and

$$\lambda_s(M) := \inf\{\mathcal{Q}_g^s(\varphi) : \varphi \in H^1(M) \setminus \{0\}\}, \tag{1.0.5}$$

where $\mathcal{Q}_g^s(\varphi) = \dfrac{\int_M a|\nabla\varphi|^2 + S\varphi^2 dV_g}{\left(\int_M \varphi^s dV_g\right)^{2/s}}$. Using classical techniques from the theory of Elliptic Partial Differential Equations and the Direct Method of the Calculus of Variations, Yamabe showed in [19] that these problems have a smooth positive solution. Using the assumption $\lambda(M) < \lambda(\mathbb{S}^n)$, one is able to show that the family of solutions $\{u_s\}$ is uniformly bounded in $L^r(M)$ for some $r > 2^*$. This was originally proved by Trudinger in [18] (where he proved the result when $\lambda(M)$ is small enough) and Aubin in [3] extended Trudinger's result for the case $\lambda(M) < \lambda(\mathbb{S}^n)$. Finally, one proceeds to apply the regularity theory followed by the Arzelà–Ascoli Theorem to show that along a subsequence the solutions $\{u_s\}$ converge in the $\mathcal{C}^2-$norm along a subsequence to a solution of the Yamabe equation. The mistake in Yamabe's attempt was to assume that the sequence $\{u_s\}$ was uniformly bounded whether or not $\lambda(M) < \lambda(\mathbb{S}^n)$.

This theorem breaks the problem of minimizing $\mathcal{Q}_g$ into two smaller problems, the first one being when the manifold we are working with is the standard sphere or conformal to the standard sphere, the second one being all the other cases.

In the conformal class of the sphere (the case where $\lambda(M) = \lambda(\mathbb{S}^n)$), we also use the family of solutions $\{u_s\}$, but a much more delicate method is needed to prove the existence of a solution of the Yamabe equation.

In the remaining cases, it turns out that, when (M, g) is not conformal to the standard sphere, we do have $\lambda(M) < \lambda(\mathbb{S}^n)$. Proving this, however, is anything but trivial. In fact, to prove this fact took the combined efforts of Aubin [2] and Schoen [16]. Their strategy is, nowadays, a classical one, to construct a test function. Their methods, however, are very different. This is, in part, because their proofs concern different "categories" of manifolds with very distinct geometries. More precisely, Aubin in 1976 showed in [2] the following result:

Theorem 1.0.2. *Let (M, g) is a compact, connected, not locally conformally flat Riemannian manifold of dimension $\dim M \geq 6$, then $\lambda(M) < \lambda(\mathbb{S}^n)$.*

Schoen then showed in 1984 in [16]:

Theorem 1.0.3. *Let (M,g) is a compact, connected, Riemannian manifold such that $\dim M = 3,4,5$ or (M,g) is locally conformally flat and M is not conformal to the standard sphere, then $\lambda(M) < \lambda(\mathbb{S}^n)$.*

Later, in 1987, John M. Lee and Thomas H. Parker published a now widely celebrated paper titled "The Yamabe Problem" [13], where they presented, for the first time, in one place, the complete solution to the Yamabe problem. However, they presented a more unified approach to showing that $\lambda(M) < \lambda(\mathbb{S}^n)$. Unlike Aubin and Schoen, who considered completely different test functions, Lee and Parker, inspired by the work of Schoen in [16], managed to show, using one test function, that $\lambda(M) < \lambda(\mathbb{S}^n)$ when (M,g) is not conformal to the standard sphere. Their method employs new tools, conformal normal coordinates, which we explore in great detail in chapter 4, that not only massively simplify the local geometry but also allow us to obtain an expansion of the Green function of the conformal Laplacian $\mathcal{L}_g$. In this dissertation, apart from the occasional detour, we will follow very closely Lee and Parker's paper [13], adding all the missing details and on occasion shortening their route.

We now give a brief description of the organisation of this dissertation, as well as the contents of each chapter. In Chapter 2 we will further explore the Yamabe invariant and prove the following theorem that is due to Aubin in [3]:

Theorem 1.0.4. *For any compact, connected Riemannian manifold (M,g) of dimension $n \geq 3$, we have $\lambda(M) \leq \lambda(\mathbb{S}^n)$.*

Furthermore, we will explore a curious connection between the sharp Sobolev constant in $\mathbb{R}^n$ and the Yamabe invariant in the standard sphere, and present a result, due to Obata [14], that completely characterises all the solutions of the Yamabe problem on the standard sphere. In other words, Obata's result gives a description of all the metrics on the sphere conformal to the standard metric that possesses constant scalar curvature.

In Chapter 3, we present the combined work of Yamabe, Trudinger and Aubin that shows that the condition $\lambda(M) < \lambda(\mathbb{S}^n)$ (in the setting described above) is sufficient to ensure the existence of a conformal metric with constant scalar curvature.

In Chapter 4, we present one of Lee and Parker's original contributions, by developing the necessary machinery for the construction of the test function done in [13]. More specifically, we present the geometrical construction of normal coordinates and conformal normal coordinates (these are normal coordinates with respect to a metric that is conformal to the original one) and some very powerful and useful properties of these coordinates; in particular, we show that for every positive integer $N \in \mathbb{N}$ there are conformal normal coordinates, $\{x^i\}$, associated to a conformal metric $\widetilde{g}$ such that

$$\det(\widetilde{g}_{ij}) = 1 + O(r^N),$$

where $r = |x|$ is the geodesic distance. Recall that a Riemannian manifold (M,g) is said to be locally conformally flat if it is locally conformal to $\mathbb{R}^n$. An immediate application of the coordinates is the following:

Theorem 1.0.5. *Let (M,g) be a compact, connected, non-locally conformally flat Riemannian manifold* *with dimension* $\dim M \geq 6$*, then* $\lambda(M) < \lambda(\mathbb{S}^n)$.

Furthermore, we will use these coordinates to obtain an asymptotic expansion of a multiple of the Green function of the conformal Laplacian, $\mathcal{L}_g$.

We also present a generalization of the well-known stereographic projection in the sphere to a class of manifolds, still called stereographic projection, and explore some of its properties.

In Chapter 5, we apply all the tools developed in Chapter 4 to the construction of the test function.

Finally, in Chapter 6 we present the culmination of all the previous chapters, the complete solution of the Yamabe problem. To do this, we introduce the concept of the mass of an asymptotically flat Riemannian manifold and present the well-known Positive Mass Theorem, relating it to the work developed in Chapter 5. Finally, we apply the aforementioned theorem to prove that $\lambda(M) < \lambda(\mathbb{S}^n)$ when either $\dim M = 3, 4, 5$ or (M, g) is locally conformally flat, and therefore get a solution to the Yamabe problem.

This works concludes with two appendices where we list all the basic facts about Riemannian Geometry and Partial Differential Equations used in the previous chapters.

Chapter 2

The Yamabe Problem and the Sphere

Let (M, g) be a compact, connected Riemannian manifold of dimension $n \geq 3$. To solve the Yamabe Problem we need to find a conformal metric with constant scalar curvature. As stated in the Introduction any conformal metric to g can uniquely be written as $\varphi^{2^*-2}g$ for some smooth positive function φ. Then the transformation laws for the scalar curvature A.0.19 implies that solving the Yamabe Problem is equivalent to finding a $\lambda \in \mathbb{R}$ for which there is a smooth positive solution of the Yamabe equation

$$a\Delta_g \varphi + S\varphi = \lambda \varphi^{2^*-1}, \text{ in } M \tag{2.0.1}$$

but the task of just finding such a constant and solution is rather complicated. However, we can formulate this problem as a minimization problem. To see this, we introduce a number that will be central in the analysis of this problem.

Definition 2.0.1. *Let (M, g) be a compact, connected Riemannian manifold of dimension $n \geq 3$. For $\widetilde{g}$ a conformal metric to g we define the Yamabe functional*

$$\mathcal{Q}(\widetilde{g}) := \frac{\displaystyle\int_M \widetilde{S} dV_{\widetilde{g}}}{\left(\displaystyle\int_M dV_{\widetilde{g}}\right)^{2/2^*}}, \tag{2.0.2}$$

where $\widetilde{S}$ is the scalar curvature of $\widetilde{g}$ and $dV_{\widetilde{g}}$ is the Riemannian volume form of $(M, \widetilde{g})$. Now define the Yamabe invariant:

$$\lambda(M) := \inf\{\mathcal{Q}(\widetilde{g}) : \widetilde{g} \text{ conformal to } g\}. \tag{2.0.3}$$

Note that, by definition $\lambda(M)$ is a conformal invariant of (M, g), and recall that every conformal metric to g has the form $\widetilde{g} = \varphi^{2^*-2}g$. Using the transformation laws in Appendix A, we can write $\lambda(M)$ as an infimum of a certain functional over Sobolev spaces. Indeed, for $\varphi > 0$ smooth and $\widetilde{g} = \varphi^{2^*-2}g$ the transformation law (A.0.19) for the scalar curvature gives

$$\widetilde{S} = \varphi^{2-2^*}\left(S + 4\frac{n-1}{n-2}\varphi^{-1}\Delta_g \varphi\right), \tag{2.0.4}$$

and thus, setting $a = 4\frac{n-1}{n-2}$

$$\begin{aligned}\mathcal{Q}(\widetilde{g}) &= \frac{\displaystyle\int_M \varphi^{2-2^*}\left(S + a\varphi^{-1}\Delta_g\varphi\right)dV_{\widetilde{g}}}{\|\varphi\|_{2^*}^2} = \frac{\displaystyle\int_M \varphi^{2-2^*}\left(S + a\varphi^{-1}\Delta_g\varphi\right)\varphi^{2^*}dV_g}{\|\varphi\|_{2^*}^2} \\ &= \frac{\displaystyle\int_M S\varphi^2 + a\varphi\Delta_g\varphi dV_g}{\|\varphi\|_{2^*}^2} = \frac{\displaystyle\int_M a|\nabla\varphi|^2 + S\varphi^2 dV_g}{\|\varphi\|_{2^*}^2}.\end{aligned}$$

Noting that $\varphi^2 \in L^{2^*/2}(M)$, Hölder's inequality implies $|\int_M S\varphi^2 dV_g|$ is bounded by a multiple of $\|\varphi\|_{2^*}^2$ and so by denoting $\mathcal{Q}_g(\varphi) = \mathcal{Q}(\varphi^{2^*-2}g)$ we have that $\mathcal{Q}_g$ is bounded from below:

$$\mathcal{Q}_g(\varphi) = \frac{\int_M a|\nabla\varphi|^2 + S\varphi^2 dV_g}{\|\varphi\|_{2^*}^2} \geq \frac{\int_M a|\nabla\varphi|^2 dV_g - C\|\varphi\|_{2^*}^2}{\|\varphi\|_{2^*}^2} \geq C,$$

for some positive constant that depends only on S. Due to the definition of $\lambda(M)$ we have:

$$\lambda(M) = \inf\{\mathcal{Q}_g(\varphi) : \varphi \in \mathcal{C}^\infty(M) \text{ positive}\}. \tag{2.0.5}$$

Furthermore, due to the density of $\mathcal{C}^\infty(M)$ in $H^1(M)$ and the fact that $\mathcal{Q}_g(|\varphi|) = \mathcal{Q}_g(\varphi)$ we conclude that

$$\lambda(M) = \inf_{\varphi \in H^1(M)\setminus\{0\}} \mathcal{Q}_g(\varphi) = \inf_{\substack{\varphi \in H^1(M) \\ \|\varphi\|_{2^*}=1}} \mathcal{Q}_g(\varphi). \tag{2.0.6}$$

Define $E(\varphi) := \displaystyle\int_M a|\nabla\varphi|^2 + S\varphi^2 dV_g$ and call $\mathcal{Q}_g$ the Yamabe functional of (M, g). For any $\varphi, \psi \in H^1(M) \setminus \{0\}$, integration by parts yields

$$\begin{aligned}\frac{d}{dt}\mathcal{Q}_g(\varphi + t\psi)_{|t=0} &= \frac{\|\varphi\|_{2^*}^2 \frac{d}{dt}_{|t=0}E(\varphi + t\psi) - E(t)\frac{d}{dt}_{|t=0}\|\varphi + t\psi\|_{2^*}^2}{\|\varphi\|_{2^*}^4} \\ &= 2\frac{\displaystyle\int_M \left(a\Delta_g\varphi + S\varphi - \frac{E(\varphi)}{\|\varphi\|_{2^*}^{2^*}}\varphi^{2^*-1}\right)\psi dV_g}{\|\varphi\|_{2^*}^2}.\end{aligned} \tag{2.0.7}$$

So we see that $\varphi \in \mathcal{C}^\infty(M) \setminus \{0\}$ is a critical point of $\mathcal{Q}_g$ if and only if φ is a solution of the Yamabe equation 2.0.1 with $\lambda = \frac{E(\varphi)}{\|\varphi\|_{2^*}^{2^*}}$. In particular, if $\lambda(M)$ is achieved by some positive smooth solution, the minimizing function is a critical point of the functional $\mathcal{Q}_g$, and so it solves the Yamabe equation with $\lambda = \lambda(M)$.

The analysis of the Yamabe problem depends upon a precise understanding of the problem in the model case of the sphere $\mathbb{S}^n$ and in the relation between $\lambda(\mathbb{S}^n)$ and $\lambda(M)$. In fact, we have the following result due to Aubin in [3] which we will prove below:

Theorem 2.0.2. *If (M, g) is any compact, connected Riemannian manifold of dimension $n \geq 3$, then $\lambda(M) \leq \lambda(\mathbb{S}^n)$.*

To prove this theorem we need to explore the Yamabe problem in the standard sphere and its relation to the sharp Sobolev constant. First, we explore the sharp constant in the Sobolev inequality.

2.1 Sharp Sobolev constant

The Yamabe Problem in the sphere is related to the problem of finding minimizers for the best constant in the Sobolev inequality on $\mathbb{R}^n$. In this section, we explore this relationship quantitatively, we also show that $\lambda(M) \leq \lambda(\mathbb{S}^n)$ for any compact, connected Riemannian manifold (M, g) of dimension n.
We start by writing the standard metric on $\mathbb{S}^n$ in local coordinates induced by the usual stereographic projection. Let $P = (0, ..., 0, 1)$ be the north pole on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. The stereographic projection

$$\sigma : \mathbb{S}^n \setminus \{P\} \to \mathbb{R}^n, \ \ \sigma(\zeta^1, \dots, \zeta^n, \xi) = \frac{(\zeta^1, \dots, \zeta^n)}{1 - \xi}$$

is a chart for $\mathbb{S}^n$ compatible with the usual differential structure on $\mathbb{S}^n$, and therefore, the stereographic projection is a conformal diffeomorphism as the next lemma shows. The inverse map to the stereographic projection, $\rho = \sigma^{-1}$ is given by $\rho(x^1, ..., x^n) = \frac{1}{|x|^2+1}(2x^1, ..., 2x^n, |x|^2 - 1)$.

Lemma 2.1.1. *Let $\overline{g}$ denote the standard metric in $\mathbb{S}^n$. Then*

$$(\rho^* \overline{g})_{ij} = 4(|x|^2 + 1)^{-2} \delta_{ij}. \tag{2.1.1}$$

Proof. Let $x \in \mathbb{R}^n$ and $V, W \in T_x\mathbb{R}^n$. By definition, $\rho^*\overline{g}(V, W) = \overline{g}(\rho_*V, \rho_*W) = \overline{g}(d\rho(V), d\rho(W))$, and $d\rho(V) = \frac{d}{dt}_{|t=0} \rho(x + tV)$. Write $V = (V^1, ..., V^n)$ (here we are using the identification $T_x\mathbb{R}^n \cong \mathbb{R}^n$).Then,

$$\begin{aligned}
d\rho(V) &= \frac{d}{dt}_{|t=0} \left[\frac{1}{|x + tV|^2 + 1} \left(2(x^1 + tV^1), \dots, 2(x^n + tV^n), |x + tV|^2 - 1 \right) \right] \\
&= \left(\frac{2V^1}{|x|^2 + 1} - \frac{4x^1\langle x, V \rangle}{(|x|^2 + 1)^2}, \dots, 2\frac{V^n}{|x|^2 + 1} - \frac{4x^n\langle x, V \rangle}{(|x|^2 + 1)^2}, 4\frac{\langle x, V \rangle}{(|x|^2 + 1)^2} \right) \\
&= \frac{2}{|x|^2 + 1}(V, 0) - \frac{4\langle x, V \rangle}{(|x|^2 + 1)^2}(x, 0) + \frac{4\langle x, V \rangle}{(|x|^2 + 1)^2}(0, \dots, , 0, 1) \\
&= \frac{2}{|x|^2 + 1}(V, 0) - \frac{4\langle x, V \rangle}{(|x|^2 + 1)^2}(x, -1),
\end{aligned}$$

where, $\langle x, V \rangle = \sum_j x^j V^j$ denotes the Euclidean inner product in $\mathbb{R}^n$. Since $\overline{g}$ is obtained from the Euclidean metric in $\mathbb{R}^{n+1}$ by restricting it to $\mathbb{S}^n$ we have that

$$\begin{aligned}
\rho^*\overline{g}(V, W) &= \overline{g}(\rho_*V, \rho_*W) \\
&= \frac{1}{(|x|^2 + 1)^2} \left[2 \begin{pmatrix} V^T & 0 \end{pmatrix} - \frac{\langle x, V \rangle}{|x|^2 + 1} \begin{pmatrix} 4x^T & -4 \end{pmatrix} \right] I_{n+1} \left[2 \begin{pmatrix} W \\ 0 \end{pmatrix} - \frac{\langle x, W \rangle}{|x|^2 + 1} \begin{pmatrix} 4x \\ -4 \end{pmatrix} \right] \\
&= \frac{1}{(|x|^2 + 1)^2} \left(4V^TW - 16\frac{\langle x, V \rangle\langle x, W \rangle}{|x|^2 + 1} + \frac{\langle x, V \rangle\langle x, W \rangle(16|x|^2 + 16)}{(|x|^2 + 1)^2} \right) \\
&= \frac{4}{(|x|^2 + 1)^2} V^TW.
\end{aligned}$$

So $\rho^*\overline{g} = \frac{4}{(|x|^2+1)^2} ds^2$. $\square$

The pullback metric computed in the previous lemma (which is just the round metric expressed in the coordinates induced by the stereographic projection) can be written as $4u_1^{2^*-2} ds^2$, where

$u_1(x) = (|x|^2+1)^{(2-n)/2}$.

Using the stereographic projection, one can see that the group of conformal diffeomorphisms of the sphere is generated by rotations, together with maps of the form $\sigma^{-1}\tau_v\sigma$ and $\sigma^{-1}\delta_\epsilon\sigma$, where $\tau_v, \delta_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ denote the translation by v and dilation by $\alpha > 0$, respectively:

$$\tau_v(x) = x - v, \quad \delta_\alpha(x) = \alpha^{-1}x.$$

Under dilations the spherical metric on $\mathbb{R}^n$ (the standard metric on the sphere) transforms as

$$\delta_\alpha^*\rho^*\overline{g} = 4u_\alpha^{2^*-2}ds^2, \text{ where } u_\alpha(x) = \left(\frac{|x|^2+\alpha^2}{\alpha}\right)^{(2-n)/2}. \tag{2.1.2}$$

It is a well-known fact the sphere with the round metric has constant scalar curvature ($S = \frac{n(n-1)}{r^2}$ for a sphere of radius r). And so the Yamabe problem in the conformal class of the standard sphere is already solved. What is not yet solved is the problem of achieving $\lambda(\mathbb{S}^n)$. What is not immediately obvious is that the standard metric is a minimiser of the Yamabe functional, this will be proved using the results from this chapter and the ones from Chapter 3. To see this, we start by relating the Yamabe functional (and its minimization) to the Sobolev inequality in $\mathbb{R}^n$. By the density of $\mathcal{C}^\infty(\mathbb{S}^n)$ in $H^1(\mathbb{S}^n)$ we can conclude that $\lambda(\mathbb{S}^n)$ is actually the infimum over $H^1(\mathbb{S}^n)\setminus\{0\}$:

$$\lambda(\mathbb{S}^n) = \inf_{\varphi\in H^1(\mathbb{S}^n)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{S}^n}\left(a|\nabla\varphi|^2 + S\varphi^2\right)dV_{\overline{g}}}{\left(\displaystyle\int_{\mathbb{S}^n}|\varphi|^{2^*}dV^{\overline{g}}\right)^{2/2^*}}. \tag{2.1.3}$$

For $\varphi \in \mathcal{C}^\infty(\mathbb{S}^n)$ let $\overline{\varphi}$ denote the weighted pull-back function on $\mathbb{R}^n$ defined by $\overline{\varphi} = u_1\rho^*\varphi$. We then have

$$\rho^*(\varphi^{2^*-2}\overline{g}) = 4\overline{\varphi}^{2^*-2}ds^2,$$

and

$$\rho^*(a\Delta_{\overline{g}}\varphi + S\varphi) = au_1^{1-2^*}\Delta\overline{\varphi}, \text{ on } \mathbb{R}^n.$$

Then

$$\int_{\mathbb{S}^n}\left(a\varphi\Delta_{\overline{g}}\varphi + S\varphi^2\right)dV_{\overline{g}} = \int_{\mathbb{R}^n}\left(au_1^{1-2^*}\Delta\overline{\varphi}\right)2^n u_1^{2^*}dx = 2^n\int_{\mathbb{R}^n}a\overline{\varphi}\Delta\overline{\varphi}dx = 2^n\int_{\mathbb{R}^n}a|\nabla\overline{\varphi}|^2dx, \tag{2.1.4}$$

where, in the last integral, ∇ denotes the usual Euclidean gradient. Moreover,

$$\int_{\mathbb{S}^n}|\varphi|^{2^*}dV_{\overline{g}} = \int_{\mathbb{R}^n}2^n|\rho^*\varphi|^{2^*}u_1^{2^*}dx = 2^n\int_{\mathbb{R}^n}|\overline{\varphi}|^{2^*}dx.$$

Now by recalling the definition of the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ (Definition B.0.17), its properties and the fact that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}^{1,2}(\mathbb{S}^n)$ and taking (2.1.4) into account we have:

Proposition 2.1.2. *The map $\mathcal{J} : H^1(\mathbb{S}^n) \to \mathcal{D}^{1,2}(\mathbb{R}^n)$ given by $\varphi \to u_1\rho^*\varphi$ is an isomorphism. And if we multiply u_1 by an appropriate constant we conclude that there is an isometric isomorphism between*

$H^1(\mathbb{S}^n)$ *and* $\mathcal{D}^{1,2}(\mathbb{R}^n)$.

Therefore,

$$\lambda(\mathbb{S}^n) = \inf_{\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} a|\nabla\varphi|^2 dx}{\left(\int_{\mathbb{R}^n} |\varphi|^{2^*}\right)^{2/2^*}} \tag{2.1.5}$$

Denoting the sharp constant of the Sobolev inequality by

$$\sigma_n = \inf_{\substack{\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^n) \\ \|\varphi\|_{2^*}=1}} \|\nabla\varphi\|_2^2, \tag{2.1.6}$$

we have, the following theorem:

Theorem 2.1.3. *The* $n-$*dimensional Sobolev constant* σ_n *is equal to* $\lambda(\mathbb{S}^n)/a$. *Thus the sharp form of the Sobolev inequality in* $\mathbb{R}^n$ *is:*

$$\|\varphi\|_{2^*}^2 \leq \frac{a}{\lambda(\mathbb{S}^n)} \|\nabla\varphi\|_2^2,$$

for all $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^n)$.

In chapter 3 we will show that the standard metric and the conformal metrics $\delta_\alpha^* \rho^* \overline{g}$ are extremal with respect to the Yamabe functional, for now we assume this fact and that Theorem 2.1.3 and proceed with the proof of Theorem 2.0.2.

Proof of Theorem 2.0.2. As we just mentioned, the functions u_α satisfy $a\|\nabla u_\alpha\|_2^2 = \lambda(\mathbb{S}^n)\|u_\alpha\|_{2^*}^2$ on $\mathbb{R}^n$. For any fixed $\epsilon > 0$, let B_ϵ denote the ball of radius ϵ centered at the origin in $\mathbb{R}^n$, and choose a smooth radial cutoff function $0 \leq \eta \leq 1$ supported in $B_{2\epsilon}$, with $\eta \equiv 1$ on B_ϵ. Consider the smooth compactly supported function $\varphi_\alpha = \eta u_\alpha$. Since φ_α is a radial function,

$$\begin{aligned} \int_{\mathbb{R}^n} a|\nabla\varphi_\alpha|^2 dx &= \int_{B_{2\epsilon}} \left(a\eta^2|\nabla u_\alpha|^2 + 2a\eta u_\alpha\langle\nabla u_\alpha, \nabla\eta\rangle + a u_\alpha^2|\nabla\eta|^2\right) dx \\ &\leq \int_{\mathbb{R}^n} a|\nabla u_\alpha|^2 dx + C\int_{A_\epsilon} \left(u_\alpha|\partial_r u_\alpha| + u_\alpha^2\right) dx, \end{aligned} \tag{2.1.7}$$

where $A_\epsilon = B_{2\epsilon} - B_\epsilon$. To estimate these terms, first note that $u_\alpha(r) \leq \alpha^{(n-2)/2} r^{2-n}$ and, since

$$\partial_r u_\alpha = (2-n)\alpha^{-1} r \left(\frac{\alpha}{\alpha^2 + r^2}\right)^{n/2}, \tag{2.1.8}$$

we also have $|\partial_r u_\alpha| \leq (n-2)\alpha^{(n-2)/2} r^{1-n}$. Thus, for fixed $\epsilon > 0$, the second term in (2.1.7) is $O(\alpha^{n-2})$ as $\alpha \to 0$:

$$\begin{aligned} \int_{A_\epsilon} \left(u_\alpha|\partial_r u_\alpha| + u_\alpha^2\right) dx \leq \alpha^{n-2} \int_{A_\epsilon} \left(r^{3-2n} + r^{4-2n}\right) dx &= \alpha^{n-2}\omega_{n-1}\int_\varepsilon^{2\varepsilon} \left(r^{3-2n+n-1} + r^{4-2n+n-1}\right) dr \\ &\leq C\alpha^{n-2}, \end{aligned}$$

for some positive constant C. As for the first term,

$$\begin{aligned}\int_{\mathbb{R}^n} a|\nabla u_\alpha|^2 dx &= \lambda(\mathbb{S}^n)\left(\int_{B_\epsilon} u_\alpha^{2^*} dx + \int_{\mathbb{R}^n\setminus B_\epsilon} u_\alpha^{2^*} dx\right)^{2/2^*} \\ &\leq \lambda(\mathbb{S}^n)\left(\|\varphi_\alpha\|_{2^*}^{2^*} + \int_{\mathbb{R}^n\setminus B_\epsilon} \alpha^n r^{-2n} dx\right)^{2/2^*} \\ &= \lambda(\mathbb{S}^n)\left(\|\varphi_\alpha\|_{2^*}^{2^*} + C\alpha^n\right)^{2/2^*} \\ &= \lambda(\mathbb{S}^n)\left(\|\varphi_\alpha\|_{2^*}^{2^*} + C\alpha^{2^*(n-2)/2}\right)^{2/2^*} \\ &\leq \lambda(\mathbb{S}^n)\|\varphi_\alpha\|_{2^*}^2 + O(\alpha^{n-2}),\end{aligned} \tag{2.1.9}$$

where we have used the triangle inequality to obtain the last inequality. So, the Sobolev quotient of φ_α on $\mathbb{R}^n$ is less than $\lambda(\mathbb{S}^n)+O(\alpha^{n-2})$. On a compact manifold M, let $p \in M$ and consider normal coordinates $\{x^i\}$ centered in p. Let $\varphi_\alpha = \eta u_\alpha$ in these coordinates, extended by zero to a smooth function on M. Since φ is a radial function and $g^{rr} \equiv 1$ in these coordinates, we have

$$|\nabla\varphi_\alpha|^2 = \langle\nabla\varphi_\alpha, \nabla\varphi_\alpha\rangle = g^{rr}|\partial_r\varphi_\alpha|^2 = |\partial_r\varphi_\alpha|^2,$$

i.e. $|\nabla\varphi|^2 = |\partial_r\varphi|^2$ (in a neighbourhood of p). We want to estimate the energy, $E(\varphi_\alpha)$. However, the current situation is not as simple as before due to the extra term of the scalar curvature and the fact that $\det(g_{ij}) = 1 + O(r^2)$ (where $r = |x|$ is the geodesic distance) in normal coordinates. The previous estimates then give

$$\begin{aligned}E(\varphi_\alpha) &= \int_{B_{2\epsilon}} a|\nabla\varphi_\alpha|^2 + S\varphi_\alpha^2 dV_g \\ &\leq (1+C\epsilon^2)\left(\lambda(\mathbb{S}^n)\|\varphi_\alpha\|_{L^{2^*}(\mathbb{R}^n)}^2 + C_1\alpha^{n-2} + \|S\|_{L^\infty(M)}\int_0^{2\epsilon}\int_{S_r} u_\alpha^2 r^{n-1} d\omega dr\right),\end{aligned} \tag{2.1.10}$$

for some positive constants C, C_1. Lemma 2.1.4 below shows, the last term is bounded by a constant multiple of α. However, we cannot just divide (2.1.10) by $\|\varphi_\alpha\|_{L^{2^*}(M)}$ and conclude the result, since in the above inequality $\|\varphi_\alpha\|_{L^{2^*}(\mathbb{R}^n)}$ is not exactly equal to $\|\varphi_\alpha\|_{L^{2^*}(M)}$. In the context of $\|\varphi_\alpha\|_{L^{2^*}(\mathbb{R}^n)}$ we are thinking of φ_α as a function in $\mathbb{R}^n$, while in $\|\varphi_\alpha\|_{L^{2^*}(M)}$ φ_α is now a function in M. So we need to estimate the ratio between $\|\varphi_\alpha\|_{L^{2^*}(\mathbb{R}^n)}$ and $\|\varphi_\alpha\|_{L^{2^*}(M)}$. To estimate $\mathcal{Q}_g(\varphi_\alpha)$ we need to get a lower bound for $\|\varphi_\alpha\|_{L^{2^*}(M)}^2$:

$$\|\varphi_\alpha\|_{L^{2^*}(M)} = \left(\int_M \varphi_\alpha^{2^*} dV_g\right)^{2/2^*} \geq \left(\int_{B_\epsilon} \varphi_\alpha^{2^*}\sqrt{\det(g_{ij})}dV_g\right)^{2/2^*} \geq (1-C\epsilon^2)\left(\int_{B_\epsilon}\varphi_\alpha^{2^*} dx\right)^{2/2^*}. \tag{2.1.11}$$

This leads us to estimate $\|\varphi_\alpha\|_{L^{2^*}(B_\epsilon)}$ (now we are thinking of φ_α as a function in $\mathbb{R}^n$):

$$\begin{aligned}\int_{B_\epsilon} \varphi_\alpha^{2^*} dx = \int_{B_\epsilon} \left(\frac{\alpha}{\alpha^2+|x|^2}\right)^n dx = \omega_{n-1}\int_0^\epsilon \left(\frac{\alpha}{\alpha^2+r^2}\right)^n r^{n-1}dr \\ = \omega_{n-1}\int_0^{\epsilon/\alpha} \frac{1}{(1+y^2)^n} y^{n-1}dy.\end{aligned} \tag{2.1.12}$$

Thus, dividing (2.1.10) by $\|\varphi_\alpha\|_{L^{2^*}(M)}$ and using (2.1.11) we get

$$\mathcal{Q}_g(\varphi_\alpha) \le \frac{1+C\epsilon^2}{(1-C\epsilon^2)^{1/2^*}}\lambda(\mathbb{S}^n) + \frac{C_1\alpha^{n-2}}{(1-C\epsilon^2)^{1/2^*}\|\varphi\|_{L^{2^*}(B_\epsilon)}} + \frac{C_2\alpha}{(1-C\epsilon^2)^{1/2^*}\|\varphi\|_{L^{2^*}(B_\epsilon)}}, \tag{2.1.13}$$

for some positive constant C_2. Looking at (2.1.12) and (2.1.11), we see that if we chose $\alpha=\epsilon$ small, the $L^{2^*}(M)-$norm of φ_α is controlled from above and below by constants independent of ϵ and α. So

$$\begin{aligned}\mathcal{Q}_g(\varphi_\alpha) &\le \frac{1+C\epsilon^2}{(1-C\epsilon^2)^{1/2^*}}\lambda(\mathbb{S}^n) + C\alpha \le \frac{1+C\epsilon^2}{1-C\epsilon^2}\lambda(\mathbb{S}^n) + C\alpha \\ &= \lambda(\mathbb{S}^n) + C\alpha,\end{aligned} \tag{2.1.14}$$

for some positive constant $C>0$ (that is independent of α) and $\epsilon>0$ sufficiently small. This proves that $\lambda(M)\le\lambda(\mathbb{S}^n)$ since we are taking the infimum. □

The following lemma will be used again in Chapter 4, and so we state and prove it in its full generality.

Lemma 2.1.4. *Fix $\epsilon>0$ and suppose that $k>-n$. Then, as $\alpha\to 0$,*

$$I(\alpha) = \int_0^\epsilon r^{k+n-1}u_\alpha^2 dr$$

is bounded above and below by positive multiples of α^{k+2} if $n>k+4$, $\alpha^{k+2}\log(1/\alpha)$ if $n=k+4$, and α^{n-2} if $n<k+4$.

Proof. The change of variable $t=r/\alpha$ yields

$$I(\alpha) = \alpha^{k+2}\int_0^{\epsilon/\alpha} \frac{t^{k+n-1}}{(1+t^2)^{n-2}}dt.$$

Now, note that for $t\ge 1$, $t^2\le t^2+1\le 2t^2$, and so we can find bounds for $I(\alpha)$:

$$\alpha^{k+2}\left(C+\int_1^{\epsilon/\alpha} 2^{-n+2}t^{k-n+3}dt\right) \le I(\alpha) \le \alpha^{k+2}\left(C+\int_1^{\epsilon/\alpha} t^{k-n+3}dt\right).$$

So, if $n>k+4$, $\int_1^{\epsilon/\alpha} t^{k-n+3}dt$ is bounded, if $n=k+4$, $\int_1^{\epsilon/\alpha} t^{k-n+3}dt = \log(\epsilon)-\log(\alpha)$, and if $n<k+4$, $\alpha^{k+2}\int_1^{\epsilon/\alpha} t^{k-n+3}dt$ is comparable to α^{n-2}. □

2.2 Characterization of the Solutions in the Sphere

An interesting feature of the Yamabe Problem in the standard sphere is that we have a complete solution for the problem in the sense that we are able to describe all the solutions.

Recall that a diffeomorphism between two Riemannian manifolds (M, g) and (N, h), $\Psi : M \to N$, is said to be a conformal diffeomorphism if the pullback of the metric h by Ψ is conformal to g.

The following is due to Obata [14]:

Proposition 2.2.1. *If g is a metric on $\mathbb{S}^n$ that is conformal to the standard metric $\overline{g}$ and has constant sectional curvature, then up to multiplication by a constant factor, g is obtained from $\overline{g}$ by a conformal diffeomorphism of the sphere.*

Proof. Our aim is to show that $\mathbb{S}^n$ equipped with g has constant scalar curvature and then apply the Killing-Hopf theorem to prove the result. So we start by showing that g is Einstein. Considering the metric g as the background metric on the sphere, we can write $\overline{g} = \varphi^{-2}g$, where $\varphi \in \mathcal{C}^\infty(M)$ is strictly positive and smooth. By recalling the local forms of the Laplace-Beltrami operator and Hessian of a function we have, by making the substitution $e^{2f} = \varphi^{-2} \iff f = -\log\varphi$,

$$\begin{aligned}(\mathcal{H}_f)_{ij} &= \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right) dx^i \otimes dx^j = \left(-\frac{\frac{\partial^2\varphi}{\partial x^i \partial x^j}}{\varphi} + \frac{\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}}{\varphi^2} + \sum_k \Gamma^k_{ij}\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^k}\right) dx^i \otimes dx^j \\ &= -\frac{1}{\varphi}(\mathcal{H}_\varphi)_{ij} + \frac{1}{\varphi^2}(d\varphi \otimes d\varphi)_{ij},\end{aligned} \tag{2.2.1}$$

and therefore

$$\Delta_g(-\log\varphi) = -\frac{1}{\varphi}\Delta_g\varphi - \frac{1}{\varphi^2}\|d\varphi\|^2.$$

The transformation laws (A.0.18), (A.0.19) imply

$$\begin{aligned}\overline{\mathrm{Ric}} &= \mathrm{Ric} - (n-2)\mathcal{H}_f + (n-2)df \otimes df + (\Delta_g f - (n-2)\|df\|^2)g \\ &= \mathrm{Ric} + \frac{n-2}{\varphi}\mathcal{H}_\varphi - \frac{n-2}{\varphi^2}d\varphi \otimes d\varphi + \frac{n-2}{\varphi^2}d\varphi \otimes d\varphi - \left(\frac{1}{\varphi}\Delta_g\varphi + \frac{1}{\varphi^2}\|df\|^2 + \frac{n-2}{\varphi^2}\|d\varphi\|^2\right)g \\ &= \mathrm{Ric} + \frac{n-2}{\varphi}\mathcal{H}_\varphi - \left(\frac{1}{\varphi}\Delta_g\varphi + \frac{n-1}{\varphi^2}\|d\varphi\|^2\right)g,\end{aligned}$$

and therefore

$$\begin{aligned}\overline{S} &= \overline{R}_{ij}\overline{g}^{ij} \\ &= \varphi^2 R_{ij}g^{ij} + (n-2)\varphi(\mathcal{H}_\varphi)_{ij}g^{ij} - n\varphi\Delta_g\varphi + (n-1)n\|d\varphi\|^2 \\ &= \varphi^2 S - 2(n-1)\varphi\Delta_g\varphi - n(n-1)\|d\varphi\|^2,\end{aligned}$$

where all operations are taken with respect to the metric g.

If $\overset{\circ}{\mathrm{Ric}} = \mathrm{Ric} - \frac{S}{n} g$ denotes the traceless Ricci tensor, then, because $\overline{g}$ is Einstein:

$$
\begin{aligned}
0 &= \overset{\circ}{\overline{R}}_{ij} = \overline{R}_{ij} - \frac{\overline{S}}{n}\overline{g}_{ij} \\
&= R_{ij} + \frac{n-2}{\varphi}(\mathcal{H}_\varphi)_{ij} - \left(\frac{1}{\varphi}\Delta_g\varphi + \frac{n-1}{\varphi^2}\|d\varphi\|^2\right) g_{ij} - \varphi^{-2}\left(\frac{1}{n}\varphi^2 S - \frac{2(n-1)}{n}\varphi\Delta_g\varphi - (n-1)\|d\varphi\|^2\right) g_{ij} \\
&= \overset{\circ}{R}_{ij} + \frac{n-2}{\varphi}(\mathcal{H}_\varphi)_{ij} + \frac{n-2}{n\varphi}\Delta_g\varphi g_{ij}
\end{aligned}
$$

Since S is constant, the contracted Bianchi's identity (A.0.13) implies that the divergence of the Ricci tensor vanishes identically. And since the divergence of the metric also vanishes identically, the divergence of the traceless Ricci tensor also vanishes identically. Since $\overset{\circ}{\mathrm{Ric}}$ is traceless, integration by parts yields

$$
\begin{aligned}
\int_{\mathbb{S}^n} \varphi |\overset{\circ}{\mathrm{Ric}}|^2 dV_g &= \int_{\mathbb{S}^n} \varphi \overset{\circ}{R}_{ij} \overset{\circ}{R}{}^{ij} dV_g = -(n-2)\int_{\mathbb{S}^n} \overset{\circ}{R}{}^{ij}\left((\mathcal{H}_\varphi)_{ij} + \frac{1}{n}\Delta_g\varphi g_{ij}\right) dV_g \\
&= -(n-2)\int_{\mathbb{S}^n}\left(\overset{\circ}{R}{}^{ij}(\mathcal{H}_\varphi)_{ij} + \frac{1}{n}\overset{\circ}{R}{}^{ij}\Delta_g\varphi g_{ij}\right) dV_g.
\end{aligned}
$$

The second term is zero because $\overset{\circ}{R}{}^{ij}\Delta_g\varphi g_{ij} = tr_g(\overset{\circ}{\mathrm{Ric}})\Delta_g\varphi$, and so we are left with

$$
\int_{\mathbb{S}^n} \varphi |\overset{\circ}{\mathrm{Ric}}|^2 dV_g = -(n-2)\int_{\mathbb{S}^n} \overset{\circ}{R}{}^{ij}(\mathcal{H}_\varphi)_{ij} dV_g = -(n-2)\int_{\mathbb{S}^n}\langle \overset{\circ}{R}, \mathcal{H}_\varphi\rangle dV_g = (n-2)\int_{\mathbb{S}^n}\langle \operatorname{div}\overset{\circ}{\mathrm{Ric}}, d\varphi\rangle dV_g = 0.
$$

This shows that $\overset{\circ}{\mathrm{Ric}}$ must vanish identically, and so g is Einstein. Since g is conformal to the round metric $\overline{g}$, which is locally conformally flat, the Weyl tensor is identically zero. And so, by Schur's Lemma, g has constant curvature, and so $(\mathbb{S}^n, g)$ is isometric to the standard sphere by the Killing-Hopf Theorem. The isometry is the desired conformal diffeomorphism. $\square$

Combining Proposition 2.2.1 with Theorem 3.1.7 we obtain a complete solution to the Yamabe problem on the sphere:

Theorem 2.2.2. *The Yamabe functional on $(\mathbb{S}^n, \overline{g})$ is minimized by constant multiples of the standard metric and its images under conformal diffeomorphisms. These are the only metrics conformal to the standard one on $\mathbb{S}^n$ that have constant scalar curvature.*

Chapter 3

Variational Methods

In the previous chapter, we introduced the Yamabe invariant of a compact Riemannian manifold and checked that, if it is reached by a smooth positive function, then the Yamabe problem is solved. We also saw that the Yamabe invariant of any compact $n-$dimensional Riemannian manifold satisfies $\lambda(M) \leq \lambda(\mathbb{S}^n)$. However, we did not explore the importance of this inequality.

Since the Yamabe problem is related to a minimization problem, a natural idea to solve the Yamabe problem could be to apply the direct method of the Calculus of Variations, that is, to construct a minimizing sequence for $\lambda(M)$, and hope to extract a convergent subsequence, followed by some regularity theory and the Strong Maximum Principle B.0.13 to show that the limit function is both smooth and positive.

However, because the exponent appearing in the definition of $\mathcal{Q}_g$ is precisely the critical exponent for the Sobolev inequality, this method, in general, does not work since the injection $H^1(M) \hookrightarrow L^{2^*}(M)$ is not compact. Yamabe knew this, and to overcome this obstacle he decided to perturb $\mathcal{Q}_g$ and the Yamabe equation (1.0.1) by considering a subcritical problem. By doing this, he obtained a sequence of problems that are easily solved using the direct method of the Calculus of Variations followed by regularity theory and the Strong Maximum Principle B.0.13. Then, he obtained a sequence of positive smooth functions that, under a certain uniform boundedness condition in some $L^r(M)$ space with $r > 2^*$, converges uniformly, up to a subsequence, to a positive, smooth solution of the Yamabe problem. Yamabe's mistake was to assume that this uniform boundedness condition is always satisfied, which, as noted by Trudinger in [18], is not always the case. It turns out that, as long as $\lambda(M) < \lambda(\mathbb{S}^n)$, the L^r uniform boundedness condition is satisfied. This detail led to a significant increase in the complexity of the problem. In later chapters, we will prove that, if (M, g) is not in the conformal class of the standard sphere, we have $\lambda(M) < \lambda(\mathbb{S}^n)$, and so it is only natural to conclude that the problem in the sphere requires a different approach.

The purpose of this chapter is to explore further the Yamabe invariant of a manifold, and its importance in the overall solution of the problem. To show that $\lambda(M) < \lambda(\mathbb{S}^n)$ is a sufficient condition for the Yamabe problem to have a solution. To show that there is a minimizer of the Yamabe functional in the standard sphere, where the sufficient condition mentioned above is clearly not satisfied (note that the standard sphere already has constant scalar curvature). Here, we closely follow section 4 of [13], adding however all the details, and prove a result regarding the invertibility of the conformal Laplacian as an operator from the Sobolev space $W^{2,q}(M)$ onto $L^q(M)$, for $q > 1$ on certain Riemannian manifolds (M, g). This result will be useful in the proof of the fact that the Yamabe invariant in the sphere is achieved.

3.1 The subcritical equation

As we have seen, the Yamabe problem is related to a minimization problem. As such, the most direct approach to try and solve the Yamabe Problem would be to apply the direct method of the Calculus of Variations, and afterwards, try to develop some regularity theory. This, however, is a failing strategy as the injection $H^1(M) \hookrightarrow L^{2^*}(M)$ lacks compactness. Indeed, if we take a minimizing sequence, $\{u_k\} \subset H^1(M)$, of $\mathcal{Q}_g$, such that $\|u_k\|_{L^{2^*}} = 1$ (which we can always assume to be true by homogeneity), then

$$\begin{aligned} \|u_k\|_{H^1} = \int_M |\nabla u_k|^2 + u_k^2 dV_g = \frac{1}{a}\mathcal{Q}_g(u_k) + \int_M \left(1 - \frac{S}{a}\right) u_k^2 dV_g \\ \leq \frac{1}{a}\mathcal{Q}_g(u_k) + C\|u_k\|_{2^*}^2 = \frac{1}{a}\mathcal{Q}_g(u_k) + C, \end{aligned} \tag{3.1.1}$$

where we have applied Hölder's inequality to arrive at the last inequality. Because $\mathcal{Q}_g(u_k) \to \lambda(M)$, we can conclude that $\{u_k\}$ is bounded in $H^1(M)$, so $\{u_k\}$ converges weakly along a subsequence to some $u \in H^1(M)$. But since 2^* is precisely the exponent for which the inclusion map fails to be compact, we cannot guarantee that the constraint $\|u_k\|_{2^*} = 1$ is preserved in the limit, all we know is $\|u\|_{2^*} \leq 1$ by Fatou's Lemma. In particular, u may be identically zero.
However, if $\lambda(M) < \lambda(\mathbb{S}^n)$ then, as we are going to see below, the normalized $L^{2^*}-$norm condition is satisfied in the limit.

A classical method to go around this obstacle is to perturb the problem in a way that we get a problem that we can solve. In our situation, the best way to do this is to perturb the exponent on the right-hand side of the Yamabe equation. In this text, we chose the latter option as the scalar curvature has a deep geometric meaning that is very important in the context of the Yamabe problem, and consider the perturbed/subcritical equation

$$\mathcal{L}_g\varphi = \lambda_s(M)\varphi^{s-1}, \tag{3.1.2}$$

with $2 \leq s \leq 2^*$, which comes with the associated family of functionals $\mathcal{Q}_g^s$:

$$\mathcal{Q}_g^s(\varphi) = \frac{E(\varphi)}{\|\varphi\|_s^2} = \frac{\displaystyle\int_M a|\nabla\varphi|^2 + S\varphi^2 dV_g}{\|\varphi\|_s^2}, \tag{3.1.3}$$

and the associated minimization problems $\lambda_s(M) = \inf_{\varphi \in H^1(M)\setminus\{0\}} \mathcal{Q}_g^s(\varphi)$.

It is a result due to Yamabe [19] that, for $s < 2^*$, this equation always has a smooth, strictly positive, solution φ_s that minimizes $\lambda_s(M)$. Indeed, when $s < 2^*$, the compactness of the injection $H^1(M) \hookrightarrow L^s(M)$ makes this result, modulo the regularity and positivity properties, very believable. This is because the direct method of the Calculus of Variations now produces a minimizer that satisfies $\|\varphi_s\|_s = 1$, as opposed to what happens in the critical case, where $s = 2^*$.

Proposition 3.1.1. *For each* $2 \leq s < 2^*$*, there exists a smooth, positive solution* φ_s *to the subcritical equation* (3.1.2)*, for which* $\mathcal{Q}^s(\varphi_s) = \lambda_s(M)$ *and* $\|\varphi_s\|_s = 1$*.*

Proof. Since $2 \leq s < 2^*$, the inclusion $H^1(M) \hookrightarrow L^s(M)$ is compact. The direct method of the Calculus of Variations works and through this method, we are able to extract a solution. To see this, take $\{u_k\} \subset H^1(M)$ to be a minimizing sequence for $\lambda_s(M)$ ($\mathcal{Q}_g^s(u_k) \to \lambda_s(M)$) such that $\|u_k\|_s = 1$ (the homogeneity of the functional $\mathcal{Q}_g^s$ allows us to make this assumption). By replacing, if necessary, u_k by its absolute value, we may assume without loss of generality that the sequence is nonnegative.Then, proceeding as in (3.1.1), we conclude that $\{u_k\}$ is a bounded sequence in $H^1(M)$:

$$\begin{aligned} \|u_k\|_{H^1} = \int_M |\nabla u_k|^2 + u_k^2 dV_g &= \frac{1}{a}\mathcal{Q}_g^s(u_k) + \int_M \left(1 - \frac{S}{a}\right) u_k^2 dV_g \\ &\leq \frac{1}{a}\mathcal{Q}_g^s(u_k) + C\|u_k\|_s^2. \end{aligned} \tag{3.1.4}$$

Since the embedding $H^1(M) \hookrightarrow L^s(M)$ is compact, up to a subsequence, $\{u_k\}$ converges weakly in $H^1(M)$ and strongly in $L^s(M)$ to some $u_s \in H^1(M)$ with $\|u_s\|_s = 1$. So, $\mathcal{Q}_g^s(u_s) \geq \lambda_s(M)$. Because $H^1(M) \hookrightarrow L^2(M)$ is also compact and $S \in L^\infty(M)$, we have that $\int_M S u_k^2 dV_g \to \int_M S u_s^2 dV_g$. The weak convergence in $H^1(M)$ gives

$$\int_M |\nabla u_s|^2 dV_g \leq \liminf_{k\to\infty} \int_M |\nabla u_k|^2 dV_g$$

and thus

$$\mathcal{Q}_g^s(u_s) \leq \limsup_{k\to\infty} \int_M a|\nabla u_k|^2 dV_g + \int_M S u_k^2 dV_g = \lambda_s(M)$$

but then $\mathcal{Q}_g^s(u_s) = \lambda_s(M)$. Hence, u_s is a weak solution of the subcritical equation (3.1.2) and so the regularity result B.0.20, implies that u_s is positive and smooth. ☐

3.1.1 A sufficient condition for the Yamabe Problem

We now present a sufficient condition for the existence of a solution to the Yamabe Problem. The problem with it is that it does not include the model case of the sphere, therefore, we will have to prove the existence of a minimizer for the Yamabe functional on the standard sphere by hand. It turns out, as we will see in the coming chapters that the referred condition is satisfied by all compact, connected Riemannian manifolds apart from when we are dealing with the conformal class of the standard sphere..

As mentioned at the beginning of this chapter, the idea is to consider the subcritical problems (3.1.2) for $s < 2^*$ and pass to the limit $s \to 2^*$. Because of this, we have to study the behaviour of $\lambda_s(M)$ as a function of s. To simplify our task, we choose a background metric such that M has volume one. Note that we can always choose such a metric by multiplying the original metric with an appropriate constant.

Under this assumption, Hölder's inequality implies that $\|u\|_s \leq \|u\|_{s'}$ whenever $s \leq s'$. This, in particular, makes the study of the constants $\lambda_s(M)$ much simpler. The next lemma, due to Aubin (see [3]) encapsulates the results we need about the behaviour of $\lambda_s(M)$ as a function of s:

Lemma 3.1.2. *If $\int_M dV_g = 1$, then $|\lambda_s(M)|$ is nonincreasing as a function of $s \in [2, 2^*]$. Moreover, if $\lambda(M) \geq 0$, then $\lambda_s(M)$ is continuous from the left.*

Proof. For any $u \in H^1(M) \setminus \{0\}$ we have by definition that

$$\mathcal{Q}_g^{s'}(u) = \frac{\|u\|_s}{\|u\|_{s'}} \mathcal{Q}_g^s. \tag{3.1.5}$$

So, given $s, s' \in [2, 2^*]$ we see that λ_s and $\lambda_{s'}$ have the same sign. Let $s, s' \in [2, 2^*]$ with $s \leq s'$. First, consider the case where $\lambda_s(M) > 0$. Then, the observation $\|u\|_s \leq \|u\|_{s'}$ implies that $\lambda_s(M) \geq \lambda_{s'}(M)$. Now consider the case where $\lambda_s < 0$. In this setting, consider $u \in H^1(M) \backslash \{0\}$ such that $\mathcal{Q}_g^s(u), \mathcal{Q}_g^{s'}(u) < 0$. In this case, we see that

$$\mathcal{Q}_g^{s'}(u) = \frac{\|u\|_s}{\|u\|_{s'}} \mathcal{Q}_g^s(u) \geq \mathcal{Q}_g^s(u),$$

and so $\lambda_s(M) \leq \lambda_{s'}(M)$. This finishes the proof that $|\lambda_s(M)|$ is nonincreasing.

Now assume that $\lambda(M) \geq 0$, this implies, via (3.1.5), that $\lambda_s(M) \geq 0$ for every $s \in [2, 2^*]$. Now fix $s \in [2, 2^*]$ and let $\varepsilon > 0$. By the properties of infimum there is $u \in H^1(M)$ such that $\mathcal{Q}_g^s(u) < \lambda_s(M) + \varepsilon$. Because $\|u\|_s$ is a continuous function of s, $\lambda_s(M) \leq \lambda_{s'}(M) \leq \mathcal{Q}_g^{s'}(u) < \lambda_s + 2\varepsilon \iff |\lambda_s - \lambda_{s'}| \leq 2\varepsilon$, for $s' \leq s$ sufficiently close to s. □

Using the properties about $\lambda_s(M)$ we have just shown, we are able to acquire a $L^r(M)$-uniform bound on the sequence $\{u_s\}$ provided that $\lambda(M) < \lambda(\mathbb{S}^n)$. The following proposition, which is due to the combined efforts of Trudinger (see [18]) and Aubin (see [3]), formalises this.

Proposition 3.1.3. *Let $\{u_s\}$ be the collection of functions given by Proposition* 3.1.1*. If $\lambda(M) < \lambda(\mathbb{S}^n)$, then there are constants $s_0 < 2^*, r > 2^*$, and $C > 0$ such that $\|u_s\|_r \leq C$ for all $s \geq s_0$.*

Proof. Let $\delta > 0$ (to be determined later). Multiply (3.1.2) by $u_s^{1+2\delta}$ and integrate by parts to get

$$\int_M a\langle \nabla u_s, (1+2\delta) u_s^{2\delta} \nabla u_s \rangle + S u_s^{2+2\delta} dV_g = \lambda_s \int_M u_s s^{s+2\delta} dV_g.$$

Setting $w_s = u_s^{1+\delta}$, we have $\nabla w_s = (1+\delta) u_s^{\delta} \nabla u_s$. So rewriting the previous equation with respect to w_s we have

$$\frac{1+2\delta}{(1+\delta)^2} \int_M a |\nabla w|^2 dV_g = \int_M \lambda_s(M) w^2 u_s^{s-2} - S w^2 dV_g.$$

The (sharp) Sobolev inequality (Theorem 2.1.3 and Proposition B.0.8) and the previous equation yield for any $\varepsilon > 0$:

$$\begin{aligned}
\|w_s\|_{2^*}^2 &\leq (1+\varepsilon) \frac{a}{\lambda(\mathbb{S}^n)} \int_M |\nabla w_s|^2 dV_g + C_\varepsilon \int_M w_s^2 dV_g \\
&= (1+\varepsilon) \frac{\lambda_s(M)}{\lambda(\mathbb{S}^n)} \frac{(1+\delta)^2}{1+2\delta} \int_M w_s^2 u_s^{s-2} dV_g + \int_M \left(C_\varepsilon - \frac{(1+\delta)^2}{1+2\delta} S \right) w_s^2 dV_g \\
&\leq (1+\varepsilon) \frac{\lambda_s(M)}{\lambda(\mathbb{S}^n)} \frac{(1+\delta)^2}{1+2\delta} \int_M w_s^2 u_s^{s-2} dV_g + C_\varepsilon' \|w_s\|_2^2 \\
&\leq (1+\varepsilon) \frac{(1+\delta)^2}{1+2\delta} \frac{\lambda_s(M)}{\lambda(\mathbb{S}^n)} \|w_s\|_{2^*}^2 \|u_s\|_{(s-2)n/2}^{s-2} + C_\varepsilon' \|w_s\|_2^2,
\end{aligned} \tag{3.1.6}$$

by Hölder's inequality.

If $\lambda(M) < 0$, we have $\lambda_s(M) < 0$ for every $s \in [2, 2^*]$, and as such we arrive at

$$\|w_s\|_{2^*}^2 \leq C\|w_s\|_2^2 = C\|u_s\|_{2(1+\delta)}^{1+\delta} \leq C\|u_s\|_s^{1+\delta} \leq C.$$

Recall, that $\|u_s\|_s = 1$. And since $\|u_s\|_{2^*(1+\delta)}^{1+\delta} = \|w_s\|_{2^*} \leq C$, we conclude that $\{u_s\}$ is bounded uniformly in $L^{2^*(1+\delta)}(M)$. In the case where $0 \leq \lambda(M) < \lambda(\mathbb{S}^n)$, concluding the result is not as straightforward. First, the fact that $s < 2^*$, implies $\frac{(s-2)n}{2} < s$, so Hölder's inequality implies $\|u_s\|_{(s-2)n/2} \leq \|u_s\|_s = 1$ (here we are again using the fact that M has volume one). Since $0 \leq \lambda(M) < \lambda(\mathbb{S}^n)$, the previous lemma implies that $\lambda_s(M)$ is (as a function of s) continuous from the left. By recalling that $\lambda(M) = \lambda_{2^*}(M)$, we can conclude that there is $s_0 < 2^*$, such that $\lambda_s(M)/\lambda(\mathbb{S}^n) \leq \lambda_{s_0}(M)/\lambda(\mathbb{S}^n) < 1$ for $s \geq s_0$. Noting that

$$(1+\varepsilon)\frac{(1+\delta)^2}{1+2\delta} = (1+\varepsilon)\frac{1+2\delta+\delta^2}{1+2\delta} = 1+\varepsilon+\frac{\delta^2+\varepsilon\delta^2}{1+2\delta},$$

we can choose first ε small, and then δ small enough so that the coefficient $(1+\varepsilon)\frac{(1+\delta)^2}{1+2\delta}\frac{\lambda_s(M)}{\lambda(\mathbb{S}^n)} < 1$, and so, using this in (3.1.6) we can conclude that

$$\|w_s\|_{2^*}^2 \leq C\|w_s\|_2^2,$$

for some constant $C > 0$. Now proceed as in the case $\lambda(M) < 0$ to conclude the result. ☐

Remark 3.1.4. *It is a curious fact that when $\lambda(M) < 0$, which is the case, for example, whenever the scalar curvature satisfies $\int_M S dV_g < 0$, the proof above implies that $\{u_s\}$ is uniformly bounded in every $L^r(M)$ ($1 \leq r < \infty$).*

Remark 3.1.5. *In the Introduction, we mentioned that Yamabe's attempt at solving the Yamabe Problem contained a mistake. This mistake was to assume that the sequence $\{u_s\}$ given by Proposition* 3.1.1 *is uniformly bounded in some $L^r(M)$ for some $r > 2^*$ (as in the previous proposition) whether or not $\lambda(M) < \lambda(\mathbb{S}^n)$. Actually, it is likely that Yamabe was not aware of the inequality $\lambda(M) \leq \lambda(\mathbb{S}^n)$ as this first appeared in the literature in [3] which was published in 1968, 8 years after Yamabe died.*

Using the previous proposition, we are able to prove that the Yamabe problem has a solution if $\lambda(M) < \lambda(\mathbb{S}^n)$.

Theorem 3.1.6. *Let $\{u_s\}$ be the collection of functions given by Proposition* 3.1.1*. Suppose that $\lambda(M) < \lambda(\mathbb{S}^n)$. Then, as $s \to 2^*$, $\{u_s\}$ converges, up to a subsequence, in the $\mathcal{C}^2-$norm to a positive function $u \in \mathcal{C}^\infty(M)$ that satisfies:*

$$\mathcal{Q}_g(u) = \lambda(M), \qquad \mathcal{L}_g u = \lambda(M)u^{2^*-1}.$$

In particular, the metric $\widetilde{g} = u^{2^*-2}g$ has constant scalar curvature $\lambda(M)$.

Proof. According to the previous proposition, there is $s_0 \in [2, 2^*]$ and $r > 2^*$ such that $\{u_s\}_{s \geq s_0}$ is uniformly bounded in $L^r(M)$. Therefore, the regularity theorem B.0.20 implies that $\{u_s\}_{s \geq s_0}$ is uniformly bounded in $\mathcal{C}^{2,\alpha}(M)$ for some $\alpha \in (0,1)$. But then the Arzelà–Ascoli theorem (note that we can apply this theorem because $\{u_s\}$ is bounded in $\mathcal{C}^{2,\alpha}(M)$) implies that there is a subsequence of $\{u_s\}$ along which we have uniform convergence with respect to the $\mathcal{C}^2-$norm to a function $u \in \mathcal{C}^2(M)$. So, we have that $\|u\|_{2^*} = \lim_{s\to 2^*} \|u_s\|_s = 1$. The limit function u must then satisfy

$$\mathcal{L}u = \lambda u^{p-1}, \qquad \mathcal{Q}_g(u) = \lambda,$$

where $\lambda = \lim_{s\to 2^*} \lambda_s(M)$.

If $\lambda(M) \geq 0$, Lemma 3.1.2 implies that $\lambda = \lambda(M)$, since $\lambda_s(M)$ is continuous from the left as function of s. On the other hand, if $\lambda(M) < 0$ Lemma 3.1.2 shows that $\lambda_s(M)$ is an increasing function of s and so we must have $\lambda \leq \lambda(M)$. Because $\|u\|_{2^*} \geq 1$, Theorem B.0.20 shows that u is $\mathcal{C}^\infty$ and strictly positive. But then, the definition of $\lambda(M)$ implies that $\lambda = \lambda(M)$. $\square$

3.1.2 Minimization on the Sphere

From Chapter 2 we know that the standard metric on the sphere, $\overline{g}$, has constant scalar curvature. However, we did not show that this metric is a minimizer for the Yamabe functional. In this section, we prove this fact. Furthermore, the standard sphere clearly does not satisfy the condition $\lambda(\mathbb{S}^n) < \lambda(\mathbb{S}^n)$, as such the above procedure fails to produce a smooth positive minimizer for the Yamabe functional.

It is the existence of the family of conformal diffeomorphisms described in Chapter 2, that enables us, with a little more effort, to prove the existence of minimizers for the Yamabe functional on the sphere. The following is taken from [13] where we have added all the missing details.

Theorem 3.1.7. *There is a positive, smooth function ψ on $\mathbb{S}^n$ that satisfies $\mathcal{Q}_{\overline{g}}(\psi) = \lambda(\mathbb{S}^n)$.*

Proof. First, recall that the standard sphere, $\mathbb{S}^n$, has constant scalar curvature, $S = n(n-1)$. For each $2 \leq s < 2^*$, let u_s be the solution of the subcritical equation whose existence is guaranteed by Proposition 3.1.1. Assume that the supremum of every u_s is achieved at the south pole (which is always true after composing with a suitable rotation).

If $\{u_s\}$ is uniformly bounded in $L^\infty(\mathbb{S}^n)$ then the method used in Theorem 3.1.6 gives the result. So from now on, we assume that $\|u_s\|_{L^\infty(\mathbb{S}^n)} \to \infty$. Note that we cannot guarantee a uniform L^r-bound ($r > 2^*$) of the sequence $\{u_s\}$ because the proof of 3.1.3 breaks down in the standard sphere. Now let $k_\alpha = \sigma^{-1} \circ \delta_\alpha \circ \sigma : \mathbb{S}^n \to \mathbb{S}^n$ be the conformal diffeomorphism induced by the dilation δ_α on $\mathbb{R}^n$, and set $g_\alpha = k_\alpha^* \overline{g}$. Using Lemma 2.1.1 we compute g_α in the local coordinates induced by the stereographic projection:

$$\rho^* k_\alpha^* \overline{g} = \delta_\alpha^* \rho^* \overline{g} = \delta_\alpha^* \left(\frac{4}{(1+|x|^2)^2} ds^2 \right) = 4\frac{\alpha^4}{(\alpha^2+|x|^2)^2} \frac{1}{\alpha^2} ds^2 = 4\frac{\alpha^2}{(\alpha^2+|x|^2)^2} ds^2.$$

Now let $\phi_\alpha(\zeta,\xi) = \left(\frac{1+\xi+\alpha^2(1-\xi)}{2\alpha}\right)^{(2-n)/2}$, where $(\zeta,\xi) \in \mathbb{S}^n$ are as in Lemma 2.1.1, i.e. $\zeta \in \mathbb{R}^n$ and

$\xi \in \mathbb{R}$. Direct computation shows:

$$\begin{aligned}
\rho^*(\phi_\alpha^{p-2}\overline{g}) &= \rho^*\left[\left(\frac{2\alpha}{1+\xi+\alpha^2(1-\xi)}\right)^2\right]\frac{4}{(1+|x|^2)^2}ds^2 \\
&= \left(\frac{4\alpha}{1+\frac{|x|^2-1}{|x|^2+1}+\alpha^2-\alpha^2\frac{|x|^2-1}{|x|^2+1}}\right)^2\frac{1}{(1+|x|^2)^2}ds^2 \\
&= \left(\frac{4\alpha(|x|^2+1)}{2\alpha^2+2|x|^2}\right)^2\frac{1}{(1+|x|^2)^2}ds^2 \\
&= \frac{4\alpha^2}{(\alpha^2+|x|^2)^2}ds^2,
\end{aligned}$$

that is, $g_\alpha = \phi_\alpha^{2^*-2}\overline{g}$. Observe that, at the south pole, $\phi_\alpha = \alpha^{(2-n)/2}$.

For each $s < 2^*$, let $\psi_s = \phi_\alpha k_\alpha^* u_s$, where $\alpha = \alpha(s)$ is chosen so that $\psi_s = 1$ at the south pole. This implies that $\alpha_s = \|u_s\|_{L^\infty(\mathbb{S}^n)}^{2/(n-2)} \to \infty$ as $s \to 2^*$, and therefore $\psi_s \le \alpha^{(n-2)/2}\phi_\alpha$.

Let $\mathcal{L}_\alpha$ denote the conformal Laplacian with respect to the metric g_α, then we have

$$\mathcal{L}_\alpha(k_\alpha^* u_s) = \lambda_s(\mathbb{S}^n)(k_\alpha^* u_s)^{s-1} = k_\alpha^*(\lambda_s(\mathbb{S}^n)u_s^{s-1}) = k_\alpha^*(\mathcal{L}_{\overline{g}} u_s).$$

The transformation law A.0.20 for the conformal Laplacian implies

$$\mathcal{L}_{\overline{g}}\psi_s = \mathcal{L}(\phi_\alpha k_\alpha^* u_s) = \phi_\alpha^{2^*-1}\mathcal{L}_\alpha(k_\alpha^* u_s) = \lambda_s(\mathbb{S}^n)\phi_\alpha^{2^*-1}(k_\alpha^* u_s)^{s-1} = \lambda_s(\mathbb{S}^n)\phi_\alpha^{2^*-s}\psi_s^{s-1}. \qquad (3.1.7)$$

Using this we find that $\{\psi_s\}$ is bounded in $H^1(\mathbb{S}^n)$:

$$\begin{aligned}
\|\psi_s\|_{H^1} &\le C\int_{\mathbb{S}^n}\psi_s\mathcal{L}_{\overline{g}}\psi_s dV_{\overline{g}} = C\int_{\mathbb{S}^n}\psi_s\phi_\alpha^{p-1}\mathcal{L}_\alpha(k_\alpha^* u_s)dV_{\overline{g}} = C\int_{\mathbb{S}^n}k_\alpha^* u_s k_\alpha^*\mathcal{L}_{\overline{g}}u_\alpha\phi_\alpha^{2^*}dV_g \\
&= C\int_{\mathbb{S}^n}u_\alpha\mathcal{L}_{\overline{g}}u_\alpha dV_{\overline{g}} \le C'\|u_s\|_{H^1},
\end{aligned}$$

so the boundedness of $\{u_s\}$ in $H^1(\mathbb{S}^n)$ implies the boundedness of $\{\psi_s\}$ in $H^1(\mathbb{S}^n)$. Since $\{u_s\}$ is bounded in $H^1(\mathbb{S}^n)$ it also is bounded in $L^{2^*}(\mathbb{S}^n)$ by the Sobolev Embedding Theorem (B.0.7). Let $\psi \in H^1(\mathbb{S}^n)$ denote the weak limit of $\{\psi_s\}$. If p denotes the north pole, on any compact subset of $\mathbb{S}^n \setminus \{p\}$ there exists a constant A such that $\phi_\alpha \le A\alpha^{(2-n)/2}$, and thus on compact sets away from p, the right-hand side of (3.1.7) is bounded by $\lambda_2(\mathbb{S}^n)A^{2^*-1}$, independently of s:

$$\begin{aligned}
\lambda_s(\mathbb{S}^n)\phi_\alpha^{2^*-1}(k_\alpha^* u_s)^{s-1} &\le \lambda_s(\mathbb{S}^n)A^{2^*-1}\alpha^{-\frac{n+2}{2}}\|k_\alpha^* u_s\|_{L^\infty(\mathbb{S}^n)}^{2^*-1} \\
&= \lambda_s(\mathbb{S}^n)A^{2^*-1}\alpha^{-\frac{n+2}{2}}\|u_s\|_{L^\infty(\mathbb{S}^n)}^{2^*-1} \\
&\le \lambda_2(\mathbb{S}^n)A^{2^*-1}\alpha^{-\frac{n+2}{2}}\|u_s\|_{L^\infty(\mathbb{S}^n)}^{2^*-1} \\
&= \lambda_2(\mathbb{S}^n)A^{2^*-1}\|u_\alpha\|_{L^\infty(\mathbb{S}^n)}^{-\frac{n+2}{n-2}}\|u_s\|_{L^\infty(\mathbb{S}^n)}^{\frac{n+2}{n-2}} \\
&= \lambda_2(\mathbb{S}^n)A^{2^*-1},
\end{aligned}$$

where we have used the fact that $\lambda_s(\mathbb{S}^n)$ is a nonincreasing function of s since $\lambda(\mathbb{S}^n) > 0$ (see Lemma 3.1.2). This implies that on any compact set away from p, the right-hand side of (3.1.7) is bounded in L^r for every r. Then arguing as in the proof of Theorem B.0.20 but using Local Elliptic Regularity (Theorem B.0.11) we conclude that $\{\psi_s\}$ is bounded in $\mathcal{C}^{2,\alpha}$ on compact sets that do not contain p. Now let $K_1 \subset K_2 \subset \cdots$ be a sequence of compact sets whose union is $\mathbb{S}^n \setminus \{p\}$. By the Arzelà-Ascoli theorem, using the same method as in the proof of Theorem 3.1.6, we can extract a subsequence of $\{\psi_s\}$ that converges in $\mathcal{C}^{2,\alpha}(K_1)$, and then from this subsequence extract a subsequence that converges in $\mathcal{C}^{2,\alpha}(K_2)$, etc. And by taking a diagonal subsequence, we see that the limit function ψ is $\mathcal{C}^2$ on $\mathbb{S}^n \setminus \{p\}$. Since (by Lemma 3.1.2) $\lambda_s(\mathbb{S}^n) \to \lambda(\mathbb{S}^n)$ as $s \to 2^*$ and $\phi_\alpha^{2^*-s} \leq 1$ away from the north pole for s near 2^*, we conclude that ψ satisfies $\mathcal{L}_{\overline{g}}\psi = f\psi^{2^*-1}$ weakly on $\mathbb{S}^n \setminus \{p\}$, for some bounded function $0 \leq f \leq \lambda(\mathbb{S}^n)$. By the Weak Removable Singularities Theorem (Theorem B.0.14), the same equation holds weakly on $\mathbb{S}^n$.

For each s,

$$\|\psi_s\|_{2^*}^{2^*} = \int_{\mathbb{S}^n} \phi_\alpha^{2^*} (k_\alpha^* u_s)^{2^*} dV_{\overline{g}} = \int_{\mathbb{S}^n} (k_\alpha^* u_s)^{2^*} k_\alpha^* dV_{\overline{g}} = \|u_s\|_{2^*}^{2^*} \geq \omega_n^{1-2^*/s} \|u_s\|_s^{2^*},$$

where we have applied Hölder's inequality to obtain the last inequality. This implies that $\|\psi\|_{2^*} \geq 1$, and therefore $\mathcal{Q}_{\overline{g}}(\psi) \leq \lambda(\mathbb{S}^n)$. But with $\lambda(\mathbb{S}^n)$ being the infimum of $\mathcal{Q}_{\overline{g}}$, we must have $\mathcal{Q}_{\overline{g}}(\psi) = \lambda(\mathbb{S}^n)$ and $\mathcal{L}\psi = \lambda(\mathbb{S}^n)\psi^{2^*-1}$.

To finish the proof we still have to show that ψ is positive and smooth. Due to the Regularity Theorem B.0.20, it is sufficient to show that $\psi \in L^r(\mathbb{S}^n)$ for some $r > 2^*$. The operator $\mathcal{L}_{\overline{g}} : W^{2,q}(\mathbb{S}^n) \to L^q(\mathbb{S}^n)$ has a bounded inverse for $q > 1$ as is shown below in Lemma 3.1.8 (just set M to be the standard sphere, $\mathbb{S}^n$). Now let $\eta \in \mathcal{C}^\infty(\mathbb{S}^n)$ be a smooth function supported in a small neighbourhood of p, and consider the perturbed operator

$$\mathcal{L}_\eta := \mathcal{L}_{\overline{g}} - \eta\lambda(\mathbb{S}^n)\psi^{2^*-2}.$$

Because the space of invertible operators is open, if the operator norm of the perturbation term $\lambda(\mathbb{S}^n)\eta\psi^{2^*-2}$ is small enough, the operator $\mathcal{L}_\eta$ will also have a bounded inverse. Now chose q such that $\frac{2n}{n+2} < q < \frac{n}{2}$ (in particular, $q > 1$) and set $r = \frac{nq}{n-2q}$, for $u \in W^{2,q}(\mathbb{S}^n)$, Hölder's inequality followed by Sobolev's inequality give us

$$\|\eta\psi^{2^*-2}u\|_q \leq \|\eta^{1/(2^*-2)}\|_{2^*}^{2^*-2}\|u\|_r \leq C\|\eta^{1/(n-2)}\psi\|_{2^*}^{2^*-2}\|u\|_{W^{2,q}},$$

therefore,

$$\|\eta\psi^{2^*-2}\|_{op} \leq C\|\eta^{1/(n-2)}\psi\|_{2^*}^{2^*-2},$$

where $\|\cdot\|_{op}$ denotes the operator norm. Thus, by shrinking the support of η and imposing $0 \leq \eta \leq 1$, we can make the operator norm of the term $\lambda(\mathbb{S}^n)\eta\psi^{2^*-2}$ as small as we want.

Now $\mathcal{L}_\eta\psi = \lambda(\mathbb{S}^n)\psi^{2^*-1} - \lambda(\mathbb{S}^n)\eta\psi^{2^*-1} = (1-\eta)\lambda(\mathbb{S}^n)\psi^{2^*-1} \in L^q(\mathbb{S}^n)$ because ψ is $\mathcal{C}^2$ away from the north pole.

Since $\mathcal{L}_\eta : W^{2,q}(\mathbb{S}^n) \to L^q(\mathbb{S}^n)$ is invertible, there exists $v \in W^{2,q}(\mathbb{S}^n)$ such that $\mathcal{L}_\eta\psi = \mathcal{L}_\eta v$, but Hölder's inequality and the Sobolev inequality yield

$$
\begin{aligned}
\|u\|_{H^1} &\leq \int_{\mathbb{S}^n} u\mathcal{L}u dV_{\overline{g}} = \int_{\mathbb{S}^n} u\mathcal{L}_\eta u dV_{\overline{g}} + \int_{\mathbb{S}^n} \lambda\eta\psi^{2^*-1} u dV_{\overline{g}} \\
&\leq \int_{\mathbb{S}^n} u\mathcal{L}_\eta u dV_{\overline{g}} + \left(\int_{\mathbb{S}^n} \eta^{2^*/(2^*-1)}\psi^{2^*} dV_{\overline{g}}\right)^{\frac{2^*-1}{2^*}} \|u\|_{2^*} \\
&\leq \int_{\mathbb{S}^n} u\mathcal{L}_\eta u dV_{\overline{g}} + C\left(\int_{\mathbb{S}^n} \eta^{2^*/(2^*-1)}\psi^{p} dV_{\overline{g}}\right)^{\frac{2^*-1}{2^*}} \|u\|_{H^1} \\
&\leq \int_{\mathbb{S}^n} u\mathcal{L}_\eta u dV_{\overline{g}} + C\varepsilon\|u\|_{H^1},
\end{aligned}
$$

where $\varepsilon > 0$ depends on η and ψ and can be as small as we want it. Hence $\mathcal{L}_\eta$ is injective on $H^1(\mathbb{S}^n)$ and since $W^{2,q}(\mathbb{S}^n) \subset H^1(\mathbb{S}^n)$ (this is due to a simple application of Hölder's inequality using the fact that $\mathbb{S}^n$ has finite volume) we must have $\psi = v \in W^{2,q}(\mathbb{S}^n) \rightharpoonup L^r(\mathbb{S}^n)$. Since $r > 2^*$, Theorem B.0.20 implies that ψ must be smooth and since ψ is one at the south pole, ψ must be strictly positive. ☐

Lemma 3.1.8. *Let (M, g) be a compact Riemannian manifold with strictly positive scalar curvature S. Then, for every $1 < q < \infty$, the operator $\mathcal{L}_g : W^{2,q}(M) \to L^q(M)$ is invertible.*

Proof. We start by showing that $\mathcal{L}_g$ is injective. Indeed if there is $u \in W^{2,q}(\mathbb{S}^n) \setminus \{0\}$ such that $\mathcal{L}_g u = 0$, then, by elliptic regularity, u is a smooth function, and is an eigenfunction of the Laplace-Beltrami operator and $-S$ is its eigenvalue. But then (since $u \in \mathcal{C}^\infty(M) \subset H^1(M)$) we have

$$
\int_M a|\nabla u|^2 dV_g = -\int_{\mathbb{S}^n} Su^2 dV_g, \tag{3.1.8}
$$

where we have used the fact that the scalar curvature of M is strictly positive. This implies that $u \equiv 0$ which contradicts the assumption that u is not identically zero. And so we have shown that $\mathcal{L}_g$ is a injective operator from $W^{2,q}(M)$ to $L^q(M)$. Now we show that $\mathcal{L}_g$ is surjective. This is equivalent to showing that for every $f \in L^q(M)$ there is $u \in W^{2,q}(M)$ such that $\mathcal{L}_g u = f$ at almost every point. Fix $f \in L^q(M)$. Let $\{U_i\}$ be a finite open covering of M such that each U_i is the domain of a chart whose image is a bounded smooth domain in $\mathbb{R}^n$ and let $\{\eta_i\}$ be a smooth partition of unity subordinate to the open covering $\{U_i\}$. Now let $f_i = \eta_i f$. Denote by V_i the (bounded) open set in $\mathbb{R}^n$ that corresponds to the open set U_i in M. For each i consider the problem

$$
\begin{cases}
-\frac{1}{\sqrt{\det(g_{ij})}}\partial_i(g^{ij}\sqrt{\det(g_{ij})}\partial_j u) + Su = f_i, & \text{in } V_i, \\
u = 0 & \text{in } \partial V_i.
\end{cases} \tag{3.1.9}
$$

The equation above is the equation $\mathcal{L}_g u = f_i$ written in local coordinates. Now, by noting that we are in the conditions of Theorem B.0.16, we have a unique solution $u_i \in W^{2,p}(V_i)$ to the problem (3.1.9). Now transport the function u_i to U_i via the chart and define u_i on the rest of $\mathbb{S}^n$ by extending it by zero. Now define $u = \sum_i u_i$, which by construction is in the Sobolev space $W^{2,q}(M)$. We claim that u is such that $\mathcal{L}_g u = f$ at almost every point. To see this fix $\varphi \in \mathcal{C}_c^\infty(M)$, and, for each i, let $\{\varphi_n^i\} \subset \mathcal{C}_c^\infty(U_i) \subset W^{2,q}(U_i)$

be a sequence of compactly supported smooth functions such that

$$\|\varphi_n^i - \varphi_{|_{U_i}}\|_{W^{2,q}(U_i)} \to 0, \text{ as } n \to \infty.$$

So we have,

$$\begin{aligned}
\int_M u\mathcal{L}_g^*\varphi dV_g &= \int_M \varphi\mathcal{L}_g u dV_g = \sum_i \int_{U_i} \varphi\mathcal{L}_g u_i dV_g \\
&= \sum_i \lim_{n\to\infty} \int_{U_i} \varphi_n^i \mathcal{L}_g u_i dV_g = \sum_i \lim_{n\to\infty} \int_{U_i} \varphi_n^i f_i dV_g \\
&= \sum_i \int_{U_i} f_i \varphi dV_g = \int_M \varphi f dV_g,
\end{aligned}$$

Where $\mathcal{L}_g^*(= \mathcal{L}_g)$ is the adjoint operator of $\mathcal{L}_g$. This shows that u is indeed a weak solution of $\mathcal{L}_g u = f$. By noting that

$$\|\mathcal{L}_g u\|_q \leq a\|\Delta u\|_q + \|S\|_{L^\infty(M)}\|u\|_q \lesssim \|u\|_{W^{2,q}},$$

we conclude that $\mathcal{L}_g : W^{2,q}(M) \to L^q(M)$ is a bounded and bijective operator, hence invertible with bounded inverse. $\square$

Chapter 4

Conformal Normal Coordinates and Stereographic Projections

In Chapter 3 we saw that if $\lambda(M) < \lambda(\mathbb{S}^n)$ then we are able to apply the direct method of the Calculus of Variations followed by some regularity theory to solve the Yamabe problem. Our objective, from now on will be to show that whenever M is not in the conformal class of the standard sphere this is indeed the case. However, this task is far from trivial. This was, originally, the result of the combined efforts of Aubin and Schoen ([2, 16]). More specifically, Aubin proved this fact when M is not conformally flat (i.e. the metric is locally conformal to the Euclidean metric) and has dimension greater or equal to 6. And Schoen covered the remaining cases, those being when either M is conformally flat or has dimension 3,4 or 5. Later on, John M. Lee and Thomas H. Parker also proved this fact using different methods in [13]. They managed, using the machinery presented in this chapter and Chapter 6, to merge the proofs of Schoen and Aubin. The common trait between the works of Aubin, Schoen, and therefore, Lee and Parker is the construction of a test function. In this chapter, we follow closely Lee and Parker's approach from sections 5 and 6 in [13] to build the necessary machinery to prove $\lambda(M) < \lambda(\mathbb{S}^n)$. The purpose of this chapter is to develop the necessary tools for this construction.

4.1 Conformal Normal Coordinates

The calculations one usually encounters in Riemannian Geometry are dense and complicated. So, it would be very useful if we had access to coordinates that greatly simplify the analysis of the local geometry of a Riemannian manifold. Said coordinates do exist, they are called normal coordinates. In this section, we describe the geometric construction of normal coordinates and proceed to describe a similar set of coordinate charts on a conformal manifold to M. These will be normal coordinates for some metric g within the conformal class of the original metric, and the freedom in the choice of metric will allow us to simplify the local geometry considerably more than the usual normal coordinates.

4.1.1 Normal Coordinates

In this subsection, we present the construction of normal coordinates. The results and proofs in this subsection were based on the excellent [11] with some details added for better comprehension.

To properly construct normal coordinates we start by introducing the notions of covariant derivative of a vector field along a curve and of a geodesic.

Definition 4.1.1. *Let (M, g) be a Riemannian manifold. A vector field defined along a differentiable curve $c : I \to M$ is a differentiable map $V : I \to TM$ such that $V(t) \in T_{c(t)}M, \forall t \in I$. If $\dot{c} \neq 0$, the covariant derivative of V along c is the vector field defined along c by*

$$\nabla_{\dot{c}(t)} V = (\nabla_X Y)_{c(t)} ,$$

for any vector fields $X, Y \in \mathfrak{X}(M)$ such that $X(c(t)) = \dot{c}(t)$ and $Y(c(s)) = V(s)$ with $s \in (t - \varepsilon, t + \varepsilon)$ for some $\varepsilon > 0$.

Since the covariant derivative of a vector field Y along a vector field X at $p \in M$ only depends on X_p and the values of Y along a curve tangent to X, the covariant derivative along a curve is well-defined. In local coordinates $\{x^i\}$ we have that

$$\nabla_{\dot{c}(t)} V = \sum_{i=1}^{n} \left(\dot{V}^i(t) + \sum_{j,k=1}^{n} \Gamma^i_{jk}(c(t)) V^j(t) \dot{x}^k(t) \right) \left(\frac{\partial}{\partial x^i} \right)_{c(t)} ,$$

where $x^i(t) = x^i(c(t))$ and Γ^i_{jk} denote the Christofel symbols as in (A.0.2). There is another way to obtain the covariant derivative of a vector field along a curve which uses the concept of vector bundles, but we will not use it in order to not overcomplicate matters.

Definition 4.1.2. *A curve $c : I \to M$ is said to be a geodesic (of the Levi-Civita connection ∇) if*

$$\nabla_{\dot{c}} \dot{c} = 0.$$

In local coordinates, this equation becomes

$$\ddot{x}^i(t) + \sum_{j,k=1}^{n} \Gamma^i_{jk}(c(t)) \dot{x}^j(t) \dot{x}^k(t) = 0, \forall i = 1, ..., n.$$

Definition 4.1.3. *Let (M, g) be a Riemannian manifold, take $p \in M$ and $v \in T_pM$. Denote by c_v the unique geodesic starting at p with $\dot{c}_v(0) = v$. Let $V_p := \{v \in T_pM : c_v$ is defined in for all $t \in [0, 1]\}$ (note V_p contains a neighborhood of $0 \in T_pM$) and define the exponential map of M at p, $\exp_p : V_p \to M$, is defined by $\exp(v) = c_v(1)$.*

Lemma 4.1.4. *The exponential map of M at p is a diffeomorphism from a neighbourhood of $0 \in T_pM$ onto a neighbourhood of $p \in M$.*

Proof. Since T_pM is a vector space, its tangent space at 0 can be identified with itself. Therefore, we can think of $d\exp_p$ at 0 as a map from T_pM onto itself. So, for $v \in T_pM$, we have (with this identification):

$$d\exp_p(0)(v) = \frac{d}{dt}c_{tv}(1)_{|t=0} = \frac{d}{dt}c_v(t)_{|t=0} = \dot{c}_v(0) = v.$$

This shows that $d\exp_p(0)$ is the identity map. So, the inverse function theorem implies that $\exp_p$ maps a neighbourhood of $0 \in T_pM$ diffeomorphically onto a neighbourhood of $p \in M$. $\square$

Now let $e_1, ..., e_n$ $(n = \dim M)$ be a basis of T_pM which is orthonormal with respect to the inner product induced on T_pM by g. Writing each vector $v \in T_pM$ with respect to this basis yields a map $\Phi : T_pM \to \mathbb{R}^n$, given by $v = v^i e_i \mapsto (v^1, ..., v^n)$. Under this identification, we conclude that $\exp_p$ induces a local coordinate chart.

Definition 4.1.5. *The local coordinates induced by the exponential map $\exp_p$ are called normal coordinates with centre p.*

Note that, under the coordinates induced by the exponential map at p, the geodesics that start at p are simply the straight line going through $0 \in T_pM$. Therefore all the Christoffel symbols are zero, at p or equivalently, all the partial derivatives of the components of the metric tensor vanish at p. The following result summarises this.

Theorem 4.1.6. *In normal coordinates, the following holds:*

$$g_{ij}(0) = \delta_{ij}, \;\; \Gamma^k_{ij}(0) = 0, \;\; \frac{\partial g_{ij}}{\partial x^k}(0) = 0, \forall i, j, k.$$

In particular, $g_{ij} = \delta_{ij} + O(r^2)$ and $\det(g_{ij}) = 1 + O(r^2)$, where $r = |x|$ denotes the geodesical distance.

Some properties of normal coordinates are more easily seen in polar coordinates rather than in normal Euclidean coordinates. Introduce standard polar coordinates on $\mathbb{R}^n$, $(r, \varphi^1, ..., \varphi^{n-1})$, where $\varphi = (\varphi^1, ..., \varphi^{n-1})$ parametrizes the unit sphere. Now, via the isomorphism Φ, transport these coordinates to T_pM. We write g_{rr}, instead of g_{11} and $g_{r\varphi^i}$ instead of g_{1i+1} and $g_{\varphi^i\varphi^j}$ instead of $g_{i-1,j-1}$. Then in the limit $v \to 0 \in T_pM$, we have $\lim_{v\to 0} g_{rr}(p) = 1$ and $\lim_{p\to 0} g_{r\varphi^j}(p) = 0$. Since the geodesics are the lines $\varphi \equiv$ const. when parametrized by arc length, we have, by definition of the Christoffel symbols (A.0.2), $\Gamma^i_{rr} = 0$ for all i, hence

$$g^{il}(2g_{rl,r} - g_{rr,l}) = 0, \forall i,$$

thus $2g_{rl,r} - g_{rr,l} = 0$, for all l, in particular, $g_{rr,r} = 0$, and therefore $g_{rr} = 1$ in a neighborhood of p. But this implies that $g_{r\varphi^i,r} = 0$ and therefore, $g_{r\varphi^i} \equiv 0$. We have shown the following.

Theorem 4.1.7. *In polar normal coordinates, the metric has the form*

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_{\varphi^1\varphi^1} & \cdots & g_{\varphi^1\varphi^{n-1}} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & g_{\varphi^{n-1}\varphi^1} & \cdots & g_{\varphi^{n-1}\varphi^{n-1}} \end{pmatrix}$$

in a neighbourhood of p.

4.1.2 Asymptotic Analysis using Conformal Normal Coordinates

The following theorem was one of the key results obtained by Lee and Parker in [13] that allowed them to unify Schoen's and Aubin's work. Indeed, Lee and Parker exploited the freedom to change the background metric to make the local geometry of the manifold we are working on very close to the geometry of the Euclidean space $\mathbb{R}^n$. This, as one expects, greatly diminishes the complexity of the analysis of the Yamabe problem. We start by exploring the power of conformal normal coordinates, these are normal coordinates with respect to a metric conformal to the original one. The usage of these coordinates in the context of the Yamabe problem was Lee and Parker's original idea. Indeed, in this subsection we closely follow Section 5 of [13] with all the missing details added. The following is due to Lee and Parker in [13].

Theorem 4.1.8 (Conformal Normal Coordinates)**.** *Let (M, g) be a Riemannian manifold and $p \in M$. For each $N \geq 2$ there is a conformal metric $\widetilde{g}$ on M such that*

$$\det(\widetilde{g}_{ij}) = 1 + O(r^N), \text{ as } r \to 0^+,$$

where $r = |x|$ in $\widetilde{g}-$normal coordinates at p. When $N \geq 5$, in $\widetilde{g}-$normal coordinates, the scalar curvature of $\widetilde{g}$, $\widetilde{S}$, satisfies $\widetilde{S} = O(r^2)$ and $\Delta_{\widetilde{g}}\widetilde{S} = \frac{1}{6}|W|^2$ at p, where W denotes the Weyl tensor (see Definition A.0.20*) that is invariant under conformal changes of metric.*

Proving this theorem requires some machinery. Therefore, we will break the proof into smaller parts that hopefully will make it simpler. At the end of this section, we will prove Theorem 4.1.8. In Theorem 4.1.20 we will use it to show that if (M, g) is a compact, connected Riemannian manifold of dimension greater or equal to 6 that is not locally conformally flat, then $\lambda(M) < \lambda(\mathbb{S}^n)$.

The proof of Theorem 4.1.8 heavily relies on the following theorem, which is due to Lee and Parker [13].

Theorem 4.1.9. *Let (M, g) be a compact, connected Riemannian manifold and take $p \in M$. Given $k \geq 0$ and T a symmetric $(k+2)-$tensor on T_pM, there is a unique homogeneous polynomial f of degree $k+2$ in $g-$normal coordinates such that the metric $\widetilde{g} = e^{2f} g$ satisfies*

$$\mathrm{Sym}(\widetilde{\nabla}^k \widetilde{R}_{ij})(p) = T,$$

where $\mathrm{Sym}(\widetilde{\nabla}^k \widetilde{R}_{ij})$ denotes the symmetric part of the $k-$th covariant derivative of the Ricci tensor the metric $\widetilde{g}$.

Proof. Let $\{x^i\}$ be normal $g-$coordinates at p, $r = |x|$ and let $\mathcal{P}_m$ denote the set of homogeneous polynomials in $\{x^i\}$ of degree m. If we define $F_g(x) = R_{ij}(x)x^i x^j$, where $x = (x^1, ..., x^n)$, then the Taylor expansion of F_g is:

$$F_g(x) = \sum_{m=2}^{k+2} F_g^m + O(r^{k+3}),$$

where

$$F_g^m = \frac{1}{(m-2)!} \sum_{|K|=m-2} \sum_{i,j=1}^{n} \partial_K R_{ij}(p) x^i x^j x^K \in \mathcal{P}_m,$$

where $n = \dim M$. Now observe that the components of the covariant derivatives of the Ricci tensor are related to the usual partial derivatives of the components of the Ricci tensor by $R_{ij,K}(p) = \partial_K R_{ij}(p) + S_{ij,K}$, where the S_{ijK} are constructed as polynomials in the curvature and its derivatives of order strictly smaller than $|k|$ at p. If $\widetilde{g} = e^{2f} g$ with $f \in \mathcal{P}_{k+3}$, we have, $\widetilde{S}_{ijK} = S_{ijK}$ when $|K| = k$ (this is because all the extra terms given by the transformation laws of the Ricci curvature are zero at p since f and its partial derivatives up to order $k+2$ are zero at p).
Proving the result is equivalent to finding $f \in \mathcal{P}_{k+2}$ such that

$$\begin{aligned} 0 &= \frac{1}{k!} \sum_{|K|=k} \sum_{i,j=1}^{n} \left(\widetilde{R}_{ij,K}(p) - T_{ijK} \right) \widetilde{x}^i \widetilde{x}^j \widetilde{x}^K \\ &= \frac{1}{k!} \sum_{|K|=k} \sum_{i,j=1}^{n} \left(\widetilde{R}_{ij,K}(p) - T_{ijK} \right) x^i x^j x^K + O(r^{k+4}) \qquad (4.1.1) \\ &= F_{\widetilde{g}}^{k+2}(x) + \sum_{i,j=1}^{n} \frac{1}{k!} \left(\widetilde{S}_{ij,K}(p) - T_{ijK} \right) x^i x^j x^K, \end{aligned}$$

where we have used the fact that $g-$normal coordinates differ from $\widetilde{g}-$normal coordinates by $O(r^{k+2})$ (see Lemma 4.1.11 below) with $\{\widetilde{x}^i\}$ being $\widetilde{g}-$normal coordinates. By Euler's formula, $x^i x^j \partial_i \partial_j f = (x^i \partial_i)^2 f - x^i \partial_i f = (k+2)(k+1)f$, and $\Delta_g f = \Delta f + O(r^{k+1})$ (see Lemma 4.1.11) where Δ denotes the

Euclidean Laplacian in $x-$coordinates. Thus the transformation law (A.0.18) for the Ricci tensor yields:

$$\begin{aligned}
F_{\widetilde{g}}^{k+2}(x) &= \frac{1}{k!}\sum_{|K|=k}\sum_{ij}\partial_K \widetilde{R}_{ij}(p)x^i x^j x^K \\
&= \frac{1}{k!}\sum_{|K|=k}\sum_{ij}\partial_K \left(R_{ij} - (n-2)(\mathcal{H}_f)_{ij} + (n-2)(df\otimes df)_{ij} + (\Delta_g f - (n-2)\|df\|^2)g_{ij}\right)(p)x^i x^j x^K \\
&= F_g^{k+2}(x) + \frac{1}{k!}\sum_{|K|=k}\sum_{ij}\partial_K \left(-(n-2)(\mathcal{H}_f)_{ij} + (\Delta f + O(r^{k+1}))(\delta_{ij} + O(r^2))\right)(p)x^i x^j x^K \\
&= F_g^{k+2}(x) + \sum_{ij}(2-n)x^i x^j \partial_i\partial_j f + +x^i x^j \Delta f \delta_{ij} \\
&= F_g^{k+2}(x) - (n-2)(k+2)(k+1)f + r^2\Delta f.
\end{aligned} \tag{4.1.2}$$

Note that there are no error terms in the previous calculations because we are evaluating all the terms at p. The following lemma shows that $r^2\Delta-(n-2)(k+2)(k+1)$ is invertible on $\mathcal{P}_{k+2}$ since $(n-2)(k+2)(k+1)$ is not an eigenvalue of $r^2\Delta$, and as a consequence there is a unique $f\in\mathcal{P}_{k+2}$ so that (4.1.1) is satisfied. □

Lemma 4.1.10. *The eigenvalues of $r^2\Delta$ on $\mathcal{P}_m$ are*

$$\{\alpha_j = -2j(n-2+2m-2j) : j = 0, ..., [m/2]\}.$$

The eigenfunctions corresponding to α_j are the functions of the form $r^{2j}u$, where $u\in\mathcal{P}_{m-2j}$ is harmonic.

Proof. The result clearly holds in the cases $m = 0$ and $m = 1$ since in both cases $[m/2] = 0$ and $r^2\Delta f = 0$ for all $f\in\mathcal{P}_m$. Now assume that $m\geq 2$ and that the result holds up to $m-1$. Let $f\in\mathcal{P}_m$ be an eigenfunction of $r^2\Delta$ corresponding to λ, i.e. $r^2\Delta f = \lambda f$. By Euler's formula $\Delta f\in\mathcal{P}_{m-2}$ satisfies

$$\begin{aligned}
\lambda\Delta f = \Delta_0(r^2\Delta_0 f) &= \Delta(r^2)\Delta f - 2\partial_i(r^2)\partial_i(\Delta f) + r^2\Delta^2 f \\
&= -2n\Delta f - 4x^i\partial_i(\Delta f) + r^2\Delta^2 f \\
&= -2n\Delta f - 4(m-2)\Delta f + r^2\Delta^2 f,
\end{aligned}$$

so $r^2\Delta(\Delta f) = (\lambda + 2n + 4m - 8)\Delta f$. Then either $\Delta_0 f = 0$, in which case $\lambda = 0$ and f is harmonic, or $\lambda + 2n + 4m - 8$ is an eigenvalue of $r^2\Delta$ on $\mathcal{P}_{m-2}$ with eigenfunction Δf. In the later case, $f = \lambda^{-1}r^2\Delta f$ and $\lambda + 2n + 4m - 8 = -2j(n-2+2m-4-2j)$ for some $j\in\{0, ..., [(m-1)/2]\}$, so $\lambda = -2n - 4m + 8 - 2j(n-2+2(m-2)-2j) = -(j+1)(n-2+2m-2(j+1))$. One easily checks that $\lambda_j\leq[(m-1)/2]$ implies that $\lambda_j + 1\leq[m/2]$. Furthermore, note that in the case where Δf is an eigenfunction of $r^2\Delta$ in $\mathcal{P}_{m-2}$ the induction hypothesis implies that $\Delta f = r^{2j}u$ for some $u\in\mathcal{P}_{m-2-2j}$, but then $f = \lambda^{-1}r^{2+2j}u$. □

Lemma 4.1.11. *If $f\in\mathcal{P}_k$ (with $k\geq 2$) and $\widetilde{g} = e^{2f}g$, then $g-$normal coordinates differ from $\widetilde{g}-$coordinates by $O(r^{k+1})$. If $k\geq 2$, $\Delta_g f = \Delta f + O(r^{k-1})$.*

Proof. Let $k \geq 2$ and $\exp_1$ and $\exp_2$ denote the exponential maps relative to the metrics g and $\widetilde{g}$, respectively. To prove the result we need to show that given $v \in T_pM$ with $\|v\| = 1$, $|x_1(t) - x_2(t)| = O(t^{k+1})$ for small t, where $x_1(t) = \exp_1(tv)$ and $x_2(t) = \exp_2(tv)$. First we note that x_1 is a geodesic for g and x_2 is a geodesic for $\widetilde{g}$, and as such (in g- normal coordinates $\{x^i\}$):

$$\ddot{x}_1^i + \sum_{i,j=1}^{n} \Gamma_{jk}^i(x_1(t))\dot{x}_1^j \dot{x}_1^k = 0, \quad i = 1, ..., n, \tag{4.1.3}$$

and

$$\ddot{x}_2^i + \sum_{i,j=1}^{n} \widetilde{\Gamma}_{jk}^i(x_2(t))\dot{x}_2^j \dot{x}_2^k = 0, \quad i = 1, ..., n, \tag{4.1.4}$$

where the $\widetilde{\Gamma}_{jk}^i$ denotes the Christoffel symbols of the metric $\widetilde{g}$. Since $\widetilde{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_{ij}\partial_k f + \delta_{ik}\partial_k f - g_{jk}\nabla^i f$, we have that $\widetilde{\Gamma}_{jk}^i = \Gamma_{jk}^i + O(t^{k-1})$. On the other hand, since $\{x^i\}$ are normal coordinates, $\Gamma_{jk}^i(x_1(t)) = O(t)$ and $\widetilde{\Gamma}_{jk}^i(x_2(t)) = O(t)$.

$$\begin{aligned}
\ddot{x}_1^i - \ddot{x}_2^i &= \sum_{j,k=1}^{n} \widetilde{\Gamma}_{jk}^i(x_2(t))\dot{x}_2^j \dot{x}_2^k - \Gamma_{jk}^i(x_1(t))\dot{x}_1^j \dot{x}_1^k = \sum_{j,k=1}^{n} O(t)(\dot{x}_2^j \dot{x}_2^k - \dot{x}_2^j \dot{x}_2^k) + O(t^{k-1}) \\
&= \sum_{j,k=1}^{n} O(t)(\dot{x}_2^j \dot{x}_2^k - \dot{x}_1^j \dot{x}_2^k + \dot{x}_1^j \dot{x}_2^k - \dot{x}_1^j \dot{x}_1^k) + O(t^{k-1}) \\
&= \sum_{j,k=1}^{n} O(t)((\dot{x}_2^j - \dot{x}_1^j)\dot{x}_2^k + (\dot{x}_1^j(\dot{x}_2^k - \dot{x}_1^k)) + O(t^{k-1}) \\
&= \sum_{j,k=1}^{n} O(t)((\dot{x}_2^j - \dot{x}_1^j)(\dot{x}_2^k + \dot{x}_1^k) + O(t^{k-1}).
\end{aligned}$$

This implies that

$$|\ddot{x}_1^i(t) - \ddot{x}_1^i(t)| \leq +O(t^{k-1}) + O(t)\sum_{j=1}^{n} |\dot{x}_1^j(t) - \dot{x}_2^j(t)|, \tag{4.1.5}$$

for some constant $C > 0$. Since we are only concerned with small t there is a constant $K > 0$ for which $|\dot{x}_1(t) - \dot{x}_2(t)| \leq K$ for t small, therefore by integrating we conclude that if $k > 1$

$$|\dot{x}_1^i(t) - \dot{x}_2^i(t)| = O(t^2), \forall i = 1, ..., n.$$

and by replacing this estimate in (4.1.5) we conclude that $|\dot{x}_1^i(t) - \dot{x}_2^i| = O(t^4), \forall i = 1, ..., n$. By iterating this process we can conclude that $|\dot{x}_1^i(t) - \dot{x}_2^i(t)| = O(t^k)$ and therefore $|x_1(t) - x_2(t)| = O(t^{k+1})$.
For the estimate on the Laplacian of f note that in $g-$normal coordinates we have

$$\Delta_g f = -g^{ij}\frac{\partial^2 f}{\partial x^i \partial x^j} + \Gamma_{ij}^k \frac{\partial f}{\partial x^k} g^{ij} = -(\delta_{ij} + O(r^2))\frac{\partial^2 f}{\partial x^i \partial x^j} + \Gamma_{ij}^k \frac{\partial f}{\partial x^k} g^{ij} = \Delta f + O(r^k)$$

$\square$

The following is due to R. Graham.

Corollary 4.1.12. *Given $p \in M$, $N \geq 0$, there exists a metric conformal to g such that all symmetrized* *covariant derivatives of the Ricci tensor of order $\leq N$ vanish at p.*

Proof. We proceed by induction in N. Choose $T = 0$ in the above theorem, and note that $f \in \mathcal{P}_{N+2}$ implies that $\widetilde{\nabla}^k \widetilde{R}_{ij} = \nabla^k R_{ij}$ for $k < N$. This is because, at p all the terms in the covariant derivatives up to order $N-1$ of $\widetilde{\mathrm{Ric}}$ that are not the covariant derivatives of $\nabla^k \mathrm{Ric}$ are zero at p, and this is because the Christoffel symbols, f and its derivative up to order $N-1$ vanish at p. □

Before continuing our path towards a proof of Theorem 4.1.8, we need to introduce the important concept of Jacobi fields.

To prove Theorem 4.1.8 we will employ Jacobi fields. As such, a brief exposition, containing the results we will use, is helpful. What follows regarding Jacobi fields is based on [11]. And all the proofs that we omit may be found there.

Definition 4.1.13. *Let $c : I \to M$ be a geodesic. A vector field X along c is called a Jacobi field if*

$$\nabla_{\dot{c}} \nabla_{\dot{c}} X + \mathrm{R}(X, \dot{c})\dot{c} = 0. \tag{4.1.6}$$

As an abbreviation, we will write when possible

$$\dot{X} = \nabla_{\dot{c}} X, \qquad \ddot{X} = \nabla_{\dot{c}} \nabla_{\dot{c}} X$$

A lot can be said about Jacobi fields, for our purposes, we only need a result that gives a characterization of Jacobi fields.

Theorem 4.1.14. *Let $c : [0, a] \to M$ be a geodesic and $c(\cdot, \cdot) : [0, a] \times (-\varepsilon, \varepsilon) \to M)$ a variation of $c(t)$ through geodesics, i.e., every curve $c(t, s) =: c_s(t)$ (for fixed s) is also a geodesic. Then*

$$X(t) := \frac{\partial}{\partial s} c(t, s)_{|_{s=0}} \tag{4.1.7}$$

is a Jacobi field along $c(t) = c_0(t)$. Furthermore, every Jacobi field can be obtained through this method.

Proof. See Theorem 6.2.1 in [11]. □

Corollary 4.1.15. *Let $c : [0, T] \to M$ be a geodesic and $p = c(0)$, i.e., $c(t) = \exp_p(t\dot{c}(0))$. For $w \in T_pM$, the Jacobi field X along c with $X(0) = 0$, $\dot{X}(0) = w$ at the point $c(t)$ is given by derivative of the exponential map $\exp_p : T_pM \to M$, evaluated at $c(t)$ and applied to the vector tw:*

$$X(t) = \left(d \exp_p\right)_{t\dot{c}(0)} (tw), \tag{4.1.8}$$

Proof. See Corollary 6.2.2 [11]. □

From the previous corollary, one concludes that the derivative of the exponential map can be computed from Jacobi fields along radial geodesics. Now, we are able to prove another Lemma from [13] that will be a key step in the proof of Theorem 4.1.8.

Lemma 4.1.16. *In $g-$normal coordinates, the determinant of the metric, $\det g_{ij}$, has the expansion*

$$\det(g_{ij}) = 1 - \frac{1}{3}R_{ac}x^a x^c - \frac{1}{6}R_{ac,k}x^a x^c x^k - \left(\frac{1}{20}R_{ac,kl} + \frac{1}{90}R_{aicj}R_{ljkd} - \frac{1}{18}R_{ac}R_{kl}\right) x^a x^c x^k x^l + O(r^5), \tag{4.1.9}$$

where $r = |x|$ and all the curvature terms are evaluated at p.

Proof. Let $\{x^i\}$ denote normal coordinates for g on a neighbourhood U of p, for simplicity we identify U with an open set in $\mathbb{R}^n$. To compute the expansion of the metric $g_{ij}(x)$ we will apply the theory we presented about Jacobi fields.
First, fix $\xi, \tau \in \mathbb{R}^n$ and consider the map $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ given, in the coordinates $\{x^i\}$, by $c(t,s) = t(\tau + s\xi)$, which is a one-parameter family of geodesics; in fact, $c(t,s)$ is a variation through geodesics of the geodesic $c(t) := c(t,0)$. The previous results about Jacobi fields imply that the variational vector field given by $X(c_s(t)) = \frac{\partial}{\partial s}c(t,s) = t\xi$ is a Jacobi field along $c(t)$. Also, define T as the vector field along $c(t,s)$ given by $T(t,s) := \frac{\partial}{\partial s}c(t,s)$. From now on, R_T will denote the curvature endomorphism $\mathrm{R}(T,\cdot)T$. Now consider the function $f(t) = |X(c(t,0))|^2$, employing the Jacobi equation and differentiating repeatedly with respect to T we can compute the Taylor series of f. We now compute the first few terms:

$$\nabla_T f(0) = 2\langle \nabla_T X(c_0(t)), X(c_0(t))\rangle_{|t=0} = 0, \tag{4.1.10}$$

because $X(0) = 0$. For the second term,

$$\begin{aligned}
\nabla_T^2 f(0) &= 2\langle \nabla_T^2 X(c_0(t)), X(c_0(t))\rangle_{|t=0} + 2\langle \nabla_T X(c_0(t)), \nabla_T X(c_0(t))\rangle_{|t=0} \\
&= 2\langle R_T X(c_0(t)), X(c_0(t))\rangle_{|t=0} + 2\langle \nabla_T X(c_0(t)), \nabla_T X(c_0(t))\rangle_{|t=0} \\
&= 2\langle \xi, \xi\rangle,
\end{aligned} \tag{4.1.11}$$

because $\nabla_T X(0) = \xi$ (we are identifying vectors in the tangent space with vectors in $\mathbb{R}^n$). From now on we write X instead of $X(c_0(t))$. For the third term,

$$\begin{aligned}
\nabla_T^3 f(0) &= 2\langle \nabla_T(R_T(X)), X\rangle_{|t=0} + 2\langle R_T(X), \nabla_T X\rangle_{|t=0} + 4\langle \nabla_T^2 X, \nabla_T X\rangle_{|t=0} \\
&= 2\langle \nabla_T R_T(X), X\rangle_{|t=0} + 2\langle R_T(\nabla_T(X)), X\rangle_{|t=0} + 2\langle R_T(X), \nabla_T X\rangle_{|t=0} + 4\langle \nabla_T^2 X, \nabla_T X\rangle_{|t=0} \\
&= 2\langle \nabla_T R_T(X), X\rangle_{|t=0} + 2\langle R_T(\nabla_T(X)), X\rangle_{|t=0} + 6\langle R_T(X), \nabla_T X\rangle_{|t=0} \\
&= 0,
\end{aligned} \tag{4.1.12}$$

because $\nabla_T(R_T(X)) = \nabla_T R_T(X) + R_T(\nabla_T X)$ and $R_T(X(c_0(t)))_{|t=0} = (tR_\tau(\xi)))_{|t=0} = 0$.

For the fourth term, we have

$$\begin{aligned}
\nabla_T^4 f(0) &= 2\langle \nabla_T(\nabla_T R_T(X)), X\rangle_{|t=0} + 2\langle \nabla_T R_T(X), \nabla_T X\rangle_{|t=0} + 2\langle \nabla_T(R_T(\nabla_T(X))), X\rangle_{|t=0} \\
&+ 2\langle R_T(\nabla_T(X)), \nabla_T X\rangle_{|t=0} + 6\langle \nabla_T(R_T(X)), \nabla_T X\rangle_{|t=0} + 6\langle R_T(X), \nabla_T^2 X\rangle_{|t=0} \\
&= 2\langle \nabla_T^2 R_T(X)), X\rangle_{|t=0} + 2\langle \nabla_T R_T(\nabla_T X)), X\rangle_{|t=0} + 2\langle \nabla_T R_T(X), \nabla_T X\rangle_{|t=0} \\
&+ 2\langle \nabla_T R_T(\nabla_T(X))), X\rangle_{|t=0} + +2\langle R_T(\nabla_T^2(X))), X\rangle_{|t=0} + 2\langle R_T(\nabla_T(X)), \nabla_T X\rangle_{|t=0} \\
&+ 6\langle \nabla_T R_T(X), \nabla_T X\rangle_{|t=0} + 6\langle R_T(\nabla_T X), \nabla_T X\rangle_{|t=0} + 6\langle R_T(X), R_T(X)\rangle_{|t=0} \\
&= 2\langle \nabla_T^2 R_T(X)), X\rangle_{|t=0} + 4\langle \nabla_T R_T(\nabla_T X)), X\rangle_{|t=0} + 8\langle \nabla_T R_T(X), \nabla_T X\rangle_{|t=0} \\
&+ 8\langle R_T(\nabla_T X), \nabla_T X\rangle_{|t=0} + 2\langle R_T(R_T(X)), X\rangle_{|t=0} + 6\langle R_T(X), R_T(X)\rangle_{|t=0} \\
&= 8\langle R_\tau \xi, \xi\rangle_0,
\end{aligned} \tag{4.1.13}$$

because $\nabla_T R_T(X)_{|t=0} = 0$ and $R_T(\nabla_T X)_{|t=0} = \langle R_\tau \xi, \xi\rangle_0$.
For the fifth term, we have

$$\begin{aligned}
\nabla_T^5 f(0) &= 2\langle \nabla_T^3 R_T(X)), X\rangle_{|t=0} + 2\langle \nabla_T^2 R_T(\nabla_T X), X\rangle_{|t=0} + 2\langle \nabla_T^2 R_T(X)), \nabla_T X\rangle_{|t=0} \\
&+ 4\langle \nabla_T^2 R_T(\nabla_T X)), X\rangle_{|t=0} + 4\langle \nabla_T R_T(\nabla_T^2 X)), X\rangle_{|t=0} + 4\langle \nabla_T R_T(\nabla_T X)), \nabla_T X\rangle_{|t=0} \\
&+ 8\langle \nabla_T^2 R_T(X), \nabla_T X\rangle_{|t=0} + 8\langle \nabla_T R_T(\nabla_T X), \nabla_T X\rangle_{|t=0} + 8\langle \nabla_T R_T(X), \nabla_T^2 X\rangle_{|t=0} \\
&+ 8\langle \nabla_T R_T(\nabla_T X), \nabla_T X\rangle_{|t=0} + 8\langle R_T(\nabla_T^2 X), \nabla_T X\rangle_{|t=0} + 8\langle R_T(\nabla_T X), \nabla_T^2 X\rangle_{|t=0} \\
&+ 2\langle \nabla_T R_T(R_T(X)), X\rangle_{|t=0} + 2\langle R_T(\nabla_T R_T(X)), X\rangle_{|t=0} + 2\langle R_T(R_T(\nabla_T X)), X\rangle_{|t=0} \\
&+ 2\langle R_T(R_T(X)), \nabla_T X\rangle_{|t=0} + 12\langle \nabla_T R_T(X), R_T(X)\rangle_{|t=0} + 12\langle R_T(\nabla_T X), R_T(X)\rangle_{|t=0} \\
&= 2\langle \nabla_T^3 R_T(X)), X\rangle_{|t=0} + 6\langle \nabla_T^2 R_T(\nabla_T X), X\rangle_{|t=0} + 10\langle \nabla_T^2 R_T(X)), \nabla_T X\rangle_{|t=0} \\
&+ 6\langle \nabla_T R_T(R_T(X)), X\rangle_{|t=0} + 20\langle \nabla_T R_T(\nabla_T X)), \nabla_T X\rangle_{|t=0} + 20\langle \nabla_T R_T(X), R_T(X)\rangle_{|t=0} \\
&+ 10\langle R_T(R_T(X)), \nabla_T X\rangle_{|t=0} + 20\langle R_T(\nabla_T X), R_T(X)\rangle_{|t=0} \\
&+ 2\langle R_T(\nabla_T R_T(X)), X\rangle_{|t=0} + 2\langle R_T(R_T(\nabla_T X)), X\rangle_{|t=0} \\
&= 20\langle \nabla_\tau R_\tau(\xi), \xi\rangle_0.
\end{aligned} \tag{4.1.14}$$

Finally, the sixth term is

$$
\begin{aligned}
\nabla_T^6 f(0) = {} & 2\langle \nabla_T^4 R_T(X)), X\rangle_{|t=0} + 2\langle \nabla_T^3 R_T(\nabla_T X)), X\rangle_{|t=0} + 2\langle \nabla_T^3 R_T(X)), \nabla_T X\rangle_{|t=0} \\
& + 6\langle \nabla_T^3 R_T(\nabla_T X), X\rangle_{|t=0} + 6\langle \nabla_T^2 R_T(\nabla_T^2 X), X\rangle_{|t=0} + 6\langle \nabla_T^2 R_T(\nabla_T X), \nabla_T X\rangle_{|t=0} \\
& + 10\langle \nabla_T^3 R_T(X)), \nabla_T X\rangle_{|t=0} + 10\langle \nabla_T^2 R_T(\nabla_T X)), \nabla_T X\rangle_{|t=0} + 10\langle \nabla_T^2 R_T(X)), \nabla_T^2 X\rangle_{|t=0} \\
& + 6\langle \nabla_T^2 R_T(R_T(X)), X\rangle_{|t=0} + 6\langle \nabla_T R_T(\nabla_T R_T(X)), X\rangle_{|t=0} + 6\langle \nabla_T R_T(R_T(\nabla_T X)), X\rangle_{|t=0} \\
& + 6\langle \nabla_T R_T(R_T(X)), \nabla_T X\rangle_{|t=0} + 20\langle \nabla_T^2 R_T(\nabla_T X)), \nabla_T X\rangle_{|t=0} + 20\langle \nabla_T R_T(\nabla_T^2 X)), \nabla_T X\rangle_{|t=0} \\
& + 20\langle \nabla_T R_T(\nabla_T X)), \nabla_T^2 X\rangle_{|t=0} + 20\langle \nabla_T^2 R_T(X), R_T(X)\rangle_{|t=0} + 20\langle \nabla_T R_T(\nabla_T X), R_T(X)\rangle_{|t=0} \\
& + 20\langle \nabla_T R_T(X), \nabla_T R_T(X)\rangle_{|t=0} + 20\langle \nabla_T R_T(X), R_T(\nabla_T X)\rangle_{|t=0} + 10\langle \nabla_T R_T(R_T(X)), \nabla_T X\rangle_{|t=0} \\
& + 10\langle R_T(\nabla_T R_T(X)), \nabla_T X\rangle_{|t=0} + 10\langle R_T(R_T(\nabla_T X)), \nabla_T X\rangle_{|t=0} + 10\langle R_T(R_T(X)), \nabla_T^2 X\rangle_{|t=0} \\
& + 20\langle \nabla_T R_T(\nabla_T X), R_T(X)\rangle_{|t=0} + 20\langle R_T(\nabla_T^2 X), R_T(X)\rangle_{|t=0} + 20\langle R_T(\nabla_T X), \nabla_T R_T(X)\rangle_{|t=0} \\
& + 20\langle R_T(\nabla_T X), R_T(\nabla_T X)\rangle_{|t=0} + 2\langle \nabla_T R_T(\nabla_T R_T(X)), X\rangle_{|t=0} + 2\langle R_T(\nabla_T^2 R_T(X)), X\rangle_{|t=0} \\
& + 2\langle R_T(\nabla_T R_T(\nabla_T X)), X\rangle_{|t=0} + 2\langle R_T(\nabla_T R_T(X)), \nabla_T X\rangle_{|t=0} + 2\langle \nabla_T R_T(R_T(\nabla_T X)), X\rangle_{|t=0} \\
& + 2\langle R_T(\nabla_T R_T(\nabla_T X)), X\rangle_{|t=0} + 2\langle R_T(R_T(\nabla_T^2 X)), X\rangle_{|t=0} + 2\langle R_T(R_T(\nabla_T X)), \nabla_T X\rangle_{|t=0}.
\end{aligned}
\tag{4.1.15}
$$

Since we are not going to compute any more terms we do not have to organise the terms in (4.1.15), we can just get rid of the terms that are zero at $t = 0$

$$
\begin{aligned}
\nabla_T^6 f(0) &= 36\langle \nabla_T^2 R_T(\nabla_T X), \nabla_T X\rangle_{|t=0} + 10\langle R_T(R_T(\nabla_T X)), \nabla_T X\rangle_{|t=0} \\
&\quad + 20\langle R_T(\nabla_T X), R_T(\nabla_T X)\rangle_{|t=0} + 2\langle R_T(R_T(\nabla_T X)), \nabla_T X\rangle_{|t=0} \\
&= 36\langle \nabla_\tau^2 R_\tau(\xi), \xi\rangle_0 + 32\langle R_\tau(\xi), R_\tau \xi\rangle_0.
\end{aligned}
\tag{4.1.16}
$$

This implies that

$$
\begin{aligned}
\langle \xi, \xi\rangle_{t\tau} &= t^{-2}|X(c_0(t))|^2 \\
&= \langle \xi, \xi\rangle_0 + \frac{t^2}{3}\langle R_\tau \xi, \xi\rangle_0 + \frac{t^3}{6}\langle \nabla_\tau R_\tau(\xi), \xi\rangle_0 + \frac{t^4}{20}\langle \nabla_\tau^2 R_\tau(\xi), \xi\rangle_0 + \frac{2t^4}{45}\langle R_\tau \xi, R_\tau \xi\rangle_0 + O(t^5).
\end{aligned}
\tag{4.1.17}
$$

Now substituting $x = t\tau$ and $\xi = \frac{\partial}{\partial x^i} \pm \frac{\partial}{\partial x^j}$ we get

$$
\begin{aligned}
g_{ii}(x) + 2g_{ij}(x) + g_{jj}(x) = {} & 2 + 2\delta_{ij} + \frac{1}{3}(R_{aici} + R_{ajci} + R_{aicj} + R_{ajcj})x^a x^c \\
& + \frac{1}{6}(R_{aici,k} + R_{ajci,k} + R_{aicj,k} + R_{ajcj,k})x^a x^c x^k \\
& + \frac{1}{20}(R_{aici,kl} + R_{ajci,kl} + R_{aicj,kl} + R_{ajcj,kl})x^a x^c x^k x^l \\
& + \frac{2}{45}(R_{aicd}R_{limd} + R_{aicd}R_{ljmd} + R_{ajcd}R_{limd} + R_{ajcd}R_{ljmd})x^a x^c x^m x^l + O(r^5)
\end{aligned}
\tag{4.1.18}
$$

and

$$\begin{aligned}
g_{ii}(x) - 2g_{ij}(x) + g_{jj}(x) = 2 - 2\delta_{ij} &+ \frac{1}{3}(R_{aici} - R_{ajci} - R_{aicj} + R_{ajcj})x^a x^c \\
&+ \frac{1}{6}(R_{aici,k} - R_{ajci,k} - R_{aicj,k} + R_{ajcj,k})x^a x^c x^k \\
&+ \frac{1}{20}(R_{aici,kl} - R_{ajci,kl} - R_{aicj,kl} + R_{ajcj,kl})x^a x^c x^k x^l \\
&+ \frac{2}{45}(R_{aicd}R_{limd} - R_{aicd}R_{ljmd} - R_{ajcd}R_{limd} + R_{ajcd}R_{ljmd})x^a x^c x^m x^l + O(r^5),
\end{aligned} \tag{4.1.19}$$

since $\nabla_\tau \tau = 0$ as $c_s(\cdot)$ is a geodesic for every s, and hence $\nabla_\tau R_\tau = \nabla_\tau R(\tau, \cdot)\tau$ and $\nabla_\tau^2 R_\tau = \nabla_\tau^2 R(\tau, \cdot)\tau$. So by subtracting (4.1.19) from (4.1.18) and dividing by 4, we get

$$\begin{aligned}
g_{ij}(x) = \delta_{ij} &+ \frac{1}{6}(R_{ajci} + R_{aicj})x^a x^c + \frac{1}{12}(R_{ajci,k} + R_{aicj,k})x^a x^c x^k \\
&+ \frac{1}{40}(R_{ajci,kl} + R_{aicj,kl})x^a x^c x^k x^l + \frac{1}{45}(R_{aicd}R_{ljmd} + R_{ajcd}R_{limd})x^a x^c x^m x^l + O(r^5),
\end{aligned} \tag{4.1.20}$$

which simplify to

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}R_{aicj}x^a x^c + \frac{1}{6}R_{aicj,k}x^a x^c x^k + \left(\frac{1}{20}R_{aicj,kl} + \frac{2}{45}R_{aicd}R_{ljkd}\right) x^a x^c x^k x^l + O(r^5). \tag{4.1.21}$$

If we define the matrix $A = (A_{ij})$ by setting

$$A_{ij} = \frac{1}{3}R_{aicj}x^a x^c + \frac{1}{6}R_{aicj,k}x^a x^c x^k + \left(\frac{1}{20}R_{aicj,kl} - \frac{1}{90}R_{aicd}R_{ljkd}\right) x^a x^c x^k x^l + O(r^5), \tag{4.1.22}$$

then

$$\begin{aligned}
\exp(A_{ij}) &= I + A + \frac{1}{2}A^2 + O(r^5) \\
&= \delta_{ij} + \frac{1}{3}R_{aicj}x^a x^c + \frac{1}{6}R_{aicj,k}x^a x^c x^k + \left(\frac{1}{20}R_{ac,kl} - \frac{1}{90}R_{aicd}R_{ljkd}\right) x^a x^c x^k x^l \\
&+ \frac{1}{18}R_{aicd}R_{ljkd}x^a x^c x^k x^l + O(r^5) \\
&= (g_{ij}).
\end{aligned} \tag{4.1.23}$$

Then, using the fact that $\det(g_{ij}) = \exp(\mathrm{tr}(A_{ij}))$, we get

$$\begin{aligned}
\det(g_{ij}) &= \exp\left(-\frac{1}{3}R_{ac}x^a x^c - \frac{1}{6}R_{ac,k}x^a x^c x^k - \frac{1}{20}R_{ac,kl} - \frac{1}{90}R_{aicd}R_{ljkd}x^a x^c x^k x^l + O(r^5)\right) \\
&= 1 - \frac{1}{3}R_{ac}x^a x^c - \frac{1}{6}R_{ac,k}x^a x^c x^k - \left(\frac{1}{20}R_{ac,kl} + \frac{1}{90}R_{aicd}R_{likd} - \frac{1}{18}R_{ac}R_{kl}\right) x^a x^c x^k x^l + O(r^5).
\end{aligned} \tag{4.1.24}$$

□

We need an extra technical lemma in order to prove Theorem 4.1.8

Lemma 4.1.17. *Let $f(t) = |X(c(t,0))|^2$ be as in the proof of Lemma* 4.1.16*. Then every term in the Taylor expansion of $\langle \xi, \xi \rangle_{t\tau}$ has the form*

$$c_k t^k \left(\langle \nabla_\tau^{k-2} R_\tau(\xi), \xi \rangle + B_k(\xi, \xi) \right), \tag{4.1.25}$$

for $k \geq 3$, where c_k is a constant and B_k is a bilinear form constructed from R_τ, and its derivatives of order less than $k-2$.

Proof. Start by noting that

$$\nabla_T^k f = \sum_{i=0}^{k} \binom{k}{i} \langle \nabla_T^{k-i} X, \nabla_T^i X \rangle, \tag{4.1.26}$$

which, for $k \geq 3$, we can write as (taking into account that $X = 0$ at $t = 0$),

$$\begin{aligned} \nabla_T^k f &= 2k \langle \nabla_T^{k-1} X, \nabla_T X \rangle + C(X,X) = 2k \langle \nabla_T^{k-3} \mathrm{R}_T(X), \nabla_T X \rangle + C(X,X) \\ &= 2k \langle \nabla_T^{k-4} \mathrm{R}_T(\nabla_T X), \nabla_T X \rangle + C_1(X,X) \end{aligned}$$

where C and C_1 are bilinear forms constructed from R_T and its derivatives of order less than $k-2$, more specifically, C is what remains from (4.1.26) when we subtract $2\langle \nabla_T^{k-1} X, \nabla_T X \rangle$ from $\nabla_T^k f$ and C_1 is what remains when we subtract $2k \langle \nabla_T^{k-4} \mathrm{R}_T(\nabla_T X), \nabla_T X \rangle$ from $\nabla_T^k f$. Now, the result follows when we replace $t = 0$. $\square$

Now we are ready to prove Theorem 4.1.8.

Proof of Theorem 4.1.8*.* We prove that for every $N \geq 2$ there is a conformal metric $\widetilde{g}$ such that $\det(\widetilde{g}_{ij}) = 1 + O(r^N)$ as $r \to 0^+$, where $r = |x|$ in $\widetilde{g}-$normal coordinates at p, by an induction argument. For $N = 2$ we know the result is true in the usual normal coordinates. Now assume by induction that g satisfies $\det(g_{ij}) = 1 + O(r^N)$, $N \geq 2$. By Lemma 4.1.17 each term in the expansion (4.1.17) of $\langle \xi, \xi \rangle_{t\tau}$ is of the form (4.1.25), therefore, using the notation from the proof of Lemma 4.1.16, we have

$$\langle \xi, \xi \rangle_{t\tau} = \langle \xi, \xi \rangle_0 + \sum_{k=1}^{N} 2k t^k \langle \nabla_\tau \mathrm{R}_\tau(\xi), \xi \rangle_0 + t^k B_k(\xi, \xi).$$

Then, proceeding as in the proof of Lemma 4.1.16 (and using the same definitions) we that the components of the matrix $A = (A_{ij})$ have the expansion

$$A_{ij} = \sum_{|K|=0}^{N} \alpha_K \left(R_{aicj,K} + S^k_{aicjK} \right) x^a x^c x^K + O(r^{N+1}),$$

where $k = |K|$, for $k \geq 2$, S_{aicjK} are the components of a symmetric tensor S^k on T_pM constructed from the curvature and its derivatives of order less than $|K| - 2$, and for $k < 2$ we get the explicit formulas for the tensors S^k in (4.1.23). Then, taking the trace of the matrix $A = (A_{ij})$ yields:

$$\mathrm{tr}(A) = \sum_{|K|=0}^{N} \alpha_K \left(R_{ac,K} + \overline{S}^k_{acK} \right) x^a x^c x^K + O(r^{N+1}),$$

where, $\overline{S}^k_{acK} = \sum_{ij} S^k_{aicjK}$. But then the formula $\det(g_{ij}) = 1+O(r^N)$ together with the formula $\det(g_{ij}) = \exp(\operatorname{tr} A)$ we see that

$$\det(g_{ij}) = 1 + \sum_{|K|=n-2} c_N(R_{ij.K} - T_{ijK})x^i x^j x^K + O(r^{N+1}), \tag{4.1.27}$$

where T_{ijK} are the coefficients of a symmetric tensor T on T_pM constructed from the curvature and its derivatives of order less than $N-2$. By Theorem 4.1.9, there is a unique $f \in \mathcal{P}_N$, for which $\operatorname{Sym}(\widetilde{\nabla}\widetilde{R}_{ij}) = T$, where $\widetilde{R}_{ij}$ denotes the Ricci tensor of $\widetilde{g} = e^{2f}g$ and $\widetilde{\nabla}$ the Levi-Civita connection of $\widetilde{g}$. However, the transformation laws for the Ricci tensor (A.0.17) show that $T = \widetilde{T}$ (where $\widetilde{T}$ is constructed in the same way as T) when $f \in \mathcal{P}_N$, so $\det(g_{ij})$ vanishes up to order $N+1$ in $\widetilde{g}-$normal coordinates, hence $\det((g_{ij})) = 1 + O(r^{N+1})$.

Now we prove the second part of the theorem. Assume that $N \geq 5$, and (for simplicity) replace $\widetilde{g}$ with g. The condition $\det(g_{ij}) = 1+O(r^N)$ implies that the symmetrization of the coefficients of (4.1.9) vanishes at p:

$$R_{ac} = 0, \tag{4.1.28}$$

$$R_{ac,k} + R_{ck,a} + R_{ka,c} = 0, \tag{4.1.29}$$

$$\operatorname{Sym}(R_{ac,kl} + \frac{2}{9}R_{aicd}R_{likd}) = 0. \tag{4.1.30}$$

Then $R_{ijkl} = W_{ijkl}$ by (4.1.28) and

$$R_{ac,kl} - R_{ac,lk} = R^m_{kla}R_{mc} + R^m_{klc}R_{am} = 0,$$

by (A.0.10). So (4.1.30) gives

$$\begin{aligned}
0 &= (R_{ac,kl} + R_{kl,ac} + R_{al,ck} + R_{ck,al} + R_{ak,cl} + R_{cl,ak})x^i x^j \\
&+ \frac{2}{9}(W_{aicd}W_{likd} + W_{aicd}W_{likd} + W_{aikd}W_{cild} + W_{aikd}W_{licd} + W_{aild}W_{cikd} + W_{aild}W_{kicd})x^a x^c \\
&= (R_{ac,kl} + R_{kl,ac} + 2R_{al,ck} + 2R_{ck,al})x^a x^c \\
&+ \frac{2}{9}(W_{aicd}W_{likd} + W_{aicd}W_{kild} + W_{aikd}W_{cild} + W_{aikd}W_{licd} + W_{aild}W_{cikd} + W_{aild}W_{kicd})x^a x^c.
\end{aligned}$$

Now contract on k, l, noting the contracted Bianchi identity A.0.13 to obtain

$$\begin{aligned}
0 &= (3S_{,ac} + R_{ac,kk}+)x^a x^c \\
&+ \frac{2}{9}(W_{aicd}W_{likd} + W_{aicd}W_{likd} + W_{aikd}W_{cild} + W_{aikd}W_{licd} + W_{aild}W_{cikd} + W_{aild}W_{kicd})x^a x^c.
\end{aligned} \tag{4.1.31}$$

The first Bianchi identity together with the fact that we are summing in the indices k, p, m yield $W_{aikd}W_{kicd} = \frac{1}{2}W_{aikd}(W_{kicd} - W_{dick}) = \frac{1}{2}W_{aikd}W_{cikd}$, so by the symmetries of the Weyl tensor:

$$0 = (3S_{,ac} + R_{ij,kk} + \frac{2}{3}W_{aikd}W_{cikd})x^a x^c + \frac{2}{9}(2W_{aicd}W_{kikd})x^a x^c, \tag{4.1.32}$$

but $\sum_k W_{kikd} = -R_{id} = 0$, and so (4.1.29) becomes

$$0 = (3S_{,ac} + R_{ij,kk} + \frac{2}{3} W_{aikd} W_{cikd}) x^a x^c. \tag{4.1.33}$$

Now contracting on i, j implies $\Delta_g S = -S_{,jj} = \frac{1}{6}|W|^2$ at p.
Finally, $S(p) = R_{jj}(p) = 0$ by (4.1.28) and $0 = (2R_{jk,k} + R_{kk,j})(p) = 2S_{,j}(p)$ by (4.1.29) and the Bianchi identity, so $S = O(r^2)$. $\square$

Remark 4.1.18. *Looking at the proofs of the lemmas that were used in the proof of Theorem* 4.1.8*, we can not only conclude that for every $N \in \mathbb{N}$ there are conformal normal coordinates, in which $\det(g_{ij}) = 1 + O(r^N)$ where $r = |x|$, but we also have $\partial_i \det(g_{ij}) = O(r^{N-1})$. One may wonder whether or not around every point $p \in M$ there are local coordinates $\{x^i\}$ centred around p in which $\det(g_{ij}) = 1$. Indeed, this is the case. It was pointed out to me by T.H. Parker in a private communication that in 1991 Jianguo Cao published a very interesting paper titled "The Existence of Generalized Isothermal Coordinates for Higher Dimensional Riemannian Manifolds" [6], where the author shows the existence of coordinates around any given point in M under which $\det(g_{ij}) = 1$. The methods used by the Cao are of a completely different nature to the ones we have exposed.*

Using the fact that we have $\det(g_{ij}) = 1 + O(r^N)$ in conformal normal coordinates, we can prove that $\lambda(M) < \lambda(\mathbb{S}^n)$ for a compact Riemannian manifold (M, g) of dimension $n \geq 6$ that is not locally conformally flat. This result was originally proved by Aubin in [2]. However, Lee and Parker in [13], showed the same result using the results above, which makes the proof a lot more straightforward than Aubin's original proof.

Definition 4.1.19. *A Riemannian manifold (M, g) is said to be locally conformally flat if it is locally conformal to the Euclidean space, $\mathbb{R}^n$, equipped with the Euclidean metric.*

Theorem 4.1.20 (Aubin)**.** *Let (M, g) be a compact, connected Riemannian manifold of dimension $n \geq 6$ that is not locally conformally flat then $\lambda(M) < \lambda(\mathbb{S}^n)$.*

Proof. Since M is not locally conformally flat, then by the Weyl-Schouten Theorem, there is $p \in M$ such that the Weyl tensor at p is nonzero, $W(p) \neq 0$. Indeed, this is the key to proving the theorem since it allows for a better estimation of the term regarding the scalar curvature in the energy functional.

Let $p \in M$ such that $W(p) \neq 0$ and let $\{x^i\}$ be conformal normal coordinates in a neighbourhood of $p \in M$. For $\varepsilon > 0$, let B_ε denote the ball of radious ε in $\mathbb{R}^n$, and η a smooth radial cutoff function, $0 \leq \eta \leq 1$, supported in $B_{2\varepsilon}$ with $\eta \equiv 1$ on B_ε, and let $\varphi_\alpha = \eta u_\alpha$, where $u_\alpha(x) = \left(\frac{|x|^2 + \alpha^2}{\alpha}\right)^{(2-n)/2}$, for $x \in \mathbb{R}^n$. Now take $\varepsilon > 0$ small enough so that $B_{2\varepsilon}$ is contained in the domain of the conformal normal coordinates. Also, assume that in the selected coordinates, we have $dV_g = (1 + O(r^N))dx$ with $N \in \mathbb{N}$ large. The estimates of Lemma 2.0.2 are modified as follows:

$$\begin{aligned} E(\varphi_\alpha) &= \int_{B_{2\varepsilon}} a|\nabla \varphi_\alpha|^2 + S\varphi_\alpha^2 dV_g = (1 + C\varepsilon^N) \int_{B_{2\varepsilon}} a|\nabla \varphi_\alpha|^2 + S\varphi_\alpha^2 dx + O(\alpha^{n-2}) \\ &\leq (1 + C\varepsilon^N) \left(\lambda(\mathbb{S}^n) \|\varphi_\alpha\|^2_{L^{2^*}(\mathbb{R}^n)} + C\alpha^{n-2} + \int_{B_{2\varepsilon}} S\varphi_\alpha^2 dx \right), \end{aligned}$$

for some positive constant C. Setting $A_\varepsilon = B_{2\varepsilon} - B_\varepsilon$ and recalling that in the conformal normal coordinates $S = O(r^2)$ and $\Delta_g S(p) = \frac{1}{6}|W(p)|^2$, so

$$
\begin{aligned}
\int_{B_{2\varepsilon}} S\varphi_\alpha^2 dx &\leq \int_{B_\varepsilon} S u_\alpha^2 dx + \int_{A_\varepsilon} u_\alpha^2 dx \\
&= \int_0^\varepsilon \int_{\partial B_r} \frac{1}{2}(S_{,ij}(p)x^i x^j + O(r^3)) u_\alpha^2 d\omega_r dr \\
&= \int_0^\varepsilon \int_{S_n} \frac{1}{2}\left(r^2 S_{,ij}(p)x^i x^j + O(r^3)\right) u_\alpha^2 r^{n-1} d\omega dr + O(\alpha^{n-2}) \\
&= \int_0^\varepsilon \frac{1}{2}\left(r^2 S_{,ij}(p)\delta_{ij} + O(r^3)\right) u_\alpha^2 r^{n-1} dr + O(\alpha^{n-2}) \\
&= \int_0^\varepsilon \left(-C_1 r^2 |W(p)|^2 + O(r^3)\right) u_\alpha^2 r^{n-1} dr + O(r^{n-2}),
\end{aligned}
$$

where $C_1 > 0$ is some positive constant. Note that to obtain the last equality we used the fact that $\int_{S_n} x^i x^j d\omega = \delta_{ij}$. So Lemma 2.1.4 shows that

$$
E(\varphi_\alpha) \leq (1 + C\varepsilon^N) \begin{cases} \lambda(\mathbb{S}^n)\|\varphi_\alpha\|^2_{L^{2^*}(\mathbb{R}^n)} - C_1|W(p)|^2\alpha^4 + o(\alpha^4) & \text{if } n > 6 \\ \lambda(\mathbb{S}^n)\|\varphi_\alpha\|^2_{L^{2^*}(\mathbb{R}^n)} - C_1|W(p)|^2 \log(1/\alpha)\alpha^4 + O(\alpha^4) & \text{if } n = 6 \end{cases}
$$

Since, $\|\varphi_\alpha\|^2_{L^{2^*}(M)} \geq (1 - C\varepsilon^N)^{1/2^*}\|\varphi_\alpha\|_{L^{2^*}(\mathbb{R}^n)}$, dividing $E(\varphi_\alpha)$ by $\|\varphi_\alpha\|^2_{L^{2^*}(M)}$ we obtain

$$
\mathcal{Q}_g(\varphi_\alpha) \leq \begin{cases} \frac{1+C\varepsilon^N}{(1-C\varepsilon^N)^{1/2^*}}\lambda(\mathbb{S}^n) - \|\varphi_\alpha\|^{-2}_{L^{2^*}(M)}(C_1|W(p)|^2\alpha^4 - o(\alpha^4)) & \text{if } n > 6 \\ \frac{1+C\varepsilon^N}{(1-C\varepsilon^N)^{1/2^*}}\lambda(\mathbb{S}^n) - \|\varphi_\alpha\|^{-2}_{L^{2^*}(M)}(C_1|W(p)|^2 \log(1/\alpha)\alpha^4 - O(\alpha^4)) & \text{if } n = 6. \end{cases} \tag{4.1.34}
$$

Now chose $\alpha > 0$ sufficiently small so that

$$
-C_1|W(p)|^2\alpha^4 + o(\alpha^4) < 0
$$

and

$$
-C_1|W(p)|^2 \log(1/\alpha)\alpha^4 + O(\alpha^4)) < 0.
$$

Since $\frac{1+C\varepsilon^N}{(1-C\varepsilon^N)^{1/2^*}} \leq 1 + C_2\varepsilon^N$ (for some positive constant C_2) we are able to conclude that

$$
\mathcal{Q}_g(\varphi_\alpha) \leq \begin{cases} \lambda(\mathbb{S}^n) + C_2\lambda(\mathbb{S}^n)\varepsilon^N - \|\varphi_\alpha\|^{-2}_{L^{2^*}(M)}(C_1|W(p)|^2\alpha^4 - o(\alpha^4)) & \text{if } n > 6 \\ \lambda(\mathbb{S}^n) + C_2\varepsilon^N\lambda(\mathbb{S}^n) - \|\varphi_\alpha\|^{-2}_{L^{2^*}(M)}(C_1|W(p)|^2 \log(1/\alpha)\alpha^4 - O(\alpha^4)) & \text{if } n = 6. \end{cases} \tag{4.1.35}
$$

Thus, by choosing N large enough, we see that

$$
\mathcal{Q}_g(\varphi_\alpha) < \lambda(\mathbb{S}^n).
$$

□

The restrictions of this approach to dimensions greater or equal to 6 arise from applying Lemma 2.1.4 with $k = 2$.

4.2 Stereographic Projections

We have seen that when M is a not locally conformally flat compact Riemannian manifold with dimension $\dim M \geq 6$, $\lambda(M) < \lambda(\mathbb{S}^n)$. Hence the results in Chapter 3 show that there is a solution to the Yamabe Problem. But what about the remaining cases? Can we show that $\lambda(M) < \lambda(\mathbb{S}^n)$ when (M, g) is not conformal to the standard sphere, $\mathbb{S}^n$? The answer is, yes we can. But to get there we need to introduce the concept of a stereographic projection of a compact Riemannian manifold. For this, we will need the Green function of the operator $\mathcal{L}_g$. We start with an existence result. The exposition in this section follows closely section 6 from [13].

Theorem 4.2.1 (Existence of Green's Function)**.** *Suppose (M, g) is a compact Riemannian manifold of dimension $n \geq 3$, and let h be a strictly positive smooth function on M. For each $p \in M$, there is a unique smooth function Γ_p on $M \setminus \{p\}$, called the Green function for $\Delta_g + h$ at p, such that $(\Delta_g + h)\Gamma_p = \delta_p$, in the distributional sense, where δ_p is the Dirac mass at p.*

Proof. See Appendix A of [7]. $\square$

Lemma 4.2.2. *Let (M, g) be a compact Riemannian manifold, such that, $\lambda(M) > 0$. Then, at each point $p \in M$, the Green function Γ_p for $\mathcal{L}_g$ exists and is strictly positive.*

Proof. Fix $l < 2^*$, and let u be the smooth positive solution to the subcritical equation $\mathcal{L}_g u = \lambda_l u^{l-1}$, given by Theorem 3.1.1, and define the metric $g' = u^{2^*-2}g$. The scalar curvature of this new metric is $S' = u^{1-2^*}\mathcal{L}_g u = \lambda_l(M)u^{1-2^*-1+l} = \lambda_l(M)u^{l-2^*}$ (by the transformation law (A.0.19)). Since $\lambda(M) > 0$, arguing as in Lemma 3.1.2, we see that $\lambda_l(M) > 0$, and consequently, $S' > 0$. So, by Theorem 4.2.1 the Green function Γ'_p for $\mathcal{L}_{g'}$ exists. If Γ'_p is nonpositive at some point, by the Strong Maximum Principle Γ'_p is constant, which is false. If Γ'_p was constant, then $\mathcal{L}_g \Gamma'_p = S'\Gamma'_p \in L^2(M)$ which implies that the delta Dirac distribution is in $L^2(M)$, but this is impossible. Now set $\Gamma_p(x) = u(p)u(x)\Gamma'_p(x)$, which is strictly positive by construction. Thus, for any $f \in \mathcal{C}^\infty(M)$ we have due to $\mathcal{L}_{g'}(u^{-1}f) = u^{1-2^*}\mathcal{L}_g f$ and Theorem 4.2.1:

$$
\begin{aligned}
u^{-1}(p)f(p) &= \int_M \Gamma'_p(x)\mathcal{L}_{g'}(u^{-1}(x)f(x))dV_{g'} = \int_M u^{-1}(p)u^{-1}(x)\Gamma_p(x)(u^{1-2^*}(x)\mathcal{L}_g(f(x)))u^{2^*}(x)dV_g \\
&= u^{-1}(p)\int_M \Gamma_p(x)\mathcal{L}_g(f(x))dV_g(x),
\end{aligned}
$$

which implies that $\mathcal{L}_g\Gamma_p(x) = \delta_p$, in the distributional sense, i.e., Γ_p is the Green function for $\mathcal{L}_g$. $\square$

The results in Chapter 3 already give the solution of the Yamabe problem whenever $\lambda(M) \leq 0$ (recall that $\lambda(\mathbb{S}^n) > 0$). Therefore, from now on we assume that $\lambda(M) > 0$.

In Chapter 2, we used the standard stereographic projection from $\mathbb{S}^n \setminus \{P\}$ (where P is the north pole) onto $\mathbb{R}^n$ to transport the Yamabe problem in the standard sphere to a problem in $\mathbb{R}^n$. And Schoen in [16], to prove that $\lambda(M) < \lambda(\mathbb{S}^n)$, when (M,g) is a compact connected Riemannian manifold of dimension $n = 4,5$ not conformal to the standard sphere, had the idea to consider the metric $G^{2^*-2}g$ on $M \setminus \{p\}$ where G is a constant multiple of the Green function of $\mathcal{L}_g$ at p. This made the problem simpler. Inspired by this, Lee and Parker in [13], were able to use conformal normal coordinates to prove that $\lambda(M) < \lambda(\mathbb{S}^n)$, whenever (M,g) is either locally conformally flat or has dimension $\dim M = 3,4,5$, and is not conformal to the standard sphere. First, they introduce the generalized notion of stereographic projection.

Definition 4.2.3. *Let (M,g) be a compact, connected Riemannian manifold with $\lambda(M) > 0$. For $p \in M$ define the metric $\widehat{g} = G^{2^*-2}g$ on $\widehat{M} = M \setminus \{p\}$, where*

$$G = (n-2)a\omega_{n-1}\Gamma_p, \tag{4.2.1}$$

where ω_{n-1} denotes the measure of the standard sphere $\S^{n-1}$. The manifold $(\widehat{M}, \widehat{g})$ together with the natural map $\sigma : M \setminus \{p\} \to \widehat{M}$ is called the stereographic projection of M from p.

Note that if in the above definition, M is the standard sphere and p is the north pole then the stereographic projection coincides with the one used in Chapter 2.
Another concept that we will need going forward is that of an asymptotically flat manifold, which we now introduce.

Definition 4.2.4 (Lee, Parker [13])**.** *A Riemannian manifold $(N,g)$is said to be asymptotically flat of order $\tau > 0$ if there exists a decomposition $N = N_0 \cup N_\infty$, with N_0 compact, and a diffeomorphism $N_\infty \to \mathbb{R}^n \setminus B_R(0)$ for some $R > 0$, satisfying*

$$g_{ij} = \delta_{ij} + O(\rho^{-\tau}), \quad \partial_k g_{ij} = O(\rho^{-\tau-1}), \quad \partial_k\partial_l g_{ij} = O(\rho^{-\tau-2}), \tag{4.2.2}$$

as $\rho = |z| \to \infty$, where $\{z^i\}$ are the coordinates induced by the diffeomorphism on N_∞. The coordinates $\{z^i\}$, will be henceforth called asymptotic coordinates.

Before proceeding, a word on notation is necessary.

Notation 4.2.5. *When we write $f = O'(r^k)$, we mean $f = O(r^k)$ and $\nabla f = O(r^{k-1})$. O'' is defined in the same way. Denote the set of smooth functions that vanish up to order k at p by $\mathcal{C}_k$. And denote the set of homogeneous polynomials of degree k by $\mathcal{P}_k$,*

An instrumental tool in [16] and [13] was an asymptotic expansion of the function G around p.

Lemma 4.2.6. *Let G be given by* (4.2.1)*. Let $\{x^i\}$ be conformal normal coordinates at p, such that $\det(g_{ij}) = 1 + O(r^N)$ with $N \geq 2\dim M$, where $r = |x|$ is the geodesical distance. Then, in these coordinates, G has an asymptotic expansion in terms of geodesic distance r from p:*

$$G(x) = r^{2-n}\left(1 + \sum_{k=4}^{n} \psi_k\right) + c\log(r) + O''(1), \tag{4.2.3}$$

$\psi_k \in \mathcal{P}_k$, $c \in \mathbb{R}$ and the $\log$ term appears only if n is even. The leading terms are:

(h_1) if $n = 3, 4, 5$, or M is conformally flat in a neighbourhood of p,

$$G = r^{2-n} + A + O''(1), \quad (A = \textit{constant});$$

(h_2) if $n = 6$,

$$G = r^{2-n} - \frac{1}{288a}|W(p)|^2 \log(r) + O''(1);$$

(h_3) if $n \geq 7$,

$$G = r^{2-n}\left[1 + \frac{1}{12a(n-4)}\left(\frac{r^4}{12(n-6)}|W(p)|^2 - S_{,ij}(p)x^i x^j r^2\right)\right] + O''(r^{7-n}).$$

Proof. Let φ be a radially symmetric function. Since in polar conformal coordinates $g^{rr} = 1$, when we apply the Laplace-Beltrami operator Δ_g to radially symmetric functions we have, since $g^{ri} = 0$ for $i \neq r$,

$$\begin{aligned}\Delta_g \varphi &= -\frac{1}{\sqrt{\det(\widetilde{g}_{ij})}}\partial_i\left(g^{ir}\sqrt{\det(\widetilde{g}_{ij})}\partial_r\varphi\right) = -\frac{1}{r^{n-1}\sqrt{\det(g_{ij})}}\partial_i\left(g^{ir}r^{n-1}\sqrt{\det(g_{ij})}\partial_r\varphi\right)\\ &= -\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\varphi\right) - \frac{\partial_r\varphi\partial_r\left(\sqrt{\det(g_{ij})}\right)}{\sqrt{\det(g_{ij})}} = \Delta_0\varphi - \frac{\partial_r\varphi\partial_r\left(\sqrt{\det(g_{ij})}\right)}{\sqrt{\det(g_{ij})}},\end{aligned}$$

where $\widetilde{g}$ is the metric g written in conformal polar coordinates. Now write $G = r^{2-n}(1+\psi)$. Using the fact that, in the distributional sense $\Delta_0 r^{2-n} = (n-2)\omega_{n-1}\delta_p$ on U (an open neighbourhood of 0 in $\mathbb{R}^n$ containing the origin), the equation $\mathcal{L}_g G = (n-2)\omega_{n-1}\delta_p$ becomes

$$\mathcal{L}_g(r^{2-n}\psi) + Sr^{2-n} - \frac{\partial_r(r^{2-n})\partial_r\left(\sqrt{\det(g_{ij})}\right)}{\sqrt{\det(g_{ij})}} = 0. \tag{4.2.4}$$

By setting $\alpha = \frac{\partial_r(r^{2-n})\partial_r\left(\sqrt{\det(g_{ij})}\right)}{\sqrt{\det(g_{ij})}}$ we get

$$\begin{aligned}\mathcal{L}_g(r^{2-n}\psi) + Sr^{2-n} - \alpha &= a\Delta_g(r^{2-n}\psi) + r^{2-n}(S\psi + S) - \alpha\\ &= -\frac{\partial_i\left(g^{ij}\sqrt{\det(g_{ij})}\partial_j(r^{2-n}\psi)\right)}{\sqrt{\det(g_{ij})}} + r^{2-n}(S\psi + S) - \alpha\\ &= -\partial_i\left(g^{ij}\partial_j(r^{2-n}\psi)\right) + r^{2-n}(S\psi + S) + \widetilde{K}(\psi),\end{aligned}$$

where $\widetilde{K}(\psi) = -\alpha - \frac{g^{ij}\partial_j(r^{2-n}\psi)\partial_i(\det(g_{ij}))}{\det(g_{ij})}$. Now by setting $K(\psi) = \partial_i((\delta_{ij} - g^{ij})\partial_j\psi)$,and multiplying equation (4.2.4) by r^n/a, setting $L = r^2\Delta_0 + 2(n-2)r\partial_r$, and assuming ψ to be $\mathcal{C}^2$, equation (4.2.4) is equivalent to

$$L\psi + \frac{r^2}{a}\left(S + S\psi + aK(\psi) + r^{n-2}\widetilde{K}(\psi)\right) = 0. \tag{4.2.5}$$

We begin by computing a formal asymptotic solution to this equation, by writing $\psi = \psi_1 + \psi_2 + \cdots + \psi_n$, with $\psi_k \in \mathcal{P}_k$, and finding each ψ_k with an induction argument. By assumption, we know, by Theorem

4.1.8 that $\det(g_{ij}) = 1+O(r^N)$ and $\partial_i \det(g_{ij}) = O(r^{N-1})$ (see Remark 4.1.18). This, in turn, implies that $r^{n-2}\widetilde{K}(\psi) = O(r^{n+1})$. Since $S = O(r^2)$ in the coordinates $\{x^i\}$, and the operators L and K preserve the degree of homogeneous polynomials, we start by setting $\psi_1 = \psi_2 = \psi_3 = 0$. This is due to the fact that for $k = 1, 2, 3$, we have

$$\frac{r^2}{a}\left(S + S\psi_k + aK(\psi_k) + r^{n-2}\widetilde{K}(\psi_k)\right) = O(r^4),$$

and $L\psi_k = O(r^k)$.

Now, suppose by induction that we have found $\overline{\psi} = \psi_1 + \cdots + \psi_{k-1}$, with $\psi_1 = \psi_2 = \psi_3 = 0$, such that

$$L\overline{\psi} + \frac{r^2}{a}\left(S + S\overline{\psi} + aK\overline{\psi} + r^{n-2}\widetilde{K}\psi\right) \in \mathcal{C}_k. \tag{4.2.6}$$

Write $L\overline{\psi} + \frac{r^2}{a}\left(S + S\overline{\psi} + aK\overline{\psi} + r^{n-2}\widetilde{K}\psi\right)$ as $b_k + \phi$, for some $b_k \in \mathcal{P}_k$ and $\phi \in \mathcal{C}_{k+1}$. Now assume that we can find $\psi_k \in \mathcal{P}_k$ such that

$$L\psi_k + b_k = 0,$$

then by noting that $r^2 S\psi_k \in \mathcal{C}_{k+4}$, $r^2 K(\psi_k) \in \mathcal{C}_{k+2}$ and the error term is of order $O(r^{n+1})$, when we replace $\overline{\psi}$ by $\psi_1 + \cdots + \psi_{k-1} + \psi_k$ (4.2.6) is satisfied when k is replaced by $k+1$, thus completing the induction step. Hence to complete the proof we need to solve the equation $L\psi_k + b_k = 0$. First, consider the case when $n = \dim(M)$ is odd. By Euler's formula, $L = r^2\Delta_0 + 2k(n-2)$ on $\mathcal{P}_k$, so Lemma 4.1.10 shows that L is always invertible on $\mathcal{P}_k$, so we can simply take $\psi_k = -L^{-1}(b_k)$. By induction there exists $\overline{\psi} = \psi_1 + \cdots + \psi_n$ such that (4.2.6) holds with $k = n+1$, which is equivalent to

$$\mathcal{L}_g\left(r^{2-n}\overline{\psi}\right) + Sr^{2-n} \in r^{-n}\mathcal{C}_{n+1} \tag{4.2.7}$$

When n is even this method works for $k < n-2$, but then breaks down when $k \geq n-2$ since L is no longer invertible in these cases. However, by noting that L is self-adjoint on $\mathcal{P}_k$ with respect to the Euclidean product $\langle \sum_I a_I x^I, \sum_J b_J x^J \rangle = \sum_I a_I b_I$ on $\mathcal{P}_k$, then $\mathcal{P}_k = \mathrm{Im}(L) \oplus \ker(L)$. For $k < n-2$ we proceed as in the odd-dimensional case. When $k \geq n-2$ and $\ker(L) \neq \{0\}$, we will try a function of the form $\psi_k = p_k + q_k \log(r)$, with $p_k, q_k \in \mathcal{P}_k$. The image via L of a function like this is

$$\begin{aligned} L\psi_k &= Lp_k + L(q_k)\log(r) + r^2 q_k \Delta_0(\log(r)) - 2r^2\partial_i(q_k)\partial_i(\log(r)) + 2(n-2)r\partial_r(\log(r))q_k \\ &= Lp_k + Lq_k\log(r) + (n-2-2k)q_k. \end{aligned}$$

Thus, if $k \geq n-2$ we can solve $L\psi_k + b_k = 0$, with $b_k \in \mathcal{P}_k$, by writing $-b_k = Lp_k + q_k$, with $q_k \in \ker(L)$ (the uniqueness of p_k and q_k is given by the decomposition $\mathcal{P}_k = \mathrm{Im}(L) \oplus \ker(L)$) and setting

$$\psi_k = p_k + (n-2-2k)^{-1}q_k\log(r).$$

In fact, using Lemma 4.1.10 again we know that $\ker(L)$ is spanned by r^{n-2}, thus there exists $c \in \mathbb{R}$ and $p_{n-2} \in \mathcal{P}_{n-2}$ such that $\psi_{n-2} = p_{n-2} + cr^{n-2}\log(r)$ satisfies $L\psi_{n-2} + b_{n-2} = 0$. Before proceeding with the remaining cases we should verify whether or not ψ_{n-2} introduces any logarithmic error terms on the right-hand side of (4.2.6). Indeed if ϕ is a radially symmetric function the definition of K and the expansion (4.1.21) of the components of the metric yield

$$K\phi = \partial_i \left(\left(\frac{1}{3} R_{kilj} x^k x^l + O(r^3) \right) \frac{x^j}{r} \partial_r \phi \right),$$

but $R_{kilj} x^k x^l x^j = 0$, by the symmetries of the Riemann curvature tensor, so if we take ψ_{n-2} as above we have $K\psi_{n-2} = Kp_{n-2} + cK(r^{n-2}\log(r)) \in \mathcal{C}_{n-2} + \mathcal{C}_{n-1}\log(r)$, since $Kp_{n-2} = O(r^{n-2})$ (i.e. $Kp_{n-2} \in \mathcal{C}_{n-2}$) and

$$K(r^{n-2}\log(r)) = \partial_i \left(\left(\delta_{ij} - g^{ij}\right) \frac{x^j}{r} \left((n-2)r^{n-3}\log(r) + r^{n-3}\right) \right) \in \mathcal{C}_{n-2} + \mathcal{C}_{n-1}\log(r).$$

So writing $\overline{\psi} = \psi_1 + \cdots \psi_{n-2}$, we get $L\overline{\psi} + \frac{r^2}{a}\left(S + S\overline{\psi} + aK\overline{\psi} + \widetilde{K}\overline{\psi}\right) \in \mathcal{C}_{n-1} + \mathcal{C}_{n+1}\log(r)$, and so ψ_{n-2} does not introduce any logarithmic errors in the right-hand side of (4.2.6).
Now we proceed with the remaining cases. By writing

$$L\overline{\psi} + \frac{r^2}{a}\left(S + S\overline{\psi} + aK\overline{\psi} + \widetilde{K}\overline{\psi}\right) = b_k + \phi_{k+1} + \phi_{n+1}\log(r),$$

where $\phi_{k+1} \in \mathcal{C}_{k+1}$ and $\phi_{n+1} \in \mathcal{C}_{n+1}$, we can solve as before for $\psi_k \in \mathcal{P}_k + \log(r)\mathcal{P}_k$, $k = n-1, n$. We end up with $\overline{\psi} = \psi_1 + \cdots + \psi_n$ such that

$$\mathcal{L}_g\left(r^{2-n}\overline{\psi}\right) + Sr^{2-n} \in r^{-n}\mathcal{C}_{n+1} + r^{-n}\log(r)\mathcal{C}_{n+1}. \tag{4.2.8}$$

This gives the desired homogeneous polynomials in the expansion (4.2.3) of G. Now for any n, write $\psi = \overline{\psi} + \varphi$. Using (4.2.4) and (4.2.7) or (4.2.8) we find

$$\mathcal{L}_g(r^{2-n}\varphi) = -\mathcal{L}_g\left(r^{2-n}\overline{\psi}\right) - Sr^{2-n} \in \mathcal{C}^{\alpha}.$$

Therefore, by elliptic regularity theory $r^{2-n}\varphi \in \mathcal{C}^{2,\alpha}$. This shows (4.2.3).
Now it only remains to check the desired expansions of G.
If M is conformally flat near p, then $S = 0$ near p, so (4.2.4) shows that $r^{2-n}\varphi$ is harmonic since $\overline{\psi} = 0$ and hence $\mathcal{C}^{\infty}$. Thus $G = r^{2-n} + A + O''(r)$ for some constant A in the case that M is conformally flat near p. If $n = 3, 4, 5$ the same result hold since then $r^{2-n}\varphi \in \mathcal{C}^{2,\alpha}$ and $r^{2-n}\overline{\psi} = O''(r)$.
Finally, if $n \geq 6$ we need to confirm the expressions for ψ_4. To that end, expanding S in its Taylor series at p, the above proof shows that ψ_4 is such that

$$L\psi_4 = -\frac{1}{2a} r^2 S_{kl}(p) x^k x^l. \tag{4.2.9}$$

When $n > 6$ we look for solutions of the form $\psi_4 = b_{kl} r^2 x^k x^l$. We have

$$Lr^4 = r^2\Delta_0(r^4) + 8(n-2)r^4 = 2r^4\Delta_0(r^2) - 2r^2\nabla_0(r^2)\cdot\nabla_0(r^2) = 4(n-6)r^4,$$

and

$$\begin{aligned}L(b_{kl}x^kx^lr^2) &= b_{kl}\left(r^2(\Delta_0(r^2)x^kx^l + r^2\Delta_0(x^kx^l) - 8x^kx^l) + 8(n-2)x^kx^lr^2\right) \\ &= b_{kl}(p)\left(-2(n+4)r^2x^kx^l - 2\delta_{kl}r^2\right) + 8(n-2)b_{kl}x^kx^lr^2 \\ &= 6(n-4)b_{kl}(p)x^kx^lr^2 - 2b_{kk}r^2.\end{aligned}$$

So if we take

$$\psi_4 = \frac{1}{12a(n+4)}\left(\frac{|W(p)|^2r^4}{12(n-6)} - S_{,kl}(p)x^kx^lr^2\right),$$

we have by linearity

$$\begin{aligned}L\psi_4 &= \frac{1}{12a(n+4)}\left(\frac{|W(p)|^2L(r^4)}{12(n-6)} - L(S_{,kl}(p)x^kx^lr^2)\right) \\ &= \frac{1}{12a(n+4)}\left(\frac{4|W(p)|^2(n-6)r^4}{12(n-6)} - 6(n-4)S_{,kl}(p)(p)x^kx^lr^2 + 2S_{,kk}(p)r^2)\right) \\ &= -\frac{1}{2a}r^2S_{,kl}(p)x^kx^l.\end{aligned}$$

On the other hand, when $n = 6$ we need to be a bit more creative and look for solutions of the form $\psi_4 = b_{kl}r^2x^kx^l + c_{kl}r^2x^kx^l\log(r)$. Direct computations show that

$$\begin{aligned}L(c_{kl}r^2x^kx^l\log(r)) &= L(c_{kl}r^2x^kx^l)\log(r) + (n-2-8)c_{kl}r^2x^kx^l \\ &= 6(n-4)c_{kl}(p)x^kx^lr^2\log(r) - 2c_{kk}r^2\log(r) - 4c_{kl}r^2x^kx^l.\end{aligned}$$

So, using the calculations above, by taking $\psi_4 = -\frac{1}{24a}\left(S_{,kl}(p)x^kx^lr^2 + \frac{|W(p)|^2r^2}{12}\log(r)\right)$ we find

$$\begin{aligned}L\psi_4 &= -\frac{1}{24a}\left(L(S_{,kl}r^2x^kx^l) + \frac{|W(p)|^2}{12}L(r^4\log(r))\right) \\ &= -\frac{1}{24a}\left(6(n-4)S_{,kl}(p)x^kx^lr^2 - 2S_{,kk}(p)r^2 + \frac{|W(p)|^2}{12}\left(6(n-4)r^4\log(r) - 2nr^4\log(r) - 4r^4\right)\right) \\ &= -\frac{1}{2a}r^2S_{,kl}(p)x^kx^l.\end{aligned}$$

Thus the Lemma is proven. □

Using this lemma, we are able to prove that $(\widehat{M}, \widehat{g})$ is asymptotically flat. To see this, let $\{x^i\}$ be conformal normal coordinates as in the previous lemma and U the corresponding neighbourhood of p in M. Now, define the inverted conformal normal coordinates $z^i = \frac{x^i}{r^2}$ on $U \setminus \{p\}$. By setting $\rho = |z| = r^{-1}$ we have

$$\frac{\partial}{\partial z^i} = \frac{\partial x^j}{\partial z^i}\frac{\partial}{\partial x^j} = \rho^{-2}\left(\delta_{ij} - 2\rho^{-2}z^iz^j\right)\frac{\partial}{\partial x^j}.$$

So, writing $\gamma = r^{n-2}G$ and remembering expansion (4.1.21) of the components of the metric components

g_{ij}, the components of $\widehat{g}$ in $z-$coordinates are

$$
\begin{aligned}
\widehat{g}_{ij}(z) &= \gamma^{n-2}\rho^4 g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) \\
&= \gamma^{2^*-2}\left(\delta_{ik} - 2\rho^{-2}z^i z^k\right)\left(\delta_{jl} - 2\rho^{-2}z^j z^l\right) g_{kl}(\rho^{-2}z) \\
&= \gamma^{2^*-2}\left(\delta_{ik} - 2\rho^{-2}z^i z^k\right)\left(\delta_{jl} - 2\rho^{-2}z^j z^l\right)\left(\delta_{kl} + O''(\rho^{-2})\right) \\
&= \gamma^{2^*-2}\left(\delta_{ik}\delta_{jl} - 2\delta_{ik}\rho^{-2}z^j z^l - 2\delta_{jl}\rho^{-2}z^i z^k + 4\delta_{ik}\delta_{jl}\rho^{-4}z^i z^k z^j z^l\right)\left(\delta_{kl} + O''(\rho^{-2})\right) \\
&= \gamma^{2^*-2}\left(\delta_{ik}\delta_{jl}\delta_{kl} + O''(\rho^{-2})\right) \\
&= \gamma^{2^*-2}\left(\delta_{ij} + O''(\rho^{-2})\right).
\end{aligned}
\tag{4.2.10}
$$

In particular, if (M, g) is conformally flat near p, then we can choose conformal normal coordinates around p such that $\gamma_{kl} = \delta_{kl}$, so this reduces to

$$
\begin{aligned}
\widehat{g}_{ij}(z) &= \gamma^{2^*-2}\delta_{kl}\left(\delta_{ik}\delta_{jl} - 2\delta_{ik}\rho^{-2}z^j z^l - 2\delta_{jl}\rho^{-2}z^i z^k + 4\delta_{ik}\delta_{jl}\rho^{-4}z^i z^k z^j z^l\right) \\
&= \gamma^{2^*-2}\delta_{ij}.
\end{aligned}
$$

Using the asymptotic expansion of G in conformal normal coordinates obtained in Lemma 4.2.6 we obtain:

Theorem 4.2.7 (Lee, Parker [13]). *Let $\{x^i\}$ be conformal normal coordinates such that $\det(g_{ij}) = 1 + O(r^N)$, with $N \geq 2\dim M$, and let $\{z^i\}$ be the corresponding inverted conformal normal coordinates. Then, the metric $\widehat{g}$ as defined in Definition 4.2.3 is asymptotically flat of order 1 if $n = 3$, order 2 if $n \geq 4$ and order $n-2$ if M is conformally flat near p. Furthermore, in these coordinates, the components of $\widehat{g}$ have the expansion*

$$
\widehat{g}_{ij}(z) = \gamma^{p-2}(z)(\delta_{ij} + O(\rho^{-2})),
\tag{4.2.11}
$$

where, as in Lemma 4.2.6,

(h_1) $\gamma(z) = 1 + A\rho^{2-n} + O''(\rho^{1-n})$ with $A =$ constant, if $\dim M = 3, 4, 5$ or (M, g) is locally conformally flat;

(h_2) $\gamma(z) = 1 + \frac{1}{288a}|W(P)|^2\rho^{-4}\log\rho + O''(\rho^{-4})$ if $\dim M = 6$ and (M, g) is not locally conformally flat;

(h_3) $\gamma(z) = 1 + \frac{1}{12a(n-4)}\rho^{-6}\left(\frac{\rho^2}{12(n-6)}|W(P)|^2 - S_{,ij}(P)z^i z^j\right) + O''(\rho^{-5})$ if $\dim M > 6$ and (M, g) is not locally conformally flat.

Chapter 5

Construction of the Test Function

The purpose of this chapter is to construct the test function on the asymptotically flat manifold $(\widehat{M}, \widehat{g})$ for which the Yamabe quotient depends on a number called the distortion coefficient, which depends on the geometry of $(\widehat{M}, \widehat{g})$. In Chapter 6 we will determine the sign of the distortion coefficient using the celebrated Positive Mass Theorem, and we will see that it is always non-negative. Furthermore, it turns out that it is zero only in the case where (M, g) is in the conformal class of the standard sphere. This, together with the contents of this chapter will enable us to prove that $\lambda(M) < \lambda(\mathbb{S}^n)$ in the cases that remain. Those cases are when (M, g) is a compact, connected Riemannian manifold that is not in the conformal class of the standard sphere and is either locally conformally flat or has dimension $\dim M = 3, 4, 5$. This was originally proved by Schoen in [16]. However, whilst in his paper Schoen proves that $\lambda(M) < \lambda(\mathbb{S}^n)$ using one method when $\dim M = 3$ or (M, g) is locally conformally flat and uses a different method to prove the same result when M has dimension 4 or 5, Lee and Parker in [13] were able to show the same result using one method. To do this, the authors use the machinery developed in the previous chapter.

In this chapter, we closely follow section 7 from [13], having added all the details.

As mentioned above, we will construct a test function on the asymptotically flat manifold $(\widehat{M}, \widehat{g})$ where the geometry is closer to that of $\mathbb{R}^n$, hence simpler. However, we need to see why this is helpful.

To that end, let (M, g) be a compact, connected Riemannian manifold of dimension $\dim M \geq 3$ with positive Yamabe invariant, $\lambda(M) > 0$, and let $(\widehat{M}, \widehat{g})$ be the stereographic projection of (M, g) from some point $p \in M$. Recall the definition of the Yamabe invariant:

$$\lambda(M) = \inf_{\substack{\varphi \in \mathcal{C}^\infty(M) \\ \varphi \neq 0}} \mathcal{Q}_g(\varphi).$$

Using standard density arguments show that

$$\lambda(M) = \inf_{\substack{\varphi \in \mathcal{C}_c^\infty(M \setminus \{p\}) \\ \varphi \neq 0}} \mathcal{Q}_g(\varphi).$$

Now let $\varphi \in \mathcal{C}_c(M \setminus \{p\})$, and note that

$$
\begin{aligned}
\mathcal{Q}_{\widehat{g}}(\varphi) := \frac{\displaystyle\int_{\widehat{M}} a|\widehat{\nabla}\varphi|^2 dV_{\widehat{g}}}{\left(\displaystyle\int_{\widehat{M}} \varphi^{2^*} dV_{\widehat{g}}\right)^{2/2^*}} &= \frac{\displaystyle\int_{M\setminus\{p\}} aG^{2-2^*}|\nabla\varphi|^2 G^{2^*} dV_g}{\left(\displaystyle\int_{M\setminus\{p\}} \varphi^{2^*} G^{2^*} dV_{\widehat{g}}\right)^{2/2^*}} \\
&= \frac{\displaystyle\int_{M\setminus\{p\}} aG^2|\nabla\varphi|^2 dV_g}{\left(\displaystyle\int_{M\setminus\{p\}} (G\varphi)^{2^*} dV_{\widehat{g}}\right)^{2/2^*}},
\end{aligned}
\tag{5.0.1}
$$

where $\widehat{\nabla}$ denotes the Levi-Civita connection of $\widehat{g}$. Now note that

$$
\begin{aligned}
\mathcal{Q}_g(\varphi G) &= \frac{\displaystyle\int_{M\setminus\{p\}} a|\nabla(\varphi G)|^2 + SG^2\varphi^2 dV_g}{\left(\displaystyle\int_{M\setminus\{p\}} (G\varphi)^{2^*} dV_g\right)^{2/2^*}} \\
&= \frac{\displaystyle\int_{M\setminus\{p\}} aG^2|\nabla\varphi|^2 + 2a\varphi G\langle\nabla\varphi, \nabla G\rangle + a\varphi^2\langle\nabla G, \nabla G\rangle + S\varphi^2 G^2 dV_g}{\left(\displaystyle\int_{M\setminus\{p\}} (G\varphi)^{2^*} dV_g\right)^{2/2^*}} \\
&= \frac{\displaystyle\int_{M\setminus\{p\}} aG^2|\nabla\varphi|^2 + a\langle\nabla(G\varphi^2), \nabla G\rangle + S\varphi^2 G^2 dV_g}{\left(\displaystyle\int_{M\setminus\{p\}} (G\varphi)^{2^*} dV_g\right)^{2/2^*}} \\
&= \frac{\displaystyle\int_{M\setminus\{p\}} aG^2|\nabla\varphi|^2 + \mathcal{L}_g(G\varphi^2) G dV_g}{\left(\displaystyle\int_{M\setminus\{p\}} (G\varphi)^{2^*} dV_g\right)^{2/2^*}} \\
&= \frac{\displaystyle\int_{M\setminus\{p\}} aG^2|\nabla\varphi|^2 dV_g}{\left(\displaystyle\int_{M\setminus\{p\}} (G\varphi)^{2^*} dV_g\right)^{2/2^*}},
\end{aligned}
$$

where we have used the fact that $\varphi \in \mathcal{C}_c^\infty(M \setminus \{p\})$ together with the fact that G is a multiple of the Green function of $\mathcal{L}_g$ at p. Because G is a strictly positive, smooth function in $M \setminus \{p\}$ we see that every compactly supported, smooth function on $M \setminus \{p\}$ can uniquely be written as $G\varphi$, for some compactly supported, smooth function ψ on $M \setminus \{p\}$. Therefore,

$$
\lambda(M) = \inf_{\varphi \in \mathcal{C}_c^\infty(\widehat{M})} \mathcal{Q}_{\widehat{g}}(\varphi). \tag{5.0.2}
$$

So, using standard density arguments, we can conclude that

$$\lambda(M) = \inf_{\substack{\varphi \in H^1(\widehat{M}) \\ \varphi \neq 0}} \mathcal{Q}_{\widehat{g}}(\varphi), \tag{5.0.3}$$

and therefore, if we find a nonzero $\varphi \in H^1(\widehat{M})$, such that $\mathcal{Q}_{\widehat{g}}(\varphi) < \lambda(\mathbb{S}^n)$, we can conclude that $\lambda(M) < \lambda(\mathbb{S}^n)$.

As mentioned above the geometry of $(\widehat{M}, \widehat{g})$ is close to the geometry of $\mathbb{R}^n$, this, together with the fact that, by construction, the scalar curvature of $\widehat{g}$, is identically zero, should, in principle, make the construction of the test function on $\widehat{M}$ to verify whether or not $\lambda(M) < \lambda(\mathbb{S}^n)$, simpler. This is, in fact, the reason why we introduced the concept of a stereographic projection.

To construct the test function consider the functions u_α defined in (2.1.2) on $\mathbb{R}^n$ and recall that $a\Delta u_\alpha = 4n(n-1)u_\alpha^{2^*-1}$ and $\lambda(\mathbb{S}^n) = 4n(n-1)\|u_\alpha\|_{2^*}^{2^*-2}$. Now fix a large $R > 0$ and let $\rho(z) = |z|$, in inverted conformal normal coordinates (extended to a smooth positive function on the whole of $\widehat{M}$) and set $\widehat{M}_\infty = \{\rho > R\}$. Define φ_α on $\widehat{M}$ by

$$\varphi_\alpha(z) = \begin{cases} u_\alpha(z), & \rho(z) \geq R, \\ u_\alpha(R), & \rho(z) \leq R, \end{cases} \tag{5.0.4}$$

where α is much larger than R (we will only have to determine α in Chapter 6).

Before we proceed, we should justify the definition of φ. First, because we will be interested in passing to the limit $\alpha \to \infty$ and u_α is close to zero for large α inside a ball of radius R, the effect in the Yamabe quotient of replacing u_α with a constant inside a ball of radius R should be inconsequential. Second, the reason why we set $\varphi(z) = u_\alpha(R)$ inside the ball $\{\rho \leq R\}$ is to make sure that $\varphi \in H^1(M)$.

Because φ_α is radially symmetric, the behaviour of $\mathcal{Q}_{\widehat{g}}(\varphi_\alpha)$ as $\alpha \to \infty$ should depend on the average behaviour of $\widehat{g}$ over spheres of large radius. Therefore, we need to study the $\widehat{g}-$volume of the spheres $\{\rho = \rho_0\}$ (for large ρ_0). Because we have a good knowledge of the volume of the Euclidean spheres a good way to study the $\widehat{g}-$volume of the spheres $\{\rho = \rho_0\}$ is to consider the ratio between the two volumes.

To do this, we first give the geometric interpretation of the scalar curvature. Let (M, g) be a Riemannian manifold and $p \in M$, and let $\{x^i\}$ be normal coordinates around a point p and $r = |x|$. The ratio between the $g-$volume of the geodesic sphere S_r (small r) around p and the Euclidean volume of the Euclidean sphere of radius r is given by the spherical density function

$$h(r) := \int_{S_r} \frac{d\widetilde{\omega}_r}{\omega_r}, \tag{5.0.5}$$

where $d\widetilde{\omega}_r$ is the induced volume form on S_r by dV_g. Since the unit normal vector to S_r is $N = \frac{\text{grad} r}{|\text{grad} r|} =$

$\frac{1}{\sqrt{g^{ij}\frac{x^i x^j}{r^2}}} g^{ij}\frac{x^i}{r}\frac{\partial}{\partial x^j} = \frac{1}{\sqrt{g^{rr}}} g^{ij}\frac{x^i x^j}{r^2}\frac{\partial}{\partial x^j}$, we have

$$\begin{aligned} d\widetilde{\omega}_r &= N \llcorner dV_g = \det(g_{ij})^{1/2}(g^{rr})^{-1/2} g^{ij}\frac{x^j}{r}\frac{\partial}{\partial x^i} \llcorner dx \\ &= \det(g_{ij})^{1/2}(g^{rr})^{-1/2} g^{ij}\frac{x^j}{r}\frac{x^i}{r}\frac{\partial}{\partial r} \llcorner dx \\ &= (\det(g_{ij})g^{rr})^{1/2} d\omega_r, \end{aligned}$$

where $d\omega_r$ denotes the Euclidean volume form of the sphere of radius r. Since $g^{rr} = 1$ in normal coordinates at p, the expansion 4.1.19 yields

$$h(r) = \int_{S_r}\left(1 - \frac{1}{6}R_{ij}(p)x^i x^j + O(r^3)\right)\frac{d\omega_r}{\omega_r} = 1 - \frac{1}{6n}S(p)r^2 + O(r^3). \tag{5.0.6}$$

Therefore, when the scalar curvature at p, $S(p)$, is positive, the volumes of geodesic spheres grow slower than the volumes of the Euclidean spheres, and they grow faster when $S(p) < 0$. So, we conclude that the scalar curvature measures the difference in the rate of growth of volumes of geodesic spheres in (M, g) when compared to the spheres in the usual Euclidean space $\mathbb{R}^n$.
As mentioned above we need to study the $\widehat{g}-$volume of the spheres $\{\rho = \rho_0\}$ (for large ρ_0) on $\widehat{M}$, so we consider the same function $h(\rho)$ for large values of $\rho = |z|$ on the manifold $(\widehat{M}, \widehat{g})$:

$$h(\rho) = \frac{1}{\omega_\rho}\int_{S_\rho} d\widehat{\omega}_\rho = \frac{1}{\omega_\rho}\int_{S_\rho}\left(g^{rr}\det(\widehat{g}_{ij})\right)^{1/2} d\omega_\rho, \tag{5.0.7}$$

where $d\widehat{\omega}_\rho$ is the volume form on S_ρ induced by $dV_{\widehat{g}}$. In inverted conformal normal coordinates $\{z^i\}$, we have $\widehat{g}^{\rho\rho} = \gamma^{2-2^*}$ and $\det(\widehat{g}_{ij}) = \gamma^{\frac{4n}{n-2}}(1 + \rho^{-N})$. To see that $\widehat{g}^{\rho\rho} = \gamma^{2-2^*}$, first note that since $\rho = r^{-1}$, we have $d\rho = -\rho^2 dr$, and therefore

$$\widehat{g}^{\rho\rho} = \widehat{g}(d\rho, d\rho) = \rho^4\widehat{g}(dr, dr) = \rho^4\widehat{g}^{rr}.$$

Using Definition 4.2.3, we know that $\widehat{g} = G^{2^*-2}g$ and hence

$$\widehat{g}^{rr} = G^{2-2^*}g^{rr},$$

where $g^{rr} = 1$ because we are in conformal normal coordinates. Recalling that $\gamma = r^{n-2}G$, we then have

$$\widehat{g}^{\rho\rho} = \rho^4 G^{2-2^*}\rho^{4+(n-2)(2-2^*)}\gamma^{2-2^*} = \gamma^{2-2^*}.$$

Thus, (5.0.6) reduces to

$$h(\rho) = \omega_\rho^{-1}\int_{S_\rho}\gamma^{1-2^*/2+2^*}(1 + O(\rho^{-N}))^{1/2} d\omega_\rho = \omega_\rho^{-1}\int_{S_\rho}\gamma^{(2+2^*)/2}(1 + O(\rho^{-N}))^{1/2} d\omega_\rho. \tag{5.0.8}$$

The expansion of γ given by Theorem 4.2.7 gives an asymptotic expansion as $\rho \to 0$:

$$h(\rho) = \begin{cases} 1 + \frac{\mu}{2-n}\rho^{2-n} + O''(\rho^{1-n}), \text{ if } \dim M = 3,4,5 \text{ or } (M,g) \text{ is conformally flat around } p, \\ 1 + \frac{\mu}{4}\rho^{-4}\log(\rho) + O''(\rho^{-4}), \text{ if } \dim M = 6 \text{ and } (M,g) \text{ is not conformally flat near } p, \\ 1 + \frac{\mu}{4}\rho^{-4} + O''(\rho^{-5}), \text{ if} \dim M > 6 \text{ and } (M,g) \text{ is not conformally flat near } p, \end{cases} \tag{5.0.9}$$

where μ is defined for each dimension. We call the constant μ, the distortion coefficient of $\widehat{g}$. The sign of this constant will be of the uttermost importance when determining the values of $\mathcal{Q}_{\widehat{g}}(\varphi)$ as is shown below. Since the $(n-1)-$form $\frac{d\omega_\rho}{\omega_\rho}$ is homogeneous of degree zero (with respect to ρ),

$$\frac{a}{2}\int_{S_\rho} \partial_\rho \gamma \frac{d\omega_\rho}{\omega_\rho} = \begin{cases} \mu\rho^{1-n} + O''(\rho^{2-n}), \text{ if } \dim M = 3,4,5 \text{ or } (M,g) \text{ is conformally flat near} p, \\ \mu\rho^{-5}\log(\rho) + O''(\rho^{-5}), \text{ if } \dim M = 6 \text{ and } (M,g) \text{ is not conformally flat near } p, \\ \mu\rho^{-5} + O''(\rho^{-6}), \text{ if} \dim M > 6 \text{ and } M \text{ is not conformally flat near } p. \end{cases} \tag{5.0.10}$$

To see this, we differentiate h. But first note that we can ignore the term related to the determinant of the metric in the definition of h due to the fact that we can absorb the error term on the right-hand side. Indeed, we have

$$\begin{aligned} h'(\rho) &= \frac{a}{2}\int_{S_\rho} \gamma^{2^*/2}\partial_\rho\gamma(1+O(\rho^{-N}))^{1/2}\frac{d\omega_\rho}{\omega_\rho} + \frac{1}{2}\int_{S_\rho}\gamma^{(2^*+2)/2}\frac{\partial_\rho \det(\widehat{g}_{ij})}{\det(\widehat{g}_{ij})}\frac{d\omega_\rho}{\omega_\rho} \\ &= \frac{a}{2}\int_{S_\rho}\gamma^{2^*/2}\partial_\rho\gamma\frac{d\omega_\rho}{\omega_\rho} + O(\rho^{-N}), \end{aligned}$$

because $\partial_\rho \det(\widehat{g}_{ij}) = O(\rho^{-N-1})$ and $(1+O(\rho^{-N}))^{1/2} - 1 = O(\rho^{-N})$. Using the expansion of γ given in Theorem 4.2.7 we have in the three cases of said theorem

$$\gamma^{2^*/2}\partial_\rho\gamma = \begin{cases} \partial_\rho\gamma(1 + \frac{\mu}{2-n}\rho^{2-n} + O''(\rho^{1-n})), \\ \partial_\rho\gamma(1 + \frac{\mu}{4}\rho^{-4}\log(\rho) + O''(\rho^{-4})), \\ \partial_\rho\gamma(1 + \frac{\mu}{4}\rho^{-4} + O''(\rho^{-5})), \end{cases} = \begin{cases} \partial_\rho\gamma + O(\rho^{3-2n}) \\ \partial_\rho\gamma + O(\rho^{-8}) \\ \partial_\rho\gamma + O(\rho^{-9}). \end{cases}$$

So, in the three cases of Theorem 4.2.7, we have

$$h'(\rho) - \frac{a}{2}\int_{S_\rho}\partial_\rho\gamma\frac{d\omega_\rho}{\omega_\rho} = \begin{cases} \partial_\rho\gamma(\frac{\mu}{2-n}\rho^{2-n} + O''(\rho^{1-n})), \\ \partial_\rho\gamma(\frac{\mu}{4}\rho^{-4}\log(\rho) + O''(\rho^{-4})), \\ \partial_\rho\gamma(\frac{\mu}{4}\rho^{-4} + O''(\rho^{-5})). \end{cases}$$

Taking into account the expansions of $\partial_\rho \gamma$ obtained by differentiating the expansions of γ we get

$$h'(\rho) - \frac{a}{2}\int_{S_\rho} \partial_\rho \gamma \frac{d\omega_\rho}{\omega_\rho} = \begin{cases} O(\rho^{3-2n}), \\ O(\rho^{-8}), \\ O(\rho^{-9})). \end{cases}$$

Hence, by differentiating the expansion of h we get (5.0.10). Using these estimates, we are able to prove the following.

Proposition 5.0.1. *Let φ_α be defined as in* (5.0.4)*. There is a positive constant C such that*

$$E(\varphi_\alpha) \le \lambda(\mathbb{S}^n)\|\varphi_\alpha\|^2_{L^{2^*}(\widehat{M})} - C\mu\alpha^{2-n} + O(\alpha^{1-n}), \tag{5.0.11}$$

if $\dim M = 3, 4, 5$ or (M, g) is conformally flat near p,

$$E(\varphi_\alpha) \le \lambda(\mathbb{S}^n)\|\varphi_\alpha\|^2_{L^{2^*}(\widehat{M})} - C\mu\alpha^{-4}\log(\alpha) + O(\alpha^{-4}), \tag{5.0.12}$$

if $\dim M = 6$ and (M, g) is not conformally flat near p,

$$E(\varphi_\alpha) < \lambda(\mathbb{S}^n)\|\varphi_\alpha\|^2_{L^{2^*}(\widehat{M})} - C\mu\alpha^{-4} + O(\alpha^{-5}), \tag{5.0.13}$$

if $\dim M > 6$ and (M, g) is not conformally flat near p.
Thus if $\mu > 0$, φ_α can be chosen so that $\mathcal{Q}_g(\varphi_\alpha) < \lambda(\mathbb{S}^n)$.

Proof. Recall that by definition $\widehat{g} = G^{2^*-2}g$, and hence the transformation law (A.0.20) yields

$$\widehat{S} = G^{1-2^*}(a\Delta_g G + SG) = 0, \text{ on } \widehat{M}, \tag{5.0.14}$$

where $\widehat{S}$ denotes the scalar curvature of $\widehat{g}$, because G is a multiple of the Green function for $\mathcal{L}_g$. So the energy $E(\varphi_\alpha)$ is

$$\begin{aligned}\int_{\widehat{M}} a|\nabla\varphi_\alpha|^2 dV_{\widehat{g}} &= \int_{\widehat{M}_\infty} a\widehat{g}^{\rho\rho}(\partial_\rho\varphi_\alpha)^2 dV_{\widehat{g}} = \int_{\widehat{M}_\infty} a\gamma^{2-2^*}(\partial_\rho u_\alpha)^2\gamma^{2^*}(1+O(\rho^{-N}))^{1/2}dz \\ &= \int_{\widehat{M}_\infty} a(\partial_\rho u_\alpha)^2\gamma^2(1+O(\rho^{-N}))^{1/2}dz \\ &= \int_{\widehat{M}_\infty} a(\partial_\rho u_\alpha)^2\gamma^2 dz + \int_{\widehat{M}_\infty} a(\partial_\rho u_\alpha)^2\gamma^2 dz O(\rho^{-N})dz.\end{aligned}$$

First note that the last integral is $O(\alpha^{-n})$:

$$\begin{aligned}\int_{\widehat{M}_\infty} (\partial_\rho u_\alpha)^2\gamma^2 dz\rho^{-N}dz &\le A_0\int_R^\infty \left(\frac{\rho}{\alpha}\right)^2\left(\frac{\alpha}{\alpha^2+\rho^2}\right)^n \gamma^2\rho^{-N+n-1}d\rho \\ &\le A_1\alpha^{-n}\int_{R/\alpha}^\infty \frac{\xi^2}{(1+\xi^2)^n}d\xi \\ &\le A_2 O(\alpha^{-n}),\end{aligned}$$

for some positive constants A_0, A_1, A_2. Let A_L denote the annulus $\{R \leq \rho \leq L\}$. Integration by parts using the Euclidean Laplacian gives

$$\begin{aligned}
\int_{A_L} a(\partial_\rho u_\alpha)^2\gamma^2 dz &= \int_{A_L} a\frac{\partial z^i}{\partial \rho}\frac{\partial u_\alpha}{\partial z^i}\partial_\rho u_\alpha \gamma^2 dz \\
&= -a\int_{A_L} \frac{\partial}{\partial z^i}\left(\frac{\partial z^i}{\partial \rho}\right) u_\alpha \partial_\rho u_\alpha \gamma^2 + \frac{\partial z^i}{\partial \rho}\frac{\partial}{\partial z^i}\left(\partial_\rho u_\alpha\right) u_\alpha \gamma^2 dz \\
&- a\int_{A_L} u_\alpha \partial_\rho u_\alpha \partial_\rho(\gamma^2) dz - a\int_{S_R\cup S_L} u_\alpha \partial_\rho u_\alpha \gamma^2 \frac{\partial z^i}{\partial \rho} n^i d\omega,
\end{aligned}$$

where n^i is the $i-$th component of the outward normal vector. Since $\frac{\partial z^i}{\partial \rho} = \frac{z^i}{\rho}$, we get

$$\begin{aligned}
\int_{A_L} a(\partial_\rho u_\alpha)^2\gamma^2 dz &= -a\int_{A_L} \frac{n-1}{\rho} u_\alpha \partial_\rho u_\alpha \gamma^2 + \partial_{\rho\rho} u_\alpha u_\alpha \gamma^2 dz \\
&- a\int_{A_L} u_\alpha \partial_\rho u_\alpha \partial_\rho(\gamma^2) dz - a\int_{S_R\cup S_L} u_\alpha \partial_\rho u_\alpha \gamma^2 \frac{z^i}{\rho} n^i d\omega \\
&= a\int_{A_L} u_\alpha \Delta u_\alpha \gamma^2 dz - a\int_{A_L} u_\alpha \partial_\rho u_\alpha \partial_\rho(\gamma^2) dz - a\int_{S_R\cup S_L} u_\alpha \partial_\rho u_\alpha \gamma^2 \frac{z^i}{\rho} n^i d\omega.
\end{aligned} \tag{5.0.15}$$

Because $\partial_\rho u_\alpha = (2-n)\frac{\rho}{\alpha}\left(\frac{\alpha}{\alpha^2+\rho^2}\right)^{n/2}$ and the fact that γ is bounded , we have that the integral over S_L is $O(L^{2-n})$ (for fixed α), thus is vanishes as $L \to \infty$. In the same way, one sees that the integral over S_R is $O(\alpha^{-n})$. We can bound the first integral in (5.0.15) using Hölder's inequality and the identity $\Delta_0 u_\alpha = n(n-2)u_\alpha^{2^*-1}$ we have

$$\begin{aligned}
\int_{A_L} a u_\alpha \Delta u_\alpha \gamma^2 dz &= 4(n-1)n\int_{A_L} u_\alpha^{2^*-2}(u_\alpha\gamma)^2 dz \\
&\leq 4(n-1)n\left(\int_{A_L} u_\alpha^{2^*} dz\right)^{1-2/2^*}\left(\int_{A_L} u_\alpha^{2^*}\gamma^{2^*} dz\right)^{2/2^*} \\
&\leq 4(n-1)n\|u_\alpha\|_{2^*}^{2^*-2}\left(\int_{\widehat{M}} \varphi_\alpha^{2^*} dV_{\widehat{g}}\right)^{2/2^*} + \frac{A}{R^{N-n+1}}\alpha^{2-n} \\
&= \lambda(\mathbb{S}^n)\|\varphi_\alpha\|_{2^*}^2 + \frac{A}{R^{N-n+1}}\alpha^{2-n}.
\end{aligned}$$

To see how the α^{2-n} term appears we have to estimate the difference between $\left(\int_{A_L} u_\alpha^{2^*} dV_{\widehat{g}}\right)^{2/2^*}$ and $\left(\int_{A_L} u_\alpha^{2^*}\gamma^{2^*} dz\right)^{2/2^*}$, which we can do using the triangle inequality:

$$\begin{aligned}
\left(\int_{A_L} u_\alpha^{2^*}\gamma^{2^*} dz\right)^{2/2^*} &\leq \left(\int_{A_L} u_\alpha^{2^*} dV_{\widehat{g}}\right)^{2/2^*} + \left(\int_{A_L} u_\alpha^{2^*}\gamma^{2^*} O(\rho^{-N}) dz\right)^{2/2^*} \\
&\leq \left(\int_{\widehat{M}} u_\alpha^{2^*} dV_{\widehat{g}}\right)^{2/2^*} + \left(\int_{A_L} u_\alpha^{2^*}\gamma^{2^*} O(\rho^{-N}) dz\right)^{2/2^*},
\end{aligned}$$

using Euclidean polar coordinates we can estimate the last term:

$$\begin{aligned}\left(\int_{A_L} u_\alpha^{2^*}\gamma^{2^*}O(\rho^{-N})dz\right)^{2/2^*} &\leq A_0\left(\int_R^\infty \frac{\alpha^n}{(\alpha^2+\rho^2)^n}\rho^{-N+n-1}d\rho\right)^{2/2^*}\\ &\leq A_1\alpha^{-n}\left(\int_0^\infty \frac{1}{(1+\rho^2)^n}d\rho\right)^{2/2^*}\\ &= A_2\alpha^{-n},\end{aligned}$$

for some positive constants A_0, A_1, A_2. In the limit $L \to \infty$ the second term in (5.0.15) becomes

$$-\int_R^\infty a u_\alpha \partial_\rho u_\alpha \int_{S_\rho} \partial_\rho(\gamma^2)d\omega_\rho d\rho. \tag{5.0.16}$$

If $\dim M = 3, 4, 5$ or M is conformally flat near p, (5.0.10) yields

$$a\int_{S_\rho}\partial_\rho(\gamma^2)d\omega_\rho = 4(h'(\rho)+O(\rho^{3-2n}))\omega_\rho = -4(\mu\rho^{1-n}+O(\rho^{2-n}))\omega_\rho. \tag{5.0.17}$$

When $\dim M > 6$ and M is not conformally flat near p we get

$$a\int_{S_\rho}\partial_\rho(\gamma^2)d\omega_\rho = 4(h'(\rho)+O(\rho^{-9}))\omega_\rho = -4(\mu\rho^{-5}+O(\rho^{-6}))\omega_\rho. \tag{5.0.18}$$

To arrive at (5.0.11) and (5.0.13) we need to estimate $\int_R^\infty \rho^{-k}\left(\frac{\alpha}{\alpha^2+\rho^2}\right)^{n-1}\rho^{n-1}d\rho$, where $2-n < k < n$. Proceeding as in the proof of Lemma 2.1.4 the change of variables $\sigma = \rho/\alpha$ gives

$$\int_R^\infty \rho^{-k}\left(\frac{\alpha}{\alpha^2+\rho^2}\right)^{n-1}\rho^{n-1}d\rho = \int_{R/\alpha}^\infty \alpha^{1-k}\frac{\sigma^{n-k-1}}{(1+\sigma^2)^{n-1}}d\sigma,$$

which is clearly bounded above and below by multiples of α^{-k+1}. Thus the second term in (5.0.15) is

$$-4\int_R^\infty \rho\alpha^{-1}\left(\frac{\alpha}{\alpha^2+\rho^2}\right)^{n-1}(\mu\rho^{1-n}+O(\rho^{2-n}))\omega_\rho d\rho \leq -C\mu\alpha^{2-n}+O(\alpha^{1-n}),$$

when $\dim M = 3, 4, 5$ or M is conformally flat near p (and (5.0.11) is proven),

$$-4\int_R^\infty \rho\alpha^{-1}\left(\frac{\alpha}{\alpha^2+\rho^2}\right)^{n-1}(\mu\rho^{-5}+O(\rho^{-6}))\omega_\rho d\rho \leq -C\mu\alpha^{-4}+O(\alpha^{-5}),$$

when $\dim M > 6$ and M is not conformally flat near p (and (5.0.13) is proven). Finally, when $\dim M = 6$ and M is not conformally flat near p, we estimate $\int_R^\infty \rho^{-4}\alpha^{-1}\log(\rho)\left(\frac{\alpha}{\alpha^2+\rho^2}\right)^5\rho^5 d\rho$. Performing the change of variables $\sigma = \rho/\alpha$ gives

$$\int_R^\infty \rho^{-4}\alpha^{-1}\log(\rho)\left(\frac{\alpha}{\alpha^2+\rho^2}\right)^5\rho^5 d\rho = \int_{R/\alpha}^\infty \alpha^{-4}\log(\alpha\sigma)\frac{1}{(1+\sigma^2)^5}d\sigma,$$

which is clearly bounded from above and below by multiples of $\alpha^{-4}\log(\alpha)$. This gives (5.0.12). □

Estimates (5.0.11)-(5.0.13) reduce the Yamabe problem in the case $\lambda(M) > 0$ to the problem of determining the sign of μ. Indeed, if $\mu > 0$ the estimates (5.0.11)-(5.0.12) together (5.0.3) with imply that $\lambda(M) < \lambda(\mathbb{S}^n)$. So, the following is proven:

Theorem 5.0.2. *Let (M, g) be a compact, connected Riemannian manifold of dimension $\dim M \geq 3$ with $\lambda(M) > 0$. Suppose that there is $p \in M$ for which there is a stereographic projection of M from p, $(\widehat{M}, \widehat{g})$, with positive distortion coefficient μ, then $\lambda(M) < \lambda(\mathbb{S}^n)$.*

Using the expansions of γ given in Theorem 4.2.7 we can explicitly determine the value of the distortion coefficient, μ, in terms of the Weyl tensor at p and the scalar curvature of g at p, when $\dim M \geq 6$ and (M, g) is not conformally flat near p. However, when (M, g) has dimension $3, 4, 5$ or is conformally flat near p the situation is not so simple. Indeed, in this case, we will, in Chapter 6, be able to relate μ with a global invariant of the Riemannian manifold $(\widehat{M}, \widehat{g})$ called "mass".

Chapter 6

The Positive Mass Theorem and the Solution of the Yamabe Problem

This chapter aims to present the final solution to the Yamabe problem. For that, we need to use the celebrated Positive Mass Theorem. The contents of this chapter are the trimmed-down version of the contents of sections 8, 9, 10 and 11 of [13], with the main difference being that in [13], Lee and Parker, provide a sketch of the proof of the positive mass theorem and the solution of the Yamabe problem, whilst in this text the main focus is solely the solution of the Yamabe problem.

We now introduce the notion of mass of an asymptotically flat manifold as presented in [13].

Definition 6.0.1 (Lee, Parker [13]). *Let (N, g) be an asymptotically flat manifold with asymptotic coordinates $\{z^i\}$. The mass of $(N.g)$ is defined as:*

$$m(g) = \lim_{R\to\infty} \omega_{n-1}^{-1} \int_{S_R} \nu \lrcorner\, dz, \tag{6.0.1}$$

when the limit exists, where ν is the mass-density vector field defined on N_∞:

$$\nu = (\partial_i g_{ij} - \partial_j g_{ii})\partial_j.$$

An immediate question is whether or not the mass of an asymptotically flat manifold is well-defined, i.e., if the limit (6.0.1) (when it exists) depends on the choice of asymptotic coordinates $\{z^i\}$. In the cases that we are interested the answer to this question is that it does not. To see this, first, we need to introduce a "good" class of metrics. For $\tau > \frac{n-2}{2}$, we define $\mathcal{M}_\tau$ as set of all smooth metrics on N such that, in some asymptotic coordinates, we have

$$g_{ij} - \delta_{ij} \in \mathcal{C}^{1,\alpha}_{-\tau}(N_\infty),\ S_g \in L^1(N),$$

where $\mathcal{C}^{1;\alpha}_{-\tau}(N_\infty)$ is the Hölder space as defined in [13] and S_g denotes the scalar curvature of g. In asymptotic coordinates, $\{z^i\}$, on N_∞:

$$S_g = g^{jk}\left(\partial_i\Gamma^i_{jk} - \partial_k\Gamma^i_{ij} + \Gamma^i_{il}\Gamma^l_{jk} - \Gamma^i_{kl}\Gamma^l_{ij}\right) = \partial_j(\partial_i g_{ij} - \partial_j g_{ii}) + O(\rho^{-2\tau-2}). \tag{6.0.2}$$

Now let (M, g) be a compact, connected Riemannian manifold that is either of dimension $3, 4, 5$ or is locally conformally flat. Consider the stereographic projection $(\widehat{M}, \widehat{g})$ of (M, g) from a point p. Using inverted conformal normal coordinates around p, Theorem 4.2.7 together with the fact that the scalar curvature of $\widehat{g}$ vanishes identically (see (5.0.14)) implies that $\widehat{g} \in \mathcal{M}_\tau$ for some $\frac{n-2}{2} < \tau < n-2$. In this setting, the mass of $\widehat{M}$ exists and is finite.

Lemma 6.0.2. *Let (M, g) be a compact, connected Riemannian manifold that either has dimensions 3, 4 or 5 or is conformally flat. Let $(\widehat{M}, \widehat{g})$ be the stereographic projection of (M, g) from p, as defined in Definition 4.2.3. Then $m(\widehat{g})$ is finite.*

Proof. Let η be a smooth cutoff function which is supported in $\widehat{M}_\infty$ and identically one for large ρ. Then by the divergence theorem,

$$\begin{aligned} m(\widehat{g}) &= \lim_{R\to\infty} \omega_{n-1}^{-1} \int_{S_R} \nu \lrcorner\, dz = \lim_{R\to\infty} \omega_{n-1}^{-1} \int_{S_R} \eta\langle \nu, n\rangle d\omega_R \\ &= \lim_{R\to\infty} \omega_{n-1}^{-1} \int_{B_R} \eta \operatorname{div}\nu + \nabla\eta\cdot\nu dz, \end{aligned} \tag{6.0.3}$$

where B_R is the open ball of radius R with centre at the origin. Since $0 = \widehat{S} = -\operatorname{div}\nu + O(\rho^{-2\tau-2})$, we have that $\eta \operatorname{div}\nu \in L^1(\widehat{M})$. Adding to this the fact that $\nabla\eta\cdot\nu \in \mathcal{C}^{0,\alpha}_c(\widehat{M}) \subset L^1(\widehat{M})$, we can conclude that the limit in (6.0.3) exists and is finite. ☐

In Chapters 3 and 4 we worked with conformal normal coordinates and inverted conformal normal coordinates without ever fixing coordinates, so it is crucial that the mass of $(\widehat{M}, \widehat{g})$ does not depend on the choice of inverted conformal normal coordinates. Indeed, this is the case as the next theorem shows.

Theorem 6.0.3. *The mass of $(\widehat{M}, \widehat{g})$ depends only on the metric $\widehat{g}$.*

Proof. See [5]. ☐

Proposition 6.0.4. *Let (M, g) be a compact, connected Riemannian manifold of dimension $\dim M \geq 3$, $(\widehat{M}, \widehat{g})$ be the stereographic projection of (M, g) from $p \in M$, and μ be the distortion coefficient as defined in (5.0.9). If $\dim M < 6$ or M is conformally flat near p, then $\mu = \frac{1}{2}m(\widehat{g})$.*

Proof. On the sphere S_ρ, we have $\partial_j \lrcorner\, dz = \frac{\partial z^j}{\partial\rho}\partial_\rho \lrcorner\, dz = \rho^{-1}z^j d\omega_\rho = \rho^{-2}z^j z^k \partial_k \lrcorner\, dz$. Therefore, the mass formula becomes

$$m(\widehat{g}) = \lim_{\rho\to\infty} \omega_{n-1}^{-1} \int_{S_\rho} (\rho^{-2}z^j z^k \partial_i\widehat{g}_{ij} - \partial_k\widehat{g}_{ii})\partial_k \lrcorner\, dz. \tag{6.0.4}$$

Now observe that

$$\widehat{g}_{\rho\rho} = \widehat{g}(\partial_\rho, \partial_\rho) = \widehat{g}\left(\frac{\partial z^k}{\partial\rho}\partial_k, \frac{\partial z^j}{\partial\rho}\partial_k\right) = \rho^{-2}z^k z^j \widehat{g}_{kj} \tag{6.0.5}$$

and

$$\partial_\rho \widehat{g}_{\rho\rho} = \rho^{-3} z^i z^j z^k \partial_i \widehat{g}_{kj}. \tag{6.0.6}$$

Moreover, the $(n-2)-$form $\eta = (z^j z^k \widehat{g}_{ij})\partial_i \lrcorner \partial_k \lrcorner dz$ satisfies

$$d\eta = (z^j z^k \partial_i \widehat{g}_{ij} - z^j z^i \partial_i \widehat{g}_{kj} + z^k \widehat{g}_{ii} - n z^j \widehat{g}_{kj})\partial_k \lrcorner dz. \tag{6.0.7}$$

And so,

$$\begin{aligned} m(\widehat{g}) &= \lim_{\rho\to\infty} \omega_{n-1}^{-1} \int_{S_\rho} \rho^{-2} d\eta + \int_{S_\rho} \left(\rho^{-2} z^j z^i \widehat{g}_{kj} - \rho^{-2} z^k \widehat{g}_{ii} + n\rho^{-2} z^j \widehat{g}_{kj} - \partial_k \widehat{g}_{ii}\right) \partial_k \lrcorner dz \\ &= \lim_{\rho\to\infty} \omega_{n-1}^{-1} \int_{S_\rho} \left(\rho^{-3} z^k z^j z^i \widehat{g}_{kj} - \rho^{-3} (z^k)^2 \widehat{g}_{ii} + n\rho^{-3} z^k z^j \widehat{g}_{kj} - z^k \partial_k \widehat{g}_{ii}\right) \partial_\rho \lrcorner dz \\ &= \lim_{\rho\to\infty} \omega_{n-1}^{-1} \int_{S_\rho} \partial_\rho(\widehat{g}_{\rho\rho} - \widehat{g}_{ii} + \rho^{-1}(n\widehat{g}_{\rho\rho} - \widehat{g}_{ii}) d\omega_\rho, \end{aligned} \tag{6.0.8}$$

where we have applied Stokes' Theorem to make the term $\rho^{-2} d\eta$ disappear.

Using inverted conformal normal coordinates, we can simplify this expression. Since $\widehat{g}_{\rho\rho} = \gamma^{2^*-2}$ and $\det \widehat{g} = \gamma^{\frac{4n}{n-2}}(1 + O(\rho^{-N})) = 1 + O(\rho^{2-n})$ (where N is large positive integer). Thus

$$\begin{aligned} n\partial_\rho \widehat{g}_{\rho\rho} &= n(2^*-2)\gamma^{2^*-3}\partial_\rho\gamma = \gamma^{2^*-2}\partial_\rho(\log(\det \widehat{g})) + O(\rho^{-N}) \\ &= \gamma^{2^*-2}\partial_\rho(\operatorname{tr}\log(\widehat{g})) = \gamma^{2^*-2}\operatorname{tr}(\widehat{g}^{-1}\partial_\rho \widehat{g}) = \gamma^{2^*-2}\widehat{g}^{ij}\partial_\rho \widehat{g}_{ij} \\ &= g^{ij}\partial_\rho \widehat{g}_{ij} = \partial_\rho \widehat{g}_{ii} + (g^{ij} - \delta_{ij})\partial_\rho \widehat{g}_{\rho\rho}, \end{aligned}$$

which using (4.1.22) and Theorem (6.0.8) becomes

$$n\partial_\rho \widehat{g}_{\rho\rho} = \partial_\rho \widehat{g}_{ii} + O(\rho^{-n-1}).$$

So, by integrating along rays from infinity and noting that $n\widehat{g}_{\rho\rho} = n = \widehat{g}_{ii}$ at infinity (by the expansion in Theorem 4.2.7), we find that $n\widehat{g}_{\rho\rho} = \widehat{g}_{ii} + O(\rho^{-n})$. Then (6.0.8) becomes

$$m(\widehat{g}) = \lim_{\rho\to\infty} \omega_{n-1}^{-1} \int_{S_\rho} \partial_\rho(\widehat{g}_{\rho\rho} - \widehat{g}_{ii}) d\omega_\rho = \lim_{\rho\to\infty} \omega_{n-1}^{-1} \int_{S_\rho} a\partial_\rho\gamma d\omega_\rho = 2\mu,$$

by (5.0.10). □

So determining the sign of the distortion coefficient, μ is equivalent to determining the sign of the mass of the asymptotically flat manifold $(\widehat{M}, \widehat{g})$ when (M, g) is in the conditions of the previous proposition. To solve the problem of determining the sign of the mass of $(\widehat{M}, \widehat{g})$, we have the following theorem, originally proved by Yau and Schoen in [17].

Theorem 6.0.5 (Positive Mass Theorem). *Let (N, g) be an asymptotically flat manifold of dimension $n \geq 3$ with metric $g \in \mathcal{M}_\tau$, with $\tau > (n-2)/2$, and nonnegative scalar curvature. Then its mass $m(g)$ is nonnegative, with $m(g) = 0$ if and only if (N, g) is isometric to $\mathbb{R}^n$ with its Euclidean metric.*

Using this theorem we can finally present a solution to the Yamabe problem.

Theorem 6.0.6. *Every compact, connected Riemannian manifold (M, g) of dimension $n \geq 3$, admits a metric $\widetilde{g}$ conformal to g with scalar curvature $\widetilde{S} = \lambda(M)$.*

Proof. If (M, g) is in the conformal class of the standard sphere, Theorem 2.2.2 together with Theorem 3.1.7 imply the result. If $\dim M \geq 6$ and (M, g) is not locally conformally flat (in particular, (M, g) is not in the conformal class of the standard sphere), then Theorem 4.1.20 and the results of Chapter 3 imply the result. If $\dim M < 6$ or (M, g) is locally conformally flat, Lemma 6.0.2 and the Positive Mass Theorem show that $\mu > 0$, unless $(\widehat{M}, \widehat{g})$ is isometric to $\mathbb{R}^n$ endowed with the Euclidean metric. In the first case, the results of chapter 3 together with Theorem 5.0.2 yield the result. In the second case, we want to show that M is conformal to the standard sphere. To that end, let β denote the isometry from $(\widehat{M}, \widehat{g})$ onto $\mathbb{R}^n$ (endowed with the standard Euclidean metric). Consider the inverse of the standard stereographic projection from $\mathbb{S}^n \setminus \{P\}$ onto $\mathbb{R}^n$ (where P denotes the north pole). Let $\Phi : \widehat{M} \to \mathbb{S}^n \setminus \{P\}$ denote the composition of the two maps. Using this map we can not only transport the round metric in the sphere, $\overline{g}$, to $\widehat{M}$ but we can also conclude, by Lemma 2.1.1 that under the isometry between $(\widehat{M}, \widehat{g})$ and $\mathbb{R}^n$ we have:

$$\frac{4}{(1+\rho^2)^2} G^{2^*-2} g = \overline{g}, \text{ in } \mathbb{R}^n,$$

in coordinates $\{z^i\}$, where $\rho = |z|$ is the Euclidean distance. So, we see that the singularity of G^{2^*-2} at infinity is cancelled by the decay of $\frac{4}{(1+\rho^2)^2}$ at infinity. Therefore, we can conclude that the metric $\Phi^* \overline{g}$ can be extended to the whole of M and is conformal to g. Since M is the one-point compactification of $\mathbb{R}^n$, it is simply connected, and because $\overline{g}$ has constant sectional curvature equal to 1, we conclude via the Killing-Hopf Theorem that $(M, \Phi^* \overline{g})$ is isometric to the standard sphere, hence (M, g) is conformal to the standard sphere.

□

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Appendix A

Review of Riemannian Geometry

The exposition that follows closely the one in Section 2 of [13] with a few extra results for better comprehension of this dissertation. All the results in this Appendix can be found in one of the following references: [4, 11–13, 15].

We start with some basic definitions.

Definition A.0.1. *A Riemannian manifold (M,g) of dimension n is a $\mathcal{C}^\infty-$differentiable manifold M of dimension n equipped with a metric tensor g_p on each tangent space T_pM such that the map $p \to g_p$ is smooth, that is, for any two smooth vector fields $X, Y \in \mathfrak{X}(M)$, the function $p \to g_p(X|_p, Y|_p)$ is smooth.*

Throughout this appendix, M will be a Riemannian manifold g will denote a metric on M. In local coordinates $(x^1, ..., x^n)$ the metric can be written as

$$g = g_{ij} dx^i \otimes dx^j,$$

where we employ the Einstein summation convention. If g_{ij} are the components of the metric tensor with respect to a coordinate system (x^i), the components of the "inverse metric tensor" are denoted by g^{ij}. The components of the metric, g_{ij}, and its inverse, g^{ij} are used to raise and lower indices in tensors. For example, if $T = T_{ij}dx^i \otimes dx^j$ is a $2-$ tensor, then $T^l_k = g^{il}T_{ik}$, where the Einstein summation convention is employed. Furthermore, we can extend the metric tensor to an inner product between any two tensors of the same type: if T, F are two (k,l)-tensor fields in M, written in local coordinates (x^i) as $T = T^{j_1,...,j_l}_{i_1,...,i_k} \frac{\partial}{\partial x^{i_1}} \otimes .... \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes dx^{j_l}$ and $F = F^{j_1,...,j_l}_{i_1,...,i_k} \frac{\partial}{\partial x^{i_1}} \otimes .... \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes dx^{j_l}$ then

$$\langle T, F \rangle := g^{i_1 s_1} ... g^{i_k s_k} g_{j_1 r_1} ... g_{j_k r_k} T^{j_1,...,j_l}_{i_1,...,i_k} F^{r_1,...,r_l}_{s_1,...,s_k}$$

Remark A.0.2. *Unlike in a general manifold where in order for integration to be well-defined we need the manifold to be orientable, in a Riemannian manifold we do not need orientability to integrate, and this is due to the Riemannian volume form (also called Riemannian density). In what, the Riemannian volume form of a Riemannian manifold (M,g) will be denoted by dV_g, in local coordinates, $dV_g = \sqrt{\det(g_{ij})}dx$.*

Definition A.0.3. *A connection on M is a smooth map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that $\forall X, Y, Z \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$:*

1. $\nabla_X(fY) = (X \cdot f)Y + f\nabla_X Y$;
2. $\nabla_{fX}Y = f\nabla_X Y$;
3. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$;
4. $\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z$.

For a local coordinate system (x_i), there are unique smooth functions $\Gamma^k_{ij}, i, j, k = 1, ..., n$ such that $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$, called the Christoffel symbols of the connection ∇ for the coordinate system (x^i) . We can extend the concept of covariant derivative to general tensors as follows: for functions, $\nabla_X f = X(f)$, ∇_X preserves the type of tensor, ∇_X commutes with contractions and satisfies the Leibniz rule ($\nabla_X(u \otimes v) = (\nabla_X u) \otimes v + u \otimes (\nabla v)$, u and v are tensor fields). If T is a $(k, m)-$tensor field, written in local coordinates (x^i) as $T = T^{j_1,\cdots,j_l}_{i_1,\ldots,i_k} \frac{\partial}{\partial x^{i_1}} \otimes .... \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes dx^{j_l}$, then for every vector field $X \in \mathfrak{X}(M)$ we have

$$\begin{aligned}
\nabla_X T = & \sum_{\substack{i_1,\ldots,i_k=1\\ j_1,\cdots,j_m=1}}^{n} X \cdot T^{j_1,\cdots,j_m}_{i_1,\ldots,i_k} dx^{i_1} \otimes ... \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes ... \otimes \frac{\partial}{\partial x^{j_m}} \\
& - \sum_{\substack{i_1,\ldots,i_k=1\\ j_1,\cdots,j_m=1\\ r,s=1}}^{n} \left(\Gamma^s_{ri_1} X^r T^{j_1,\cdots,j_m}_{s,i_2,\ldots,i_k} + \Gamma^s_{ri_k} X^r T^{j_1,\cdots,j_m}_{i_1,\ldots,i_{k-1},s}\right) dx^{i_1} \otimes ... \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes ... \otimes \frac{\partial}{\partial x^{j_m}} \\
& + \sum_{\substack{i_1,\ldots,i_k=1\\ j_1,\cdots,j_m=1\\ r,s=1}}^{n} \left(\Gamma^{j_1}_{rs} X^r T^{s,j_2,\ldots,j_m}_{i_1,\ldots,i_k} + \Gamma^{j_1}_{rs} X^r T^{j_1,\ldots,j_{m-1},s}_{i_1,\ldots,i_k}\right) dx^{i_1} \otimes ... \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes ... \otimes \frac{\partial}{\partial x^{j_m}}
\end{aligned} \tag{A.0.1}$$

Before we proceed, a word on notation is necessary. If T is a tensor, the $m-th$ covariant derivative of T is denoted by $\nabla^m T$, and if the components of T are denoted by $T^{j_1,\cdots,j_m}_{i_1,\ldots,i_k}$, then the components of $\nabla^m T$ are denoted by $T^{j_1,\cdots,j_m}_{i_1,\ldots,i_k;s_1,\ldots,s_m}$. Finally, note that if T is a $(k, l)-$tensor field, then $\nabla^m T$ is a $(k, l + m)-$tensor field and $\nabla^m_{X_1,\ldots,X_m} T = \nabla^m T(..., X_1, ..., X_m)$, for all vector fields $X_1, ..., X_m$.

Definition A.0.4. *The torsion of a connection ∇ is the map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $T(X, Y) = \nabla_X Y - D_Y X - [X, Y]$. The connection is said to be torsion-free if $T \equiv 0$.*

It is easy to see that T is indeed a tensor and that in local coordinates (x^i) we have $T_{ij} = T(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \left(\Gamma^k_{ij} - \Gamma^k_{ji}\right) \frac{\partial}{\partial x^k}$, and so we can conclude that a connection is torsion-free if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$ for all $i, j, k = 1, ..., n$.

Definition A.0.5. *A connection ∇ is said to be compatible with the metric g if*

$$X \cdot \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Theorem A.0.6 (Levi-Civita)**.** *There is a unique connection ∇ on M which is torsion-free and compatible with g, for which, in local coordinates (x^i), the Christoffel symbols are*

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right). \tag{A.0.2}$$

Definition A.0.7. *Given a connection ∇ on M the curvature tensor of the connection is the map R : $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all $X, Y, Z \in \mathfrak{X}(M)$. In local coordinates (x^i), the curvature tensor can be written as

$$R = \sum_{i,j,k,l=1}^n R_{ijk}^l dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}.$$

Direct computation shows that

$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x^i} - \frac{\partial \Gamma_{ik}^l}{\partial x^j} + \sum_{m=1}^n \Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l.$$

We further define the Riemann curvature tensor as the $(0, 4)-$tensor field given by

$$Rm(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle,$$

for all smooth vector fields $X, Y, Z, W \in \mathfrak{X}(M)$. In local coordinates, it takes the form

$$Rm = \sum_{i,j,k,l=1}^n R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l,$$

where $R_{ijkl} = g_{lm} R_{ijk}^m$.

Proposition A.0.8 (Symmetries of the Riemann Curvature Tensor)**.** *Let (M, g) be a Riemannian manifold. The Riemann curvature tensor has the following symmetries for all vector fields X, Y, Z, W:*

(a) $Rm(X, Y, Z, W) = -Rm(Y, X, Z, W)$,

(b) $Rm(X, Y, Z, W) = -Rm(X, Y, W, Z)$,

(c) $Rm(X, Y, Z, W) = Rm(Z, W, X, Y)$,

(d) (The first Bianchi identity) $Rm(X, Y, Z, W) + Rm(Y, Z, X, W) + Rm(Z, X, Y, W)$.

In local coordinates, components wise, these symmetries take the form:

(a') $R_{ijkl} = -R_{jikl}$,

(b') $R_{ijkl} = -R_{ijlk}$,

(c') $R_{ijkl} = R_{klij}$,

(d') $R_{ijkl} + R_{jkil} + R_{kijl} = 0.$

Proposition A.0.9 (Second Bianchi identity). *The covariant derivative of the Riemann curvature tensor satisfies the following identity:*

$$\nabla Rm(X, Y, Z, V, W) + \nabla Rm(X, Y, V, W, Z) + \nabla Rm(X, Y, W, Z, V) = 0, \tag{A.0.3}$$

or, in components,

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0. \tag{A.0.4}$$

Given vector fields X, Y, let $R(X,Y)^* : T^*M \to T^*M$ denote the dual map to $R(X,Y)$, defined by $(R(X,Y)^*\eta)(Z) = \eta(R(X,Y)Z)$.

Theorem A.0.10 (Ricci Identities). *On a Riemannian manifold M, the second covariant derivatives of vector and tensor fields satisfy the following identities.*
If Z is a vector field,

$$\nabla^2_{X,Y} Z - \nabla^2_{X,Y} Z = R(X,Y)Z. \tag{A.0.5}$$

If η is a $1-$form,

$$\nabla^2_{X,Y}\eta - \nabla^2_{X,Y}\eta = -R(X,Y)^*\eta. \tag{A.0.6}$$

And if β is a smooth $(k,l)-$tensor field,

$$\begin{aligned}(\nabla^2_{X,Y}\beta - \nabla^2_{X,Y}\beta)(\omega^1, ..., \omega^k, V_1, ..., V_l) = \beta(R(X,Y)^*\omega^1, \omega^2, ..., \omega^k, V_1, ..., V_l) + ...\\ + \beta(\omega^1, ..., R(X,Y)^*\omega^k, V_1, ..., V_l)\\ - \beta(\omega^1, ..., \omega^k, R(X,Y)V_1, ..., V_l) - ...\\ - \beta(\omega^1, ..., \omega^k, V_1, ..., R(X,Y)V_l),\end{aligned} \tag{A.0.7}$$

For all covectors fields ω^i and vector fields V_j. In terms of any local frame, the component versions of these formulas read

$$Z^i_{;pq} - Z^i_{qp} = -R^i_{pqm} Z^m, \tag{A.0.8}$$

$$\eta_{j;pq} - \eta_{j;qp} = R^m_{pqj}\eta_m, \tag{A.0.9}$$

$$\begin{aligned}\beta^{i_1,...,i_k}_{j_1,...,j_l;pq} - \beta^{i_1,...,i_k}_{j_1,...,j_l;qp} = -R^{i_1}_{pqm}\beta^{m,i_2,...,i_k}_{j_1,...,j_l} - R^{i_k}_{pqm}\beta^{i_1,...,i_{k-1},m}_{j_1,...,j_l}\\ + R^m_{pqj_1}\beta^{i_1,...,i_k}_{m,j_2...,j_l} + R^m_{pqj_l}\beta^{i_1,...,i_k}_{j_1,...,j_{l-1},m}.\end{aligned} \tag{A.0.10}$$

Definition A.0.11. *Let (M, g) be a Riemannian manifold. We define the Ricci curvature or Ricci tensor, Rc, as the covariant $2-$tensor field given as the trace of the curvature tensor on its first and last indices.*

$$Rc(X,Y) = tr(Z \to R(Z,X)Y), \forall X, Y \in \mathfrak{X}(M).$$

Therefore, in local coordinates,

$$Rc = \sum_{ij=1}^{n} R_{ij} dx^i \otimes dx^j,$$

where $R_{ij} = R^k_{kij}$. Also define the scalar curvature of (M, g) as the trace of the Ricci tensor,

$$S = tr_g Rc = R^i_i = g^{ij} R_{ij}.$$

Finally, define the traceless Ricci tensor of g as the symmetric $2-$tensor $\overset{\circ}{Rc} = Rc - \frac{S}{n} g$.

A Riemannian manifold whose Ricci tensor is a scalar multiple of the metric is said to be Einstein. It is easy to see that a Riemannian manifold is Einstein if and only if, its traceless Ricci tensor vanishes identically.

If T is a smooth $2-$tensor field on a Riemannian manifold, we define the exterior covariant derivative of T to be the $3-$tensor field DT defined by

$$(DT)(X, Y, Z) = -(\nabla T)(X, Y, Z) + (\nabla T)(X, Z, Y), \tag{A.0.11}$$

components wise this definition translates to:

$$(DT)_{ijk} = -T_{ij;k} + T_{ik;j}.$$

Remark A.0.12. *This operator is a generalization of the ordinary exterior derivative of a $1-$form, which can be expressed as $(d\eta)(X, Y) = -(\nabla\eta)(X, Y) + (\nabla\eta)(Y, X)$.*

Proposition A.0.13 (Contracted Bianchi identities)**.** *Let (M, g) be a Riemannian manifold. The covariant derivatives of the Riemann, Ricci, and scalar curvatures of g satisfy the following identities:*

$$tr_g(\nabla \,\mathrm{Rm}) = -D(\mathrm{Ric}), \tag{A.0.12}$$

$$tr_g(\nabla \,\mathrm{Ric}) = \frac{1}{2} dS, \tag{A.0.13}$$

where the trace, in each case, is on the first and last indices. In components, this is

$$R^i_{ijkl;} = R_{jk;l} - R_{jl;k}, \tag{A.0.14}$$

$$R_{il;}{}^{i} = \frac{1}{2} S_{;l}. \tag{A.0.15}$$

Proposition A.0.14 (Schur's Lemma)**.** *Suppose (M, g) is a connected Riemannian manifold of dimension $n \geq 3$ whose Ricci tensor satisfies $Rc = fg$ for some smooth real-valued function. Then, f is constant, satisfies $f = \frac{1}{n} S$ and (M, g) is Einstein.*

Proof. For a proof of this fact see Proposition 7.19 in [12]. □

Definition A.0.15. *Let (M, g) be a Riemannian manifold. A normal coordinate system at $P \in M$ is a local coordinate system $(U, (x^1, ..., x^n))$, for which the components of the metric tensor at P satisfy: $g_i j(P) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial x^k}(P) = 0$, for all i, j, k. Note that the condition on the partial derivatives of the components of the metric is equivalent to all Christoffel symbols with respect to normal coordinates being zero.*

Proposition A.0.16. *Let (M,g) be a Riemannian manifold. At each point $p \in M$, there exists a normal coordinate system.*

Definition A.0.17. *Let (M,g) be a Riemannian manifold and f a smooth function on M. We define the Laplace-Beltrami operator, Δ_g, by setting $\Delta_g f = -tr_g \nabla df$ (∇df is the second covariant derivative of f, also called the Hessian of f). In local coordinates:*

$$\nabla df = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right) dx^i \otimes dx^j$$

and so,

$$\Delta_g f = -g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x^k} g^{ij}.$$

Another way to define the Laplace-Beltrami operator is with the divergence operator, ∇^*. The divergence operator is the formal adjoint of ∇, given on $1-$forms by $\nabla^* \omega = -\omega_i^i = -\omega_{i;j} g^{ij}$. On a compact Riemannian manifold (M,g) with boundary, the divergence theorem holds:

$$\int_M \nabla^* \omega dV_g = -\int_{\partial M} \omega(N) dV_{\widetilde{g}}, \tag{A.0.16}$$

where $\widetilde{g}$ is the induced metric on ∂M and N is the outward unit normal. (When M is oriented this is just Stokes' Theorem). Furthermore, on a compact manifold without boundary, the integration by parts formula holds:

$$\int_M \langle \nabla u, \nabla v \rangle dV_g = \int_M v \Delta u dV_g.$$

Lemma A.0.18 (Transformation laws)**.** *Let (M,g) be a Riemannian manifold and $\widetilde{g}$ metric on M conformal to g, given by $\widetilde{g} = e^{2f} g$ for some $f \in \mathcal{C}^\infty(M)$. Let $\widetilde{S}$, $\widetilde{Rc}$ denote the scalar curvature and Ricci tensor of $(M,\widetilde{g})$, respectively. Then*

$$\widetilde{R}_{ij} = R_{ij} - (n-2)(\mathcal{H}_f)_{ij} + (n-2)(df \otimes df)_{ij} + (\Delta f - (n-2)\|df\|^2) g_{ij}, \tag{A.0.17}$$

and

$$\widetilde{S} = e^{-2f}(S + 2(n-1)\Delta f - (n-1)(n-2)\|df\|^2), \tag{A.0.18}$$

where $\mathcal{H}_f$ denotes the hessian of f.

The transformation formula of the scalar curvature can be significantly simplified id we make the substitution $\varphi^{p-2} = e^{2f}$, A.0.18 becomes

$$\widetilde{S} = \varphi^{2-p}(S - 4\frac{n-1}{\varphi(n-2)}\Delta\varphi) \iff a\Delta\varphi + S\varphi = \widetilde{S}\varphi^{2^*-1}. \tag{A.0.19}$$

The operator $\mathcal{L}_g := a\Delta_g + S$ is usually called the conformal Laplacian of (M,g). It is conformally invariant in the sense that if $\widetilde{g} = \varphi^{2^*-2} g$ is a conformal metric to g, and $\mathcal{L}_{\widetilde{g}}$ is defined similarly with respect to $\widetilde{g}$, then computing $\Delta_{\widetilde{g}}$ in terms of Δ_g and using the transformation laws for scalar curvature, we find that

$$\mathcal{L}_{\widetilde{g}}(\varphi^{-1}u) = \varphi^{1-2^*}\mathcal{L}_g u. \tag{A.0.20}$$

Definition A.0.19. *Given two $2-$tensor fields T, L on M define the Kulkarni-Nomizu product of T and L by the formula*

$$T \circ L(X,Y,Z,W) := T(X,W)L(Y,Z) + T(Y,Z)L(X,W) - T(X,Z)L(Y,W) + T(Y,W)L(X,Z), \tag{A.0.21}$$

where X, Y, Z, W are vector fields in M.

Definition A.0.20. *Let (M,g) be a Riemannian manifold. The Weyl tensor is defined as*

$$W = Rm - \frac{1}{n-2}Rc \circ g + \frac{S}{2(n-1)(n-2)} g \circ g, \tag{A.0.22}$$

in terms of components,

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il})\frac{S}{2(n-1)(n-2)}(g_{ik})g_{jl} - g_{jl}g_{ik}).$$

A manifold is said to be locally conformally flat if it is locally conformal to the Euclidean space $\mathbb{R}^n$ endowed with the standard Euclidean metric.

Theorem A.0.21 (Weyl-Schouten)**.** *Suppose (M,g) is a Riemannian manifold of dimension $n \geq 4$, then (M,g) is locally conformally flat if and only if the Weyl tensor vanishes identically. Furthermore, if $\dim M = 3$ and (M,g) is locally conformally flat, then the Weyl tensor also vanishes identically.*

The previous theorem is not presented in its full generality so as not to make this exposition immeasurably dense. To see the previous theorem in its full generality the reader may consult Theorem 7.37 in [12].

Theorem A.0.22 (Killing-Hopf)**.** *Let (M,g) be a simply connected, geodesically complete $n-$dimensional ($n \geq 2$)Riemannian manifold, with constant sectional curvature K, then (M,g) is isometric to one of the model spaces $\mathbb{S}^n$ (if $K > 0$), $\mathbb{R}^n$ (if $K = 0$), $\mathcal{H}^n$ (if $K < 0$)*

Proof. See Theorem 12.4 in [12]. □

Appendix B

Review of Analysis and Partial Differential Equations in Riemannian Manifolds

The exposition that follows closely follows the one in Section 2 of [13] with a few extra results for completeness' sake. The aim of what follows is to present all the results used throughout the dissertation. All the results may be found in [4, 8, 10, 13]. We start with a few basic definitions.

Definition B.0.1. *For $q \geq 1$, the space $L^q(M)$ is the set of locally integrable functions on (M,g) for which the L^q norm*

$$\|u\|_q := \left(\int_M |u|^q dV_g\right)^{1/q}$$

is finite.

Definition B.0.2. *Consider the space $\mathcal{C}^{k,p}(M)$ of $\mathcal{C}^\infty(M)$ functions such that $|\nabla^l \varphi| \in L^p(M)$ for all $0 \leq l \leq k$, where k, l are integers and $p \geq 1$. The Sobolev space $W^{k,q}(M)$ is defined as the completion of $\mathcal{C}^{k,q}(M)$ with respect to the norm*

$$\|u\|_{W^{k,q}(M)} := \sum_{l=0}^{k} \|\nabla^l u\|_q,$$

and the Sobolev space $W_0^{k,q}(M)$ as the closure of $\mathcal{C}_c^\infty(M)$ in $W^{k,q}(M)$.

Note that when M is complete, $\mathcal{C}_c^\infty(M)$ is dense in $W^{k,q}(M)$.

Proposition B.0.3. *For any k integer, $H^k(M) := W^{k,2}(M)$ is an Hilbert space when equipped with the equivalent norm*

$$\|u\| = \sqrt{\sum_{l=0}^{k} \int_M |\nabla^l u|^2 dV_g}.$$

The inner product $\langle \cdot, \cdot \rangle_{H^k}$ associated to $\| \cdot \|$ is defined by

$$\langle u, v \rangle_{H^k} = \sum_{l=0}^{k} \int_M \langle \nabla^l u, \nabla^l v \rangle dV_g,$$

where the inner product inside the integrals is the inner product of tensors with respect to g.

If M is a compact manifold, endowed with two metric tensors g and $\widetilde{g}$, it is not hard to see that there is $C > 1$ such that

$$\frac{1}{C} g \leq \widetilde{g} \leq C g,$$

on M, where the inequalities are to be understood in the sense of bilinear forms. This leads to the following:

Proposition B.0.4. *If M is compact, $W^{k,q}(M)$ does not depend on the metric.*

Definition B.0.5. *The space $\mathcal{C}^k(M)$ is the set of k times continuously differentiable functions on M, for which the norm*

$$\|u\|_{\mathcal{C}^k} := \sum_{l=0}^{k} \sup_M |\nabla^l u|$$

is finite, where $|\nabla^l u|$ is the norm induced by the inner product of two tensor fields of the same type as defined in Appendix A. *The Hölder space $\mathcal{C}^{k,\alpha}(M)$ is defined for $0 < \alpha < 1$ as the set of functions $u \in \mathcal{C}^k(M)$ for which the norm*

$$\|u\|_{\mathcal{C}^{k,\alpha}} = \|u\|_{\mathcal{C}^k} + \sup_{x,y} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}$$

is finite, where the supremum is over all $x \neq y$ such that y is contained in normal coordinate neighbourhood of x, and $\nabla^k u(y)$ is taken to mean the tensor at x obtained by parallel transport along the radial geodesic from x to y.

There is a relation between these spaces:

Theorem B.0.6 (Sobolev Embedding theorems for $\mathbb{R}^n$)**.** *(a) Suppose $\frac{1}{r} = \frac{1}{q} - \frac{k}{n}$. Then $W^{k,q}(\mathbb{R}^n)$ is continuously embedded in $L^r(\mathbb{R}^n)$. In particular, for $q = 2$, $k = 1$, $r = p = \frac{2n}{n-2}$, we have the following Sobolev inequality:*

$$\|\varphi\|_p^2 \leq \sigma_n \int_{\mathbb{R}^n} |\nabla \varphi|^2 dx, \quad \varphi \in H^1(\mathbb{R}^n). \tag{B.0.1}$$

We call the smallest constant σ_n for which the above inequality is valid the $n-$dimensional Sobolev constant.

(b) Suppose $0 < \alpha < 1$, and $\frac{1}{q} \leq \frac{k-\alpha}{n}$. Then $W^{k,q}(\mathbb{R}^n)$ is continuously embedded in $\mathcal{C}^\alpha(\mathbb{R}^n)$.

One of the advantages of working on compact manifolds is that we can cover the manifold with a finite number of small coordinate patches, apply the known theorems in Euclidean spaces and sum the results using partitions of unity and obtain similar results in the context of the manifold. Using this method we can arrive at the following theorem.

Theorem B.0.7 (Sobolev Embedding Theorems for Compact Manifolds)**.** *Let (M, g) is a compact Riemannian manifold of dimension n.*

(a) If $\frac{1}{r} \leq \frac{1}{q} - \frac{k}{n}$, then $W^{k,q}(M)$ is continuously embedded in $L^r(M)$.

(b) (Rellich-Kondrakov Theorem) Suppose $\frac{1}{r} < \frac{1}{q} - \frac{k}{n}$. Then the inclusion $W^{k,q}(M) \subset L^r(M)$ is a compact operator.

(c) Suppose $0 < \alpha < 1$, and $\frac{1}{q} \leq \frac{k-\alpha}{n}$. Then $W^{k,q}(M)$ is continuously embedded in $\mathcal{C}^\alpha(M)$.

One may wonder whether the Sobolev inequality still holds in compact manifolds and if the best Sobolev constant remains unaltered, and it turns out that it does not, but due to Aubin we get a kind of perturbed Sobolev inequality, this is the content of the next proposition due to Aubin in [1]:

Proposition B.0.8 (Aubin)**.** *Let (M, g) be a compact Riemannian manifold, $2^* = \frac{2n}{n-2}$ and denote by σ_n the sharp constant in the Sobolev inequality in $\mathbb{R}^n$. Then, for every $\epsilon > 0$ there is a positive constant C_ϵ, such that for all $\varphi \in \mathcal{C}^\infty(M)$,*

$$\|\varphi\|_{2^*}^2 \leq (1+\epsilon)\sigma_n \int_M |\nabla\varphi|^2 dV_g + C_\epsilon \int_M \varphi^2 dV_g.$$

We also need another class of spaces.

Definition B.0.9. *Let (N, g) be an asymptotically flat manifold (as defined in Definition 4.2.4), with asymptotic coordinates $\{z^i\}$ on N_∞. Let $\rho(z) = |z|$ on N_∞, extended to a smooth positive function on all of N. For $\beta \in \mathbb{R}$, $k \in \mathbb{N}_0$, define the weighted $\mathcal{C}^k$ space $\mathcal{C}^k_\beta(N)$ as the set of $\mathcal{C}^k$ functions u for which the norm*

$$\|u\|_{\mathcal{C}^k_\beta} = \sum_{i=0}^{k} \sup_{x \in N} \rho^{-\beta+i} |\nabla^i u(x)|$$

is finite. Also, define the weighted Hölder space $\mathcal{C}^{k,\alpha}_\beta(N)$ $(\alpha \in (0,1))$ as the set of $u \in \mathcal{C}^k_\beta(N)$ for which the norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_\beta} = \|u\|_{\mathcal{C}^k_\beta} + \sup_{x,y \in N} (\min \rho(x), \rho(y))^{-\beta+k+\alpha} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}$$

is finite. (The supremum, as in the definition of the Hölder spaces $\mathcal{C}^{k,\alpha}(M)$, is over all points $x \neq y$, such that y is contained in a normal coordinate neighbourhood of x, $\nabla^k u(y)$ is the tensor at x obtained by parallel transport along the radial geodesic from x to y.)

So far we have made precise what is the natural (coordinate-free) notion of derivative in Riemannian manifolds, in particular, we have seen that in compact Riemannian manifolds, the Sobolev spaces are well-defined in the sense that they do not depend on the choice of metric, and are, therefore, natural to the manifold. Like in the Euclidean case there is, in the Riemannian manifold setting, a notion of a weak solution to a partial differential equation. Now we have the necessary material to define what is a weak or distributional solution to a partial differential equation on a Riemannian manifold. We start with the definition of a linear differential operator.

Definition B.0.10. *Let (M, g) be a compact Riemannian manifold. A linear differential operator A of order m on M, written in local coordinates $\{x^i\}$, is an expression of the form:*

$$A(u) = \sum_{l=0}^{m} a_l^{\alpha_1,\ldots,\alpha_l} \nabla_{\alpha_1\alpha_2\ldots\alpha_l} u,$$

where a_l are $l-$tensors, $u \in \mathcal{C}^m(M)$, and $\nabla_{\alpha_1\alpha_2\ldots\alpha_l} u$ is short for $\nabla_{\frac{\partial}{\partial x^{\alpha_1}}} \ldots \nabla_{\frac{\partial}{\partial x^{\alpha_l}}} u$. The terms of the highest order, m, are called the leading part (assuming a_l is nonzero). The operator is said to be elliptic at $x \in U$, if there is $\lambda = \lambda(x) \geq 1$ such that, for all vectors ξ:

$$\|\xi\|^m \lambda(x)^{-1} \leq a_m^{\alpha_1,\ldots,\alpha_m} \xi_{\alpha_1} \ldots \xi_{\alpha_m} \leq \|\xi\|^m \lambda(x).$$

We say that the operator is uniformly elliptic in U if there exist λ_0 and $\lambda(x)$ ($\lambda(x)$ is for each x and λ_0 is independent of the point in U), with $1 \leq \lambda(x) \leq \lambda_0$, such that

$$\|\xi\|^m \lambda_0^{-1} \leq a_m^{\alpha_1,\ldots,\alpha_m} \xi_{\alpha_1} \ldots \xi_{\alpha_m} \leq \|\xi\|^m \lambda_0, \quad \forall \xi \in \mathbb{R}^m, \forall x \in U.$$

Now that we know what a partial differential operator on M is, we can generalise the notion of a weak solution known in the field of partial differential equations in the setting of $\mathbb{R}^n$.
Let A be a linear differential operator of order $2m$ defined in a Riemannian manifold M. Until now, by a solution of $A(u) = f$ the only thing that made sense is a function $u \in \mathcal{C}^{2m}(M)$, such that the equation is satisfied pointwise, now we generalise (or weaken) the notion of solution. If $f \in L^q(M)$ and if the coefficients of A are measurable and locally bounded, we say that $u \in W^{2m,q}(M)$ is a strong solution in the L^q sense of $A(u) = f$ if there is a sequence $\{\varphi_i\}$ of $\mathcal{C}^\infty$ functions such that

$$\varphi_i \to u, \ \text{ in } \ W^{2m,q}(M), \ \text{ and } \ A(\varphi_i) \to f, \ \text{ in } \ L^q(M).$$

Indeed, in this case, the weak derivatives of u up to order $2m$ are functions in $L^q(M)$ and $A(u) = f$ almost everywhere.
Now let $A(u) = a_l \nabla^l u$. If the tensors $a_l \in \mathcal{C}^l(M)$ for $0 \leq l \leq 2m$, then we define the formal adjoint of A by

$$A^*(\varphi) = (-1)^l \nabla^l (\varphi a_l).$$

We say that $u \in L^1(M)$ is a solution, in the sense of distributions, of $A(u) = f$ if for all $\varphi \in \mathcal{C}_c^\infty(M)$:

$$\int_M u A^*(\varphi) dV_g = \int_M f \varphi dV_g.$$

If the coefficients $a_l \in \mathcal{C}^\infty(M)$, then a distribution u satisfies $A(u) = f$ if for all $\varphi \in \mathcal{C}_c^\infty(M)$:

$$\langle u, A^*(\varphi) \rangle = \langle f, \varphi \rangle.$$

If the operator can be written in divergence form, i.e., if we can write $A(u)$ as

$$A(u) = \sum_{\substack{0\leq k\leq m\\0\leq l\leq m}} \nabla_{\alpha_1\dots\alpha_k}(a_{k,l}^{\alpha_1\dots\alpha_k\beta_1\dots\beta_l}\nabla_{\beta_1\dots\beta_l}u) + \sum_{l=0}^{m} b_l\nabla^l u,$$

where $a_{k,l}$ are $k+l-$tensors and b_l are $l-$tensors. Then $u \in W^{m,q}(M)$ is said to be a weak solution of $A(u) = f$ with $f \in L^1(M)$ if for all $\varphi \in \mathcal{C}_c^\infty(M)$:

$$\sum_{\substack{0\leq k\leq m\\0\leq l\leq m}} (-1)^k \int_M a_{k,l}\nabla^l u\nabla^k\varphi dV_g + \sum_{l=0}^{m}\int_M \varphi b_l\nabla^l u dV_g = \int_M f\varphi dV_g.$$

Here we only need $a_{k,l}$ to be measurable and locally bounded for all pairs (k,l). The following is from [8].

Theorem B.0.11 (Local Elliptic Regularity)**.** *Let Ω be an open set in $\mathbb{R}^n$, Δ be the Laplace-Beltrami operator with respect to any metric on Ω, and $u \in L^1_{loc}(\Omega)$ a weak solution to $\Delta u = f$.*

(a) If $f \in W^{k,q}(\Omega)$, then $u \in W^{k+2,q}(K)$, for any compact set $K \Subset \Omega$, and if $u \in L^q(\Omega)$ then

$$\|u\|_{W^{k+2,q}(K)} \leq C(\|\Delta u\|_{W^{k,q}(\Omega)} + \|u\|_{L^q(\Omega)}).$$

(b) (Schauder estimates) If $f \in \mathcal{C}^{k,\alpha}(\Omega)$, then $u \in \mathcal{C}^{k+2,\alpha}(K)$, for any compact subset $K \Subset \Omega$, and if $u \in \mathcal{C}^\alpha(\Omega)$ then

$$\|u\|_{\mathcal{C}^{k+2,\alpha}(K)} \leq C(\|\Delta u\|_{\mathcal{C}^{k,\alpha}(\Omega)} + \|u\|_{\mathcal{C}^\alpha(\Omega)}).$$

Using the procedure described above with partitions of unity, one is able to prove the following.

Theorem B.0.12 (Global Elliptic Regularity)**.** *Let (M,g) be a Riemannian manifold, and suppose $u \in L^1_{loc}(M)$ is a weak solution to $\Delta u = f$.*

(a) If $f \in W^{k,q}(M)$, then $u \in W^{k+2,q}(M)$, and

$$\|u\|_{W^{k+2,q}(M)} \leq C(\|\Delta u\|_{W^{k,q}(M)} + \|u\|_{L^q(M)}).$$

(b) If $f \in \mathcal{C}^{k,\alpha}(M)$, then $u \in \mathcal{C}^{k+2,\alpha}(M)$, and

$$\|u\|_{\mathcal{C}^{k+2,\alpha}(M)} \leq C(\|\Delta u\|_{\mathcal{C}^{k,\alpha}(M)} + \|u\|_{\mathcal{C}^\alpha(M)}).$$

Theorem B.0.13 (Strong Maximum Principle)**.** *Let h be a nonnegative, smooth function on a connected Riemannian manifold (M,g), and $u \in \mathcal{C}^2(M)$ satisfies $\Delta_g u + hu \geq 0$. If u achieves its minimum, and this minimum is nonpositive, then u is constant on M.*

The following, as seen in [13], will be useful to solve the Yamabe problem in the standard sphere.

Theorem B.0.14 (Weak Removable Singularities)**.** *Let U be an open set in M and $P \in U$. Suppose u is a weak solution of $\Delta u + hu = 0$ on $U \setminus \{P\}$, with $h \in L^{n/2}(U)$ and $u \in L^q(U)$ for some $q > p/2$. Then u satisfies $\Delta u + hu = 0$ weakly on all of U.*

Proof. For a proof of this theorem see [13]. □

The following definition and theorem, as seen in [8], will be useful in the analysis of the Yamabe problem in the sphere.

Definition B.0.15. *Let $\Omega \subset \mathbb{R}^n$ be a domain. And let L be a differential operator of the form:*

$$Lu = a_{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + b_i(x)\frac{\partial u}{\partial x^i} + c(x)u, \tag{B.0.2}$$

with coefficients a_{ij}, b_i, c, where $i, j = 1, ..., n$. Consider the equation

$$Lu = f,$$

where f is a function on Ω. We say that u is a strong solution to the previous equation if u is a twice weakly differentiable function that satisfies the equation at almost every point.

Theorem B.0.16 (Proposition 9.15 in [8])**.** *Let $\Omega \subset \mathbb{R}^n$ be a $\mathcal{C}^{1,1}$ domain. Let L be a strictly elliptic differential operator of the form* (B.0.2) *in Ω with coefficients $a_{ij} \in \mathcal{C}(\overline{\Omega})$, $b_i, c \in L^\infty$, with $i, j = 1, ..., n$ and $c \leq 0$. Then if $f \in L^p(\Omega)$ and $\varphi \in W^{2,p}(\Omega)$, with $1 < p < \infty$, the Dirichlet problem $Lu = f$ in Ω, $u - \varphi \in W_0^{2,p}(\Omega)$ has a unique solution $u \in W^{2,p}(\Omega)$.*

Definition B.0.17. *Let $n \geq 3$. We define the space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ to be the subspace of $L^{2^*}(\mathbb{R}^n)$ of functions that have $L^2(\mathbb{R}^n)$ gradient:*

$$\mathcal{D}^{1,2}(\mathbb{R}^n) := \{u \in L^{2^*}(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n)\}$$

Lemma B.0.18. *Let $n \geq 3$. The space $\mathcal{D}^{1,2}(\mathbb{R}^n)$ with inner product*

$$\langle u, v\rangle = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx, \tag{B.0.3}$$

is a Hilbert space.

Proof. In order to prove that $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is a Hilbert space with respect to the norm induced by (B.0.3), we prove it is a Banach space, and to that end, we prove that the space of compactly supported smooth functions is dense in $\mathcal{D}^{1,2}(\mathbb{R}^n)$.

Let $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_1(0)$ and $\varphi \equiv 0$ in $B_2(0)^c$. For $R > 0$ define the function $\varphi_R(x) = \varphi(x/R)$. We claim that $u\varphi_R \in H^1(\mathbb{R}^n)$. Firstly, $u\varphi_R \in L^2(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} u^2\varphi_R^2 dx \leq \left(\int_{\mathbb{R}^n} u^{2^*} dx\right)^{\frac{n-2}{n}} \left(\int_{\mathbb{R}^n} \varphi_R^n dx\right)^{\frac{2}{n}} < \infty.$$

As for the square integrability of the gradient, we have the following

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla(u\varphi_R)|^2 dx &= \int_{\mathbb{R}^n} u^2|\nabla\varphi_R|^2 + \varphi_R^2|\nabla u|^2 + 2u\varphi_R \nabla u \cdot \nabla\varphi_R dx \\ &\leq \|\nabla\varphi_R\|_n^2\|u\|_{2^*}^2 + \|\varphi_R\|_\infty^2\|\nabla u\|_2^2 + 2\|u\varphi_R\|_2\||\nabla u||\nabla\varphi_R|\|_2 \leq \infty.\end{aligned}$$

So that $\nabla(u\varphi_R) \in L^2(\mathbb{R}^n)$, and subsequently, $u\varphi_R \in H^1(\mathbb{R}^n)$.

Having now shown that we can, in some sense, descend from $\mathcal{D}^{1,2}(\mathbb{R}^n)$ into $H^1(\mathbb{R}^n)$, we make the stronger claim that $H^1(\mathbb{R}^n)$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. In fact, we want to show that $u\varphi_R \to u$, as $R \to +\infty$, in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. We denote by $\|\cdot\|$ the norm in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ induced by the inner product (6.2). Now we have

$$\begin{aligned} \|u\varphi_R - u\|^2 &= \int_{\mathbb{R}^n} \nabla(u(\varphi_R - 1)) \cdot \nabla(u(\varphi_R - 1))dx \\ &= \int_{\mathbb{R}^n} |\nabla u|^2(\varphi_R - 1)^2 + u^2|\nabla\varphi_R|^2 + 2u(\varphi_R - 1)\nabla u \cdot \nabla\varphi_R dx, \end{aligned} \tag{B.0.4}$$

to see that $\|u\varphi_R - u\|^2 \to 0$, as $R \to +\infty$, we analyse each of the terms in (B.0.4) independently. The first term converges to zero by dominated convergence. For the second term, we have

$$\begin{aligned} \int_{\mathbb{R}^n} u^2|\nabla\varphi_R|^2 dx = \int_{\{R\le|x|\le 2R\}} u^2|\nabla\varphi_R|^2 dx &\le \left(\int_{\{R\le|x|\le 2R\}} u^{2^*} dx\right)^{\frac{n-2}{n}} \left(\int_{\{R\le|x|\le 2R\}} |\nabla\varphi_R|^n dx\right)^{\frac{2}{n}} \\ &= \left(\int_{\{R\le|x|\le 2R\}} u^{2^*} dx\right)^{\frac{n-2}{n}} \left(\int_{\{1\le|x|\le 2\}} |\nabla\varphi|^n dx\right)^{\frac{2}{n}}, \end{aligned}$$

which converges to zero as R approaches infinity because $u \in L^{2^*}(\mathbb{R}^n)$. Finally, for the third term, we have

$$\int_{\mathbb{R}^n} 2u(\varphi_R - 1)\nabla u \cdot \nabla\varphi_R dx \le \||\nabla u|(\varphi_R - 1)\|_2 \|u|\nabla\varphi_R|\|_2,$$

which, by the arguments used for the second term converges to zero by dominated convergence as $R \to \infty$. This allows us to conclude that $u\varphi_R \to u$, as $R \to \infty$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Hence, $H^1(\mathbb{R}^n)$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. Furthermore, the density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n)$ allows us to conclude that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}^{1,2}(\mathbb{R}^n)$.

Now let $\{u_m\}$ be a Cauchy sequence in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, and let $\{v_m\}$ be a sequence in $\mathcal{D}(\mathbb{R}^n)$ such that $\|u_m - v_m\| \le \frac{1}{2^m}$ for all $m \in \mathbb{N}$. Then $\{v_m\}$ is also a Cauchy sequence in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, then by the Gagliardo-Nirenberg inequality we have

$$\|v_m - v_k\|_{2^*} \le C\|\nabla(v_m - v_k)\|_2 = C\|v_m - v_k\|,$$

and therefore, $\{v_m\}$ is a Cauchy sequence in $L^{2^*}(\mathbb{R}^n)$. Now let $u \in L^{2^*}(\mathbb{R}^n)$ be the limit of the sequence $\{v_m\}$ in $L^{2^*}(\mathbb{R}^n)$. Moreover, we have that the sequences of partial derivatives of $\{v_m\}$, $\{\frac{\partial v_m}{\partial x_i}\}$ $(i = 1, ..., n)$, are Cauchy sequences in $L^2(\mathbb{R}^n)$, and subsequently, there are $g_1, ..., g_n \in L^2(\mathbb{R}^n)$ such that

$$\left\|\frac{\partial v_m}{\partial x_i} - g_i\right\|_2 \to 0, \text{ as } m \to +\infty,$$

for all $i = 1, ..., n$.

It is easy to see that the (weak) partial derivatives of u are the functions g_i, that is $\frac{\partial u}{\partial x_i} = g_i$ for all $i = 1, ..., n$. And so

$$v_m \to u, \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^n), \tag{B.0.5}$$

which implies, via the triangle inequality, that $u_m \to u$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. $\square$

Remark B.0.19. *A consequence of the previous proof is the fact that the Sobolev inequality still holds in* $\mathcal{D}^{1,2}(\mathbb{R}^n)$.

We finish with a crucial regularity result as seen in [13]

Theorem B.0.20 (Regularity)**.** *Suppose* $\varphi \in H^1(M)$ *is a nonnegative weak solution of* (3.1.2) *with* $2 \leq s \leq 2^*$*, and* $|\lambda_s(M)| \leq K$ *for some constant* $K \in \mathbb{R}$*. If* $\varphi \in L^r(M)$ *for some* $r > (s-2)n/2$ *(in particular, if* $r = s < 2^*$*, or if* $s = 2^* < r$*), then* φ *is either identically zero or strictly positive and smooth, and* $\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leq C$*, for some* $\alpha \in (0,1)$*, with* $C = C(M, g, K, \|\varphi\|_r)$.

Proof. For a proof see [13]. $\square$