Two dimensional Tangent Euler Top: classical and relativistic dynamics

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Abstract

We start by reviewing the classical Euler Top and introduce the Tangent Euler Top system as an extension for curved spaces. We study the Liouville integrability of this system in two dimensional manifolds and find that the system is integrable if there is at least one Killing vector field on the manifold. Additionally, we show that this system defines a metric on the the bundle of positive orthonormal frames. We also explore the analogy between this problem and the motion of a charged particle on a magnetic field, the dynamics of this system when the manifold is embedded and study the isometry groups of the maximally symmetric manifolds: $\mathbb{R}^2$, $\mathbb{H}^2$ and S^2. We extend our results to the relativistic top and we find that in the relativistic case there is a coupling between the centre of mass velocity and the angular velocity of the top. Finally, we analyse the integrability in higher dimensional systems. We conclude that the classical system is integrable at least up to three dimensions and we argue that the main difficulty in higher dimensions arises from $SO(n)$ dimension compared to the number of first integrals related to this group.

Contents

I. INTRODUCTION

As pointed out in ([1]), if one considers the motion of an arbitrary rigid body on a curved space, it would be impossible to keep the distances between the different particles which make up the body constant (hence, the rigid body condition would be violated). For instance, if one considers the free motion of a body composed of three point masses on the surface of a sphere, it easy to see that it is always possible to maintain the distances between the points as the body moves on the surface. However, if one considers an arbitrary surface and one increases the number of points that make up the body, this assertion fails and the motion is impossible.

One way to solve the problem is to approximate the motion of the rigid body on a curved space by considering the motion of an orthonormal frame which is tangent to the surface at each point. The motion can be understood therefore as the motion of the centre of mass plus the rotation of the frame around that point. Obviously, this automatically solves the problem pointed out above.

We begin by reviewing the classical Euler top system ([2],[3],[4]) and then present the model proposed by ([1]). Next, we study the system's integrability and further explore this model.

One of the main interests of this approximation is to study the motion of a rigid body in general relativity. Therefore, we extend our model to a relativistic tangent Euler top, as seen by a distant observer. We explore the motion for different metrics.

Finally, we try to identify the key points when considering an extension for an higher dimensional problem. To do this we have to resort to more general arguments than used before and the assertion is harder to prove, mainly because the dimension of $SO(n)$ and the number of first integrals do not scale up at the same rate.

We make use of the Einstein's summation convention (except when "no sum" is indicated).

II. FROM THE EULER TOP TO THE TANGENT EULER TOP

A. The Euler Top

Let us first review the classical Euler top. We will consider the motion the free rigid body, of mass m, rotating around its centre of mass. Since the system is finite and we are in the centre of mass frame we get

$$\begin{cases} \int_{\mathbb{R}^3} \xi dm = 0 \\ \int_{\mathbb{R}^3} ||\xi||^2 dm < \infty \end{cases} \tag{1}$$

We can further define the Euler tensor (I), which is represented by a symmetric matrix given by

$$I_{ij} = \int_{\mathbb{R}^3} \xi^i \xi^j dm \tag{2}$$

We recall that a rigid body with a fixed point leads to a holonomic constraint given by

$$N = \left\{(x_1, ..., x_k) \in \mathbb{R}^{3k} | x_1 = 0 \text{ and } ||x_i - x_j|| = d_{ij} \text{ for } 1 \leq i < j \leq k\right\} \tag{3}$$

for a system of k particles with constant distance between them d_{ij}. It easy to see that if at least three particles are not collinear, then given a fixed point in N every other point in N can be written as $(S\xi_1, ..., S\xi_k)$ for a unique $S \in SO(3)$. This implies that rigid body motion can be seen as a curve on $SO(3)$. In order to consider a continuum rigid body (i.e. for an infinite number of particles) it is necessary to make the following generalisation:

Definition II.1. *A rigid body with a fixed point is any mechanical system of the form* $(SO(3), \langle\langle ., . \rangle\rangle, F)$*, with*

$$\langle\langle X, Y \rangle\rangle = \int_{\mathbb{R}^3} \langle X\xi, Y\xi \rangle dm \tag{4}$$

where $X, Y \in T_S SO(3)$ *and* $S \in SO(3)$*, taking* $\langle ., . \rangle$ *to be the usual Euclidean metric on* $\mathbb{R}^3$*. In the case of the Euler top,* F *vanishes since it is a free motion.*

A useful proposition is the following

Proposition II.1. *The metric defined by the rigid body with a fixed point on* $SO(3)$ *is given by*

$$\langle\langle X, Y \rangle\rangle = \operatorname{tr}(XIY^t) \tag{5}$$

Proof. We can directly expand the inner product between vectors X, Y using the rigid body metric obtaining

$$\langle\langle X, Y \rangle\rangle = \int_{\mathbb{R}^3} X_{ij} \xi^j Y_{ik} \xi^k dm \tag{6}$$

Only the ξ variables have to be integrated. Therefore, we get the desired result

$$\langle\langle X, Y \rangle\rangle = X_{ij} I_{jk} Y_{ik} \tag{7}$$

□

Since the Euler top system defines a curve on $SO(3)$, the Lagrangian of the system has to be defined on $TSO(3)$ and it is given by

$$L(S,\dot{S}) = \frac{1}{2}\langle\langle \dot{S},\dot{S}\rangle\rangle = \frac{1}{2}\,\mathrm{tr}(\dot{S}I\dot{S}^t) \tag{8}$$

The Euler top is a completely integrable system ([2],[3],[4]). In addition to the system's energy ($H = L$), there are first integrals coming from the conservation of angular momentum. In particular, the square of the angular momentum is conserved as well as the values in each component. Since the components of angular momentum do not commute under the Poisson bracket, we get two first integrals which commute plus the Hamiltonian, making the system integrable.

B. The Tangent Euler Top on curved space

We now consider the tangent Euler top. As stated before, we consider the motion of a rigid body over a curved manifold, approximating the body by a tangent frame which rotates around the centre of mass.

For this reason, in order to describe our system we need to specify the position of the centre of mass of the body as it moves on the manifold (M) and at the same time we can describe the angular part by defining at each point of the trajectory of the body on M an orthonormal frame. The configuration space will have to store information from both the curve of the centre of mass on M and vector fields which at each point of that curve define an orthonormal frame.

For these reason, we define OM as being the bundle of orthonormal frames on an 3-dimensional Riemannian manifold $(M, \langle .,.\rangle)$. A curve $E : \mathbb{R} \to OM$ on this space is given by

$$E(t) = (c(t), E_1(t), E_2(t), E_3(t)) \tag{9}$$

where $c : \mathbb{R} \to M$ is a curve on M and the fields $E_j(t)$ define an orthonormal frame at $c(t)$ for each $t \in \mathbb{R}$.

The Lagrangian is defined in TOM and the motion of the tangent Euler top is an Euler-Lagrange curve for the Lagrangian $L : TOM \to R$ given by

$$L(E,\dot{E}) = \frac{1}{2}\int_{\mathbb{R}^3} \langle \dot{c} + \xi^i \nabla_{\dot{c}} E_i, \dot{c} + \xi^j \nabla_{\dot{c}} E_j\rangle dm \tag{10}$$

where ∇ is the Levi-Civita connection on M.

The tangent Euler top Lagrangian is more complex than the Lagrangian for the classical Euler top. Apart from the $\dot{c}$ term, which only takes into account the fact that we are considering the motion of the centre of mass, we see that we now have to consider the covariant derivative of the vector fields which give the orthonormal frame along $c(t)$. Clearly, for flat space this is just equal to the classical Euler top, but for an arbitrary space the Lagrangian will now take into account the way the vector fields change from point to point due to the geometry of M.

The Lagrangian of the system might be written in a more explicit way. For that, we follow the steps presented in ([1]) when deriving the equation of motion of the system. The first step is to consider a trivialisation $OM|_U \cong U \times SO(3)$ by choosing a local orthonormal frame $\{\hat{E}_1, \hat{E}_2, \hat{E}_3\} \subset \mathfrak{X}(U)$ on a sufficiently small open set $U \subset M$. We describe the rotation of the moving frame, denoted by the E, with respect the frame $\hat{E}$. This can be expressed as

$$E_i(t) = S_{ji}(t)(\hat{E}_j)_{c(t)} \tag{11}$$

We can now compute the covariant derivative of the field $E_i(t)$ as

$$\nabla_{\dot{c}} E_i = \dot{S}_{ji}(\hat{E}_j)_{c(t)} + S_{ji}\hat{\omega}_j^k(\dot{c})\hat{E}_k \tag{12}$$

where $\hat{\omega}_j^k$ are the connection forms associated to our local frame. This implies that the Lagrangian of the system can be written in a simple fashion

$$L = T + K + C + F \tag{13}$$

The first term T corresponds to the usual translational kinetic energy and is given by

$$T = \frac{1}{2}\int \langle \dot{c}, \dot{c}\rangle dm = \frac{1}{2}m\langle \dot{c}, \dot{c}\rangle \tag{14}$$

The K term represents the usual rotational kinetic energy and can be expressed by

$$K = \frac{1}{2}\int \xi^i \dot{S}_{ki}\xi^j \dot{S}_{lj}\langle \hat{E}_k, \hat{E}_l\rangle dm = \frac{1}{2}\,\mathrm{tr}(\dot{S}I\dot{S}^t) \tag{15}$$

The two following terms are due to the fact that the space is not flat.

$$C = \int \xi^i \dot{S}_{ki}\xi^j S_{lj}\hat{\omega}_l^m(\dot{c})\langle \hat{E}_k, \hat{E}_m\rangle dm = \mathrm{tr}(\dot{S}IS^t\hat{\omega}(\dot{c})) \tag{16}$$

$$F = \frac{1}{2}\int \xi^i \dot{S}_{ki}\hat{\omega}_k^l(\dot{c})\xi^j S_{mj}\hat{\omega}_m^n(\dot{c})\langle \hat{E}_l, \hat{E}_n\rangle dm = \frac{1}{2}\,\mathrm{tr}(\hat{\omega}(\dot{c})SIS^t\hat{\omega}(\dot{c})) \tag{17}$$

By inspection of these equations it is easy to verify that if the reference frame happens to be parallel transported along c, then the connection forms must vanish and therefore we are left with T and K. This shows that the orthonormal frame defined by the E_i vector fields rotates exactly as an Euler top with respect to the reference frame.

C. The Tangent Euler Top metric on OM

In this section we will study how the Lagrangian of the system changes under the choice of reference frame to see whether or not the Lagrangian defines a metric on OM. We will prove these results for an n-dimensional Lagrangian.

If we consider a change of choice of the reference frame, this will affect the S, $\dot{S}$ and $\hat{\omega}$ terms of the Lagrangian. Firstly we define a new reference frame given by the vector fields $\tilde{E}_i$. This reference frame is related to the $\hat{E}_i$ frame by a rotation, therefore we can write

$$\hat{E}_i = R_{ji}\tilde{E}_j \iff \tilde{E}_j = R_{jk}\hat{E}_k \tag{18}$$

where R is the orthogonal transformation matrix. In relation to the rotating E_i frame we can write

$$E_i = S_{ji}\hat{E}_j = \tilde{S}_{ji}\tilde{E}_j \tag{19}$$

This automatically implies

$$E_i = S_{ji}\hat{E}_j = S_{ji}R_{kj}\tilde{E}_k = R_{jk}S_{ki}\tilde{E}_j \tag{20}$$

and so $\tilde{S}$ has to be given by

$$\tilde{S} = RS \text{ and } \dot{\tilde{S}} = \dot{R}S + R\dot{S} \tag{21}$$

Additionally, we also need to know how the connection forms change. The general expression for the transformation of the connection forms between two frames is given by

$$\tilde{\omega} = R\hat{\omega}R^t + RdR^t \tag{22}$$

Notice that

$$RR^t = Id \Rightarrow dRR^t + RdR^t = 0 \iff RdR^t = -dRR^t \tag{23}$$

In our case we will have terms $dR(\dot{c}) = \dot{R}$.

We have now all the necessary ingredients to write the Lagrangian of the system when using the tilde reference frame, which we will name $\tilde{L}$. Let us just remember that the Lagrangian are equivalent if they differ from one another by a total time derivative. In this case we can write that

$$\tilde{L} = L + \frac{dG}{dt} \tag{24}$$

We start by expanding the $\tilde{K}$, $\tilde{C}$ and $\tilde{F}$ terms in $\tilde{L}$.

$$\tilde{K} = \mathrm{tr}(\dot{\tilde{S}} I \dot{\tilde{S}}^t) = K + \frac{1}{2}\,\mathrm{tr}(\dot{R} S I \dot{S}^t R^t) + \frac{1}{2}\,\mathrm{tr}(R \dot{S} I S^t \dot{R}^t) + \frac{1}{2}\,\mathrm{tr}(\dot{R} S I S^t \dot{R}^t) \tag{25}$$

$$\tilde{C} = C + \mathrm{tr}(\dot{S} I S^t R^t \dot{R}) + \mathrm{tr}(\dot{R} S I S^t \hat{\omega}^t(\dot{c}) R^t) + \mathrm{tr}(\dot{R} S I S^t R^t \dot{R} R^t) \tag{26}$$

$$\tilde{F} = F + \frac{1}{2}\,\mathrm{tr}(\hat{}(\dot{c}) S I S^t R^t \dot{R}) + \frac{1}{2}\,\mathrm{tr}(\dot{R}^t R S I S^t \hat{\omega}^t(\dot{c})) + \frac{1}{2}\,\mathrm{tr}(\dot{R}^t R S I S^t R^t \dot{R}) \tag{27}$$

Thus, if we add the first term in each equation we get L. We will now prove that the remaining terms vanish.

We will make extensive use of following identity

$$\dot{R}^t R = -R^t \dot{R} \iff R \dot{R}^t = -\dot{R} R^t \tag{28}$$

which was proven above.

We sum the last term of each equation. The second term can be written as $-\,\mathrm{tr}(\dot{R} S I S^t \dot{R}^t)$ and the two other terms add up to $\mathrm{tr}(\dot{R} S I S^t \dot{R}^t)$, so the net result is zero. Now, we add up the three terms which contain the connection forms. The two terms coming from the $\tilde{F}$ add up to $\mathrm{tr}(\dot{R}^t R S I S^t \hat{\omega}^t)$, while the term above is symmetric to it so, again, this contribution disappears. By making the same manipulation as before, the remaining terms add up to zero.

This result means that $\tilde{L} = L$. Therefore, the Lagrangian is a geodesic Lagrangian, which means that we are just studying geodesics for the metric defined by the tangent Euler top on OM (which is given in equation (47), in two dimensions).

D. Motion in embedded manifolds

So far, we have only considered the motion on a manifold which is not embedded in a bigger space. The embedding will alter the Lagrangian of the system. In fact, it is easy to see that there will be extra terms which will account for the fact that we are now using the connection of the embedding manifold. We can write

$$\tilde{\nabla}_X E = (\tilde{\nabla}_X E)^\top + (\tilde{\nabla}_X E)^\perp = \nabla_X E + B(X, E) \tag{29}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the ambient manifold, ∇ is the Levi-Civita connection induced on the submanifold and B is the second fundamental form. It is easy to prove that the second fundamental form does not depend on the extensions of the fields X and Y. Furthermore, given a point p on the submanifold (N) the tangent space of the embedding manifold (M) can be decomposed as

$$T_pM = T_pN \oplus (T_pN)^\perp \tag{30}$$

This implies that any vector can be decomposed in a component which is in T_pN and a normal component to that tangent space. The second fundamental form $B(X,Y)_p \in (T_pN)^\perp$ for each $p \in V$ depends only on the values of X and Y at p, and it is billinear and symmetric.

Using the second fundamental form is also possible to define for each vector $n_p \in (T_pN)^\perp$, a symmetric billinear map $H_p : T_pN \times T_pN \to \mathbb{R}$ which is given by

$$H_p(X,Y) = \langle B_p(X,Y), n_p \rangle \tag{31}$$

We can now rewrite the Lagrangian for the tangent Euler top system. As stated above, the new terms will come from the fact that we are considering that our manifold is embedded in a bigger space, and as just shown now, the only difference comes from the fact that we have to consider the connection of the bigger space, which we have just shown can be written in terms of the submanifold connection and the second fundamental form:

$$L(E,\dot{E}) = \frac{1}{2}\int_{\mathbb{R}^3} \langle \dot{c} + \xi^i(\nabla_{\dot{c}}E_i + B(\dot{c},E_i)), \dot{c} + \xi^j(\nabla_{\dot{c}}E_j + B(\dot{c},E_j))\rangle dm \tag{32}$$

We will now follow the same procedure as in ([1]) and expand our Lagrangian. Our task is greatly simplified firstly because most terms will give the same contributions as before.

$$L(E,\dot{E}) = L_0 + \frac{1}{2}\int \langle \xi^i B(\dot{c},E_i), \xi^j B(\dot{c},E_j)\rangle + \int \langle \dot{c} + \xi^i \nabla_{\dot{c}}E_i, B(\dot{c},E_j)\rangle \tag{33}$$

where L_0 is the Lagrangian we obtained in the previous sections. This result can be further simplified since the third term is clearly equal to zero due to the orthogonal relation between the second fundamental form and the vector fields which define the frames at each point.

This implies that the embedding of the manifold in another space only contributes with one term to the Lagrangian.

We are now ready to further examine this Lagrangian. For that we note that

$$B(\dot{c}, E_i) = S_{ji} B(\dot{c}, \hat{E}_j) \tag{34}$$

We can always write the vector that defines the reference frame as $\hat{E}^\alpha_j \frac{\partial}{\partial x^\alpha}$, which enables us to write

$$B(\dot{c}, E_i) = S_{ji} \hat{E}^\alpha_j(x) B_{\mu\alpha}(x) \dot{x}^\mu \tag{35}$$

Returning to the new term in the Lagrangian we now have

$$\frac{1}{2}\int \langle \xi^i B(\dot{c}, E_i), \xi^j B(\dot{c}, E_j)\rangle = \frac{1}{2}\int_R \xi^i S_{ki} \hat{E}^\alpha_k(x) B_{\mu\alpha}(x) \dot{x}^\mu \xi^j S_{lj} \hat{E}^\beta_l(x) B_{\nu\beta}(x) \dot{x}^\nu dm \tag{36}$$

All terms except for the ξ are independent of the integration, thus the integral will only contribute with a term corresponding to the Euler tensor. Manipulating the terms above it is easy to get to the final expression

$$\frac{1}{2}\int \langle \xi^i B(\dot{c}, E_i), \xi^j B(\dot{c}, E_j)\rangle = \frac{1}{2}(SIS^t)_{kl} \hat{E}^\alpha_k(x) \hat{E}^\beta_l(x) B_{\mu\alpha}(x) B_{\nu\beta}(x) \dot{x}^\mu \dot{x}^\nu \tag{37}$$

So, as expected, the Lagrangian for the tangent Euler top moving on an embedded manifold will be different compared to the case where the manifold is not embedded. In particular, we see that the integrability of the system is now much more complicated because it depends on the way the embedding is done.

It is interesting to look at the particular case of a spherically symmetric body, i.e. the Euler tensor is represented by a matrix proportional to the the identity $I = I_0 I_d$. Additionally, we can make the following simplification

$$\hat{E}^\alpha_k \hat{E}^\beta_k = g^{\alpha\beta} \tag{38}$$

which holds since $(\hat{E}^\alpha_k \hat{E}^\beta_k)(\hat{\omega}^i_\alpha, \hat{\omega}^j_\beta) = \delta^k_i \delta^k_j = \delta_{ij}$. Using the result previously obtained we then conclude that for the symmetric top the additional term in the Lagrangian is given by

$$\frac{1}{2}\int \langle \xi^i B(\dot{c}, E_i), \xi^j B(\dot{c}, E_j)\rangle \Big|_{sym.\ top} = \frac{I_0}{2} g^{\alpha\beta}(x) B_{\alpha\mu}(x) B_{\beta\nu}(x) \dot{x}^\mu \dot{x}^\nu \tag{39}$$

III. MOTION IN 2-MANIFOLDS

A. The Lagrangian and a sufficient condition for integrability

We will start by considering the case where the top moves on some arbitrary surface (i.e. a 2-manifold). This case is especially simple, because we can easily picture the motion as being as small disk with a certain mass m, which is tangent to the surface at each point, and we let this disk move around freely. Moreover, in this case it is simple to expand the Lagrangian in its components and to solve the problem explicitly (as we will promptly do) while at the same time gaining insight into the problem.

We begin by writing the Lagrangian of the system in its components. Since we are considering a two-dimensional space we will need three variables to describe the system: x^1, x^2 and θ, where the first two are local coordinates on the surface and θ corresponds to a parametrisation of $SO(2)$. The translational kinetic term is simply written as

$$T = \frac{1}{2} m g_{ij} \dot{x}^i \dot{x}^j \tag{40}$$

where g_{ij} is the ij component of the metric. To describe the next terms we need to define S,$\dot{S}$, I and $\hat{\omega}(\dot{c})$ in terms of coordinates. We obtain the following result

$$S = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad ; \quad \dot{S} = \dot{\theta} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{bmatrix} \tag{41}$$

$$I = \begin{bmatrix} I_1 & I_3 \\ I_3 & I_2 \end{bmatrix} \quad ; \quad \mathbb{I} = tr(I) \tag{42}$$

$$\hat{\omega} = \begin{bmatrix} 0 & \alpha(x^1, x^2) dx^1 + \beta(x^1, x^2) dx^2 \\ -\alpha(x^1, x^2) dx^1 - \beta(x^1, x^2) dx^2 & 0 \end{bmatrix} \tag{43}$$

where we introduced the constant $\mathbb{I}$ which is the moment of the inertia of the body. Additionally, we have expressed the connection matrix in terms of unknown functions $\alpha(x^1, x^2)$ and $\beta(x^1, x^2)$. Having this expressions we can compute the remaining terms in the Lagrangian to obtain

$$K = \frac{1}{2} \mathbb{I} \dot{\theta}^2 \tag{44}$$

$$C = -\mathbb{I} \dot{\theta} (\dot{x}^1 \alpha + \dot{x}^2 \beta) \tag{45}$$

$$F = \frac{1}{2}\mathbb{I}(\dot{x}^1\alpha + \dot{x}^2\beta)^2 \tag{46}$$

Combining all the terms we arrive at the desired expression for the Lagrangian

$$L = \frac{1}{2}\dot{\eta}^t \begin{bmatrix} mg_{11} + \mathbb{I}\alpha^2 & mg_{12} + \mathbb{I}\alpha\beta & -\mathbb{I}\alpha \\ mg_{12} + \mathbb{I}\alpha\beta & mg_{22} + \mathbb{I}\beta^2 & -\mathbb{I}\beta \\ -\mathbb{I}\alpha & -\mathbb{I}\beta & \mathbb{I} \end{bmatrix} \dot{\eta} = \frac{1}{2}g_{AB}\dot{\eta}^A\dot{\eta}^B \tag{47}$$

where $\eta = [x^1, x^2, \theta]^t$. It is interesting to notice that this Lagrangian is quadratic in $\dot{\eta}$. Another remarkable feature of the Lagrangian is that $\frac{\partial L}{\partial \theta} = 0$ which, using the Euler-Lagrange equations, automatically implies that there is already an additional non-trivial conserved quantity (which is the angular momentum of the body). In the case of 2-manifolds, we are working in a 3 dimensional system for which we already have two commuting and independent quantities: H and p_θ. Therefore, to have a completely integrable system we only need another commuting and independent first integral. By inspection of the Lagrangian, we verify that the only relevant quantities are α, β and g_{ij}. Thus, to guarantee integrability we just have to prove that none of these three quantities is a function of either x^1 or x^2.

So far we have not imposed any condition on the surface. Nonetheless it wouldn't be very reasonable to expect to have integrable motion for surfaces without any symmetry. Therefore, we will now restrict ourselves to a smaller class of spaces.

Definition III.1. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇, and let $\phi_t : M \to M$ be a 1-parameter group of isometries. The vector field $X \in \mathfrak{X}(M)$ defined by*

$$X_p := \left(\frac{d}{dt}\right)_{t=0} \phi_t(p) \tag{48}$$

is called a Killing vector field associated to ϕ_t.

We will use a simple proposition about Killing vectors (which is sometimes used as a definition of Killing vector field) to prove our result

Proposition III.1. *Let (M, g) be a Riemannian manifold such that exist $n < dim(M)$ non-vanishing independent and commuting Killing vector fields. Then, there exists a local coordinate system such that the components g_{ij} of the metric are independent of n coordinates. The converse also holds.*

Proof. We can choose local coordinates such that the Killing vector fields K_i satisfy

$$K_i = \frac{\partial}{\partial x^i} \tag{49}$$

since the Killing vector fields commute and are independent. Therefore,

$$L_K g_{ab} = \partial_i g_{ab} = 0 \tag{50}$$

To prove converse we notice that equation (50) yields our hypotheses and the result follows.

□

From this point on, we will only consider surfaces which have at least one Killing vector field in the condition of the previous proposition. We will now prove that the existence of only one Killing field is enough to guarantee complete integrability of the system.

We start by making use of the previous result: if our surface has one Killing field one can choose the coordinates as in the proposition. We then choose x^1 to be defined such that the Killing vector field is $\frac{\partial}{\partial x^1}$ and x^2 to be defined in such a way that $\frac{\partial}{\partial x^2}$ is an orthogonal field to the Killing field.

To do this, we consider the orthogonal distribution to the Killing vector field Σ, which is given by the kernel of the 1-form ω, i.e. $\Sigma = ker(\omega)$. Since this is a 2-dimensional manifold, then the distribution is integrable, as $\omega \wedge d\omega = 0$. We can choose one of the leafs of the distribution and use x^2 as a coordinate on it. To construct the coordinates of the form (x^1, x^2) on the surface we use the flow of the Killing vector field. Considering a point x^2 on the chosen leaf, we can use the integral curve associated to K and choose x^1 to be the time along the curve, i.e. if we take a point with coordinates (x^1, x^2) we can move "backwards" along integral curve of K passing by the point (x^1, x^2) for a time x^1, ending up with a point on the leaf we chose with coordinate x^2, on the leaf (figure 1). Since the integral curves can not cross, this system of coordinates is well defined. By construction we have that $K = \frac{\partial}{\partial x^1}$. We now have to check that K and $L = \frac{\partial}{\partial x^2}$ are always orthogonal. To ensure that, we just have to verify that

$$K \cdot \left\langle K, \frac{\partial}{\partial x^2} \right\rangle = 0 \tag{51}$$

This holds since

$$\begin{aligned} K \cdot \left\langle K, \frac{\partial}{\partial x^2} \right\rangle &= \left\langle \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\rangle + \left\langle \frac{\partial}{\partial x^1}, \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^2} \right\rangle \\ &= \langle \nabla_K K, L \rangle + \langle K, \nabla_K L \rangle \end{aligned} \tag{52}$$

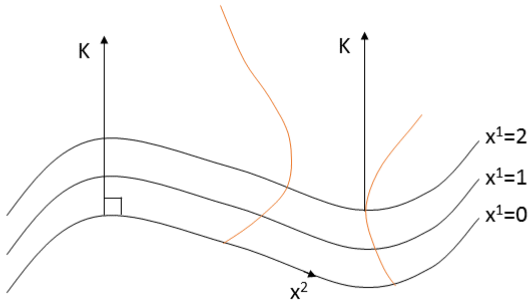

FIG. 1: Depiction of the above argument.

Since $[K, L] = 0$ we have that $\nabla_K L = \nabla_L K$. This implies

$$\begin{aligned}\langle \nabla_K K, L\rangle + \langle K, \nabla_K L\rangle &= \langle \nabla_K K, L\rangle + \langle K, \nabla_L K\rangle \\ &= \nabla_\beta K_\alpha (L^\alpha K^\beta + K^\alpha L^\beta) \\ &= 0\end{aligned} \tag{53}$$

where in the last step we used the fact that the term outside the brackets is anti-symmetric while the term inside the brackets is symmetric.

The above argument implies that the metric tensor will be diagonal and the only remaining entries g_{11} and g_{22} can only be functions of x^2, according to the previous proposition. Therefore, we only have to study the functions α and β to get our result.

In order to do that we choose a particular reference frame $\{\hat{E}_1, \hat{E}_2\}$. We choose $\hat{E}_1$ to be given by the normalised Killing vector field

$$\hat{E}_1 = \frac{K}{||K||} \tag{54}$$

where K is the Killing vector field. We do the same for $\hat{E}_2$, using the orthogonal field. We also define the associated dual forms in such a way that $\hat{\omega}^j(E_i) = \delta_i^j$, as usual. We notice that the frame fields will be equal to the original fields apart from a normalisation function which will only depend on the metric tensor. Therefore, we can write the dual forms in a simple manner: $\hat{\omega}^i = f^i(x^2)dx^i$ (no sum).

We can now write $\hat{\omega}_2^1$ in terms of these forms as

$$\hat{\omega}_2^1 = d\hat{\omega}^1(E_2, E_1)\hat{\omega}^1 + d\hat{\omega}^2(E_2, E_1)\hat{\omega}^2 \tag{55}$$

Finally, we notice that the $d\omega^i$ are the α and β functions defined above and we can compute them directly obtaining

$$\begin{cases} d\hat{\omega}^1 = \frac{\partial f^1(x^2)}{\partial x^2} dx^2 \wedge dx^1 \\ d\hat{\omega}^2 = 0 \end{cases} \tag{56}$$

When applied to E_1 and E_2 it implies that $\beta = 0$ and α is just a function of x^2.

Therefore, the matrix of the connection forms will not depend on x^1 and therefore the Lagrangian will satisfy $\frac{\partial L}{\partial x^1} = 0$, which generates a new conserved quantity which we will call p_{x^1}.

This argument proves that the motion of the tangent Euler top is completely integrable in the case where we have a two dimensional manifold with a Killing vector field. Nonetheless, during this argument we had to fix a reference frame $\{\hat{E}_1, \hat{E}_2\}$. This means that we have to further prove that the solution is invariant under the reference frame we end up choosing. But this is exactly what we did when proving that the tangent Euler Top defines a metric on OM. Therefore the result is proven regardless of the frame we choose.

B. Charged Particle Analogy

We will now prove that the 2-dimensional tangent Euler top system can be understood as the motion of a charged particle in a stationary magnetic field over the same surface.

The classical electromagnetic Lagrangian for a free particle moving under the influence of a constant magnetic field is given by

$$L_{em} = \frac{1}{2} m \hat{g}_{ij} \dot{x}^i \dot{x}^j + q A_i \dot{x}^i \tag{57}$$

where $\hat{g}$ stands for the new metric on M, A is the magnetic potential, which is a 1-form given by $A = A_i dx^i$ (i.e. in the Lagrangian we have the contraction of the magnetic potential with $\dot{c}$) and q is the electric charge. By expanding the general Lagrangian obtained for the motion of the tangent Euler top on a surface we can regroup the terms of the Lagrangian in the following manner

$$\begin{cases} L = L_K + L_m \\ L_K = \frac{1}{2} m \dot{x}^t \begin{bmatrix} g_{11} + \frac{\mathbb{I}\alpha^2}{m} & g_{12} + \frac{\mathbb{I}\alpha\beta}{m} \\ g_{12} + \frac{\mathbb{I}\alpha\beta}{m} & g_{22} + \frac{\mathbb{I}\beta^2}{m} \end{bmatrix} \dot{x} \\ L_m = \frac{1}{2} \dot{\theta} \mathbb{I} (-\dot{x}^1 \alpha - \dot{x}^2 \beta + \dot{\theta}) \end{cases} \tag{58}$$

where x represents $[x^1, x^2]^t$. By inspection, we conclude that the $\hat{g}$ is given by the matrix in L_K and the magnetic field term implies that the charge is analogue to $\frac{1}{2}\dot{\theta}\mathbb{I}$. The magnetic potential is specified by the connection forms, i.e. $A = -\alpha dx^1 - \beta dx^2$. Therefore, we showed that the motion is equivalent to that of an electric charge moving on the surface M, where at each point of M we define the magnetic potential (hence a magnetic field). Additionally, the motion is governed by the component of the magnetic field which is orthogonal to the surface at each point, according with the Laplace-Lorentz equation. Furthermore, the metric $\hat{g}$ depends on the connection forms. Since the connection forms define the magnetic field, this implies that in this case we have to prescribe a space with an intrinsic magnetic field, which also defines the metric.

Using the Euler Lagrange equations is also possible to write the equations of motion, obtaining the equations of a charge on a magnetic field and the conservation of $\dot{\theta}$. It is the fact that $\dot{\theta}$ is conserved that proves to be crucial in order to have the analogy. It is easy to verify this statement: for instance we could conduct the same analysis in the relativistic Lagrangian for the metric with a gravitational field, and the fact that the $\dot{\theta}$ is not separately conserved, destroys this analogy, despite the Lagrangians being similar.

C. Isometry groups

In this section we will consider the 2-dimensional motion of the tangent Euler top in $\mathbb{R}^2$, $\mathbb{H}^2$ and S^2, with the usual metrics. These manifolds happen to be maximally symmetric, since the metrics with the isometry groups of the largest dimension are metrics of constant sectional curvature. We recall some known results

Definition III.2. *A metric $\langle ., . \rangle$ on a Lie group G is said to be left-invariant if and only if*

$$\langle u, v \rangle_b = \langle (dL_a)_b u, (dL_a)_b v \rangle_{ab} \tag{59}$$

for all $a, b \in G$ and all $u, v \in T_b G$, where we used the map L_g (for $g \in G$) defined as

$$\begin{aligned} L_g : G &\to G \\ h &\to g \cdot h \end{aligned} \tag{60}$$

which corresponds to the left multiplication by $g \in G$.

Theorem III.1. *(Myers–Steenrod) Every distance-preserving and surjective map $\phi : M \to N$ between two connected Riemannian manifolds is a smooth isometry.*

Theorem III.2. *(Myers–Steenrod) The isometry group of a Riemanian manifold is a Lie group.*

By the previous theorem the isometry groups of the spaces we will consider are Lie groups. Using the tangent Euler top, we define a metric which we can pull-back to the isometry group of each manifold. For that we have to define a diffeomorphism $\Gamma : G \to TOM$ ((TOM, g) and (G, h)) such that $\Gamma^* g = h$. We will be able to do this since we can identify the parametrisations of each space, and thus obtain a metric on the Lie group. We will further prove that this metric is left-invariant.

1. $\mathbb{R}^2$ with the flat metric

In this case we recover the classical Euler top, since we have zero curvature. The Euler top yields the following metric on TOM

$$g = \frac{1}{2}m((dx^1)^2 + (dx^2)^2) + \frac{1}{2}\mathbb{I}d\theta^2 \tag{61}$$

We will now consider the group of isometries of $\mathbb{R}^2$, $SO(2) \ltimes \mathbb{R}^2$ acting on the element $(x, y) \in \mathbb{R}^2$ as

$$(x, y) \to R(\theta) \cdot (x, y) + (a, b) \tag{62}$$

where $R(\theta)$ is the orthogonal rotation (by θ) matrix in $\mathbb{R}^2$ applied to the column vector $[x, y]$ and the parameters θ, a, b describe the transformation. In order to compute the pull-back of the metric we need to define the left-invariant map on the group ($L_{a,b,c}$). We can think of an element of the group (g) as a complex function acting on a complex number z such that

$$g(z) = e^{i\theta}z + z_0 \tag{63}$$

where $z_0 = a + ib$.

Using composition as the group operation we get

$$(g\circ)(z) = e^{i\theta}e^{i\phi}z + e^{i\theta}w_0 + z_0 \tag{64}$$

which yields

$$L_{(\phi,u,v)}(\theta, x, y) = (\phi, u, v)\circ(\theta, x, y) = (\theta+\phi, u+\cos(\phi)x-\sin(\phi)y, v+\sin(\phi)x+\cos(\phi)y) \tag{65}$$

where L_k is the left-translation map by k (as above). Notice that we parametrise the group by θ, x, y which we identify with parametrisation of TOM. The pull-back yields the wanted result

$$\begin{aligned} L^*_{(\phi,u,v)}\left(\frac{m}{2}(dx^2+dy^2)+\frac{\mathbb{I}}{2}d\theta^2\right) &= \frac{\mathbb{I}}{2}d\theta^2 \\ &+\frac{m}{2}(\cos^2(\phi)dx^2+\sin^2(\phi)dy^2+\sin^2(\phi)dx^2+\cos^2(\phi)dy^2) \\ &= \frac{m}{2}(dx^2+dy^2)+\frac{\mathbb{I}}{2}d\theta^2 \end{aligned} \tag{66}$$

where the cross terms were discarded since they cancel out.

2. $\mathbb{H}^2$ with the hyperbolic (Poincaré) metric

The Poincaré metric on the half-plane is given by

$$g_{\mathbb{H}^2} = \frac{dx^2+dy^2}{y^2} = \frac{dz d\bar{z}}{y^2} \tag{67}$$

where $dz = dx + idy$. The isometry group of this Riemanian manifold is the real special linear group $SL(2,\mathbb{R})$. Using this metric we can define the metric tensor generated by the tangent Euler top obtaining

$$g = \frac{m}{2y^2}(dx^2+dy^2)+\frac{\mathbb{I}}{2}\left(\frac{dx}{y}+d\theta\right)^2 \tag{68}$$

We will study each term separately. The group action on an element of $\mathbb{H}^2$ can be understood as a Mobius transformation on the complex plane given by

$$g(z) = \frac{az+b}{cz+d} \tag{69}$$

where $z = x+iy$ and x, y parametrise $\mathbb{H}^2$; the real numbers a, b, c, d define the transformation. The inverse map is

$$g^{-1}(z) = \frac{dz-b}{a-cz} \tag{70}$$

which implies

$$dg(z) = \left(\frac{a}{cz+d}-\frac{c(az+b)}{(cz+d)^2}\right)dz = \frac{ad-bc}{(cz+d)^2}dz \tag{71}$$

Thus we can do the pull back of the first term of the metric tensor by writing it as a function of the transformed parameters

$$\begin{aligned}\frac{du^2+dv^2}{v^2} &= \frac{4dgd\overline{g}}{|g-\overline{g}|^2}\\ &= \frac{4(ad-bc)dzd\overline{z}}{|(az+b)(c\overline{z}+d)-(a\overline{z}+b)(cz+d)|^2}\\ &= \frac{4(ad-bc)^2dzd\overline{z}}{|(ad-bc)(z-\overline{z})|^2}\\ &= \frac{dx^2+dy^2}{y^2}\end{aligned} \tag{72}$$

which is the desired result (recall that since this is the special linear group, the determinant of this matrix group has to be one, thus $ad-bc=1$).

The second term requires more cumbersome calculations. To begin with, we expand the Mobius transform in terms of the real and imaginary components (u and v, respectively) and obtain

$$\begin{cases} u(x,y) = \frac{ac(x^2+y^2)+x(ad+bc)+bd}{(cx+d)^2+(cy)^2} \\ v(x,y) = \frac{y(ad-bc)}{(cx+d)^2+(cy)^2} \end{cases} \tag{73}$$

Additionally, we have to define how the θ parameter transforms. We require $\theta \to \theta + \phi(x,y)$ such that the group operation is given by

$$(z, e^{i\theta}) \to \left(g(z), \frac{g'(z)}{|g'(z)|}e^{i\theta}\right) \tag{74}$$

which implies the following relation

$$\theta \to \theta - i\log\left(\frac{g'(z)}{|g'(z)|}\right) \tag{75}$$

The derivative of the map g is given by

$$\begin{aligned}g'(z) &= \frac{1}{(cz+d)^2}\\ &= \left(\frac{1}{xc+d+iyc}\right)^2\\ &= \frac{1}{(xc+d)^2+(yc)^2}e^{i2\arctan\left(\frac{y}{x+\frac{d}{c}}\right)}\end{aligned} \tag{76}$$

which gives us the explicit form of ϕ

$$\phi(x,y) = -2\arctan\left(\frac{y}{x+\frac{d}{c}}\right) \tag{77}$$

To perform the pull-back we just have to compute the metric in terms of $x \to u$, $y \to v$ and $\theta \to \theta + \phi$, as before.

$$\begin{aligned}\frac{\mathbb{I}}{2}\left(\frac{du}{v} + d\theta\right)^2 &= \frac{\mathbb{I}}{2}\left(\left(\frac{\partial u}{\partial x}\frac{1}{v} + \frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\frac{1}{v} + \frac{\partial \phi}{\partial y}\right)dy + d\theta\right)^2 \\ &= \frac{\mathbb{I}}{2}\left(\frac{dx}{y} + d\theta\right)^2\end{aligned} \tag{78}$$

3. S^2 with the Euler top metric

We will now consider the motion on S^2. The metric tensor in this case is given by

$$\begin{aligned}g &= \frac{1}{2}(m d\theta^2 + d\phi^2(m\sin^2(\theta) + \mathbb{I}\cos^2(\theta)) + 2d\theta d\phi \mathbb{I}\cos(\theta) + \mathbb{I}d\theta^2) \\ &= \frac{m}{2}(d\theta^2 + d\phi^2\sin^2(\theta)) + \frac{\mathbb{I}}{2}(d\phi\cos(\theta) + d\psi)^2\end{aligned} \tag{79}$$

where we have used ψ, ϕ as coordinates on S^2. We recall that the Euler top defines a metric on $SO(3)$ which is left-invariant. When the top has two principal directions $1, 2$ for which the moment of inertia tensor satisfies $I_1 = I_2$ the kinetic energy of the top is given by

$$K = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2(\theta)) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos(\theta))^2 \tag{80}$$

where $[\theta, \psi, \phi]$ are the Euler angles. This is the same metric obtained for the tangent Euler top.

IV. RELATIVISTIC 2-DIMENSIONAL MOTION

A. The Lagrangian

We will now study the motion of the relativistic tangent Euler top. In this case our problem is further complicated since in relativity there is lack of a global notion of simultaneity, which does not allow us to define the distance between two points. This implies that for instance the motion of the tangent Euler top is observer dependent: the observer sitting on top of the centre of mass who sees a rigid body rotating around him will not see the same motion as a distant observer. Therefore we have to specify which observer we want to identify the rigid body. From this point on, we will admit that we are observing the rigid body motion as if we were a distant inertial observer.

The motion of the tangent Euler top defines a curve $E : \mathbb{R} \to OM$ on the manifold OM (which is the bundle of future-pointing orthonormal frames on a 4-dimensional Lorentzian manifold). The curve E is given by

$$E(t) = (c(t), E_0(t), E_1(t), E_2(t)E_3(t)) \tag{81}$$

where $c : \mathbb{R} \to M$ is a future-directed timelike curve on M and $E_0, E_1, E_2, E_3 : \mathbb{R} \to TM$ are vector fields along the curve c, which at every point define a orthonormal frame (we take $E_0 = \frac{\partial}{\partial t}$, since we are considering the motion as seen by a distant inertial observer).

Definition IV.1. *The motion of the tangent Euler top on $(M, \langle .,. \rangle)$ is a Euler-Lagrange curve for the Lagrangian $L : TOM \to \mathbb{R}$ given by*

$$L(E, \dot{E}) = \int_{\mathbb{R}^3} |\langle \dot{c} + \xi^i \nabla_{\dot{c}} E_i, \dot{c} + \xi^j \nabla_{\dot{c}} E_j \rangle|^{\frac{1}{2}} dm \tag{82}$$

where ∇ is the Levi-Civita connection and m is the mass distribution of the tangent Euler top.

In this case we will try again to tackle the problem of integrability, but note that in this case the task is further complicated due to the fact that now we have to deal with time and curvature at the same time. As a result we will start be studying the problem using a particular family of metrics given by

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j \tag{83}$$

Note that this is just a natural generalisation of the classical setup, since the spatial part of the metric is equal to the one considered before. In particular the tangent vector to the curve c can be expanded as

$$\dot{c} = \dot{t} \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} \tag{84}$$

which implies

$$\langle \dot{c}, \dot{c} \rangle = -\dot{t}^2 + \gamma_{ij} \dot{x}^i \dot{x}^j \tag{85}$$

If we use t as a parameter the first term reduces to -1. Notice that for this metric the connection forms are simplified due to the fact that

$$\hat{\omega}_0^i = \hat{\omega}_i^0 = 0 \iff \nabla_X \hat{E}_0 = 0 \tag{86}$$

which can be obtained from the Cartan connection equations.

1. The Minkowski space-time

We will further analyse the special case of a flat universe where the above metric is reduced to

$$ds^2 = -dt^2 + dx^i dx^i \tag{87}$$

which is just the natural extension of $\mathbb{R}^n$ to relativity (special relativity). As above, we will only consider two spatial dimensions. Let us write the fields E_j relative to a fixed reference frame. As before we obtain $E_i = S_{ji}\hat{E}_j$, where S is a rotation matrix between the two frames. The connection term is simply written as $\nabla_{\dot{c}} E_i = \dot{S}_{ji}\hat{E}_j$, since the connection forms vanish in Minkowski space-time. Therefore we obtain

$$\langle \dot{c}, \dot{c}\rangle + 2\langle \dot{c}, \xi^i \dot{S}_{ji}\hat{E}_j\rangle + \langle \xi^i \dot{S}_{ji}\hat{E}_j, \xi^j \dot{S}_{kj}\hat{E}_k\rangle = -1 + \delta_{ij}\dot{x}^i\dot{x}^j + 2\dot{x}^j\xi^j\dot{S}_{ji} + \xi^i\xi^k\dot{S}_{ji}\dot{S}_{jk} \tag{88}$$

Compared to the classical Lagrangian, this case is more complicated because we have a square-root inside the Lagrangian. For this reason, we Taylor expand the the inner product, by taking ξ to be small compared to $b = |\langle \dot{c}, \dot{c}\rangle|$ (we expand $(b + x)^{\frac{1}{2}}$, where $x = f(\xi, \xi^2, \xi^3, ...))$. We will only keep track of terms which under the integral are proportional to even powers of ξ, since the odd terms vanish after integration.

The previous calculation, after taking the absolute value, reduces to

$$(1-\delta_{ij}\dot{x}^i\dot{x}^j - 2\dot{x}^j\xi^i\dot{S}_{ji} - \xi^i\xi^j\dot{S}_{ji}\dot{S}_{jk})^{\frac{1}{2}} \overset{2^{nd}ord.}{=} (1-v^2)^{\frac{1}{2}} - \frac{1}{2}(1-v^2)^{-\frac{1}{2}}\xi^i\xi^j\dot{S}_{ji}\dot{S}_{jk} - \frac{1}{2}(1-v^2)^{-\frac{3}{2}}\dot{x}^j\xi^i\dot{S}_{ji}\dot{x}^l\xi^k\dot{S}_{lk} \tag{89}$$

and so after integration we obtain

$$L = m(1-v^2)^{\frac{1}{2}} - \frac{1}{2}(1-v^2)^{-\frac{1}{2}}\operatorname{tr}(\dot{S}I\dot{S}^t) - \frac{1}{2}(1-v^2)^{-\frac{3}{2}}(\dot{S}I\dot{S}^t)\dot{x}^i\dot{x}^j \tag{90}$$

where we used $v^2 = \dot{x}^2 + \dot{y}^2$.

This Lagrangian exhibits a term which links the top velocity with its spin. If we consider that the motion of the centre of mass is given (i.e. the Lagrangian is now only a function of θ and its derivative) we can study the motion along a curve. If we consider the special case where $\dot{y} = 0$ and $\dot{x} = constant$, we can derive the following expression

$$L = m(1-v^2)^{\frac{1}{2}} - \frac{1}{2}\dot{\theta}^2(1-v^2)^{-\frac{1}{2}}\mathbb{I} - \frac{1}{2}\dot{\theta}^2(1-v^2)^{-\frac{3}{2}}\dot{x}^2(I_1\sin^2(\theta) + I_2\cos^2(\theta) - 2I_3\sin(\theta)cos(\theta)) \tag{91}$$

The last term of the Lagrangian depends on θ, which will be true for most systems which consider an arbitrary top. Apart from the mass term, the Lagrangian can be written as $L = \dot{\theta}^2 F(\theta) = H$. We can integrate this Lagrangian by setting

$$\dot{\theta}^2 = \frac{E}{F(\theta)} \iff \dot{\theta} = \pm\sqrt{\frac{E}{F(\theta)}}\ ,\ E > 0 \tag{92}$$

We can set $V_{eff} = \frac{E}{F}$, which will be a periodic function (with period 2π). In this case, given $x(t)$ and $y(t)$, this will act as an effective potential, which basically says that, unlike the classical top, the relativistic top sometimes rotates faster and other times slower.

As stated before, in general the term in $(1 - v^2)^{-\frac{3}{2}}$ is rather complicated to evaluate for a general top and it will be an explicit function of θ. In fact, in general this term will make the system non-integrable. Nonetheless, if we consider the two dimensional system for a symmetric top, the previous system reduces to ($2I = \mathbb{I}$)

$$L = m(1 - v^2)^{\frac{1}{2}} - (1 - v^2)^{-\frac{1}{2}}\dot{\theta}^2 I - \frac{1}{2}v^2\dot{\theta}^2 I(1 - v^2)^{-\frac{3}{2}} \tag{93}$$

which is clearly the Lagrangian of an integrable system. In particular it is possible to prove that the $\dot{x}$, $\dot{y}$ and $\dot{\theta}$ are conserved quantities. For that we notice that starting from the canonical moments one can write

$$m\gamma v\left(1 + \frac{4J^2}{2Im}\frac{(1 - v^2)(4 - v^2)}{(2 - v^2)^2}\right) = cte. \tag{94}$$

where γ is the usual relativistic factor and J is a constant. One can check that the function in v given by the left hand side of (94) is never constant in any interval of its domain. Therefore, for each value we choose for the right hand side, there can only be one possible value of v. Therefore, v is conserved. For θ we have

$$\dot{\theta} = \frac{J}{I\gamma(1 + \frac{v^2}{2}\gamma^2)} \tag{95}$$

which proves that $\dot{\theta}$ is also conserved.

For the Hamiltonian of the system we obtain

$$H = -m\gamma - I\dot{\theta}^2\left(\gamma + \gamma^3\left(\frac{v^2}{2} + 2v^2\right) + \gamma^5\left(\frac{3}{2}v^4\right)\right) \tag{96}$$

where the first two terms are the usual terms, and the others correspond to relativistic corrections which vanish if we take the classical limit.

Therefore, by studying the flat metric we conclude that we will have to consider symmetric tops in order to obtain integrability.

2. Curved space time

Lets know consider a more general relativistic metric given by

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^i \tag{97}$$

which was presented above.

We again derive the Lagrangian of this system. We get

$$\langle \dot{c}, \dot{c} \rangle = -1 + \gamma_{ij} \dot{x}^i \dot{x}^j \tag{98}$$

$$\begin{aligned} 2\langle \dot{c}, \xi^i \nabla_{\dot{c}} E_i \rangle &= 2\langle \dot{c}, \xi^i \dot{S}_{ji} \hat{E}_j \rangle + 2\langle \dot{c}, \xi^i S_{ji} \omega_j^k \hat{E}_k \rangle \\ &= 2\xi^i \dot{S}_{ji} \dot{x}^\alpha g_{\alpha\beta} \hat{E}_j^\beta + 2\xi^j S_{ji} \omega_j^k \dot{x}^\alpha g_{\alpha\beta} \hat{E}_k^\beta \end{aligned} \tag{99}$$

$$\begin{aligned} \langle \xi^i \nabla_{\dot{c}} E_i, \xi^j \nabla_{\dot{c}} E_j \rangle &= \langle \xi^i \dot{S}_{mi} \hat{E}_m + \xi^i S_{mi} \omega_m^k \hat{E}_k, \xi^j \dot{S}_{aj} \hat{E}_a + \xi^j S_{aj} \omega_a^b \hat{E}_b \rangle \\ &= \xi^i \xi^j \dot{S}_{bi} \dot{S}_{aj} \langle \hat{E}_b, \hat{E}_a \rangle + 2\xi^i \xi^j \dot{S}_{bi} S_{aj} \omega_a^c \langle \hat{E}_b, \hat{E}_c \rangle + \xi^i \xi^j S_{aj} S_{bi} \omega_b^k \omega_a^c \langle \hat{E}_k, \hat{E}_c \rangle \end{aligned} \tag{100}$$

We can write the integrand as

$$\begin{aligned} I = (1 - \gamma_{ij} \dot{x}^i \dot{x}^j - 2\xi^i \dot{S}_{ji} \dot{x}^\alpha g_{\alpha\beta} \hat{E}_j^\beta - 2\xi^j S_{ji} \omega_j^k \dot{x}^\alpha g_{\alpha\beta} \hat{E}_k^\beta - \xi^i \xi^j \dot{S}_{bi} \dot{S}_{aj} \langle \hat{E}_b, \hat{E}_a \rangle \\ - 2\xi^i \xi^j \dot{S}_{bi} S_{aj} \omega_a^c \langle \hat{E}_b, \hat{E}_c \rangle - \xi^i \xi^j S_{aj} S_{bi} \omega_b^k \omega_a^c \langle \hat{E}_k, \hat{E}_c \rangle)^{\frac{1}{2}} \end{aligned} \tag{101}$$

The second order approximation is then given by

$$\begin{aligned} L \approx m(1 - \gamma_{ij} \dot{x}^i \dot{x}^j) - \frac{1}{2} \frac{1}{(1 - \gamma_{ij} \dot{x}^i \dot{x}^j)^{\frac{1}{2}}} (\operatorname{tr}(\dot{S} I \dot{S}^t) + 2 \operatorname{tr}(\dot{S} I S^t \omega^t) + \operatorname{tr}(\omega S I S^t \omega^t)) \\ - \frac{1}{2} \frac{1}{(1 - \gamma_{ij} \dot{x}^i \dot{x}^j)^{\frac{3}{2}}} \langle \dot{c}, \hat{E}_j \rangle \langle \dot{c}, \hat{E}_m \rangle ((\dot{S} I \dot{S}^t)_{jm} + (\omega S I S^t \omega^t)_{jm} + 2(\dot{S} I S^t \omega^t)_{jm}) \end{aligned} \tag{102}$$

The first two terms are the relativistic correspondent to what we had derived before. In the particular case of a symmetric top we get

$$\begin{aligned} L \approx m(1 - \gamma_{ij} \dot{x}^i \dot{x}^j) - \frac{I}{2} \frac{1}{(1 - \gamma_{ij} \dot{x}^i \dot{x}^j)^{\frac{1}{2}}} (\operatorname{tr}(\dot{S} \dot{S}^t) + 2 \operatorname{tr}(\dot{S} S^t \omega^t) + \operatorname{tr}(\omega \omega^t)) \\ - \frac{I}{2} \frac{1}{(1 - \gamma_{ij} \dot{x}^i \dot{x}^j)^{\frac{3}{2}}} \langle \dot{c}, E_j \rangle \langle \dot{c}, E_m \rangle ((\dot{S} \dot{S}^t)_{jm} + (\omega \omega^t)_{jm} + 2(\dot{S} S^t \omega^t)_{jm}) \end{aligned} \tag{103}$$

We can check that this Lagrangian is invariant under a change of frame. For that we use

$$\begin{aligned}\dot{\tilde{S}}\dot{\tilde{S}}^t &= (\dot{R}S + R\dot{S})(\dot{R}S + R\dot{S})^t \\ &= (\dot{R}S + R\dot{S})(S^t\dot{R}^t + \dot{S}^tR^t) \\ &= \dot{R}\dot{R}^t + \dot{R}S\dot{S}^tR^t + R\dot{S}S^t\dot{R}^t + R\dot{S}\dot{S}^tR^t\end{aligned} \tag{104}$$

$$\begin{aligned}\tilde{\omega}\tilde{\omega}^t &= (R\omega R^t + R\dot{R}^t)(R\omega^tR^t + \dot{R}R^t) \\ &= R\omega^tR^t + RwR^t\dot{R}R^t + R\dot{R}^tRw^tR^t + R\dot{R}^t\dot{R}R^t\end{aligned} \tag{105}$$

$$\begin{aligned}\dot{\tilde{S}}\tilde{S}^t\tilde{\omega}^t &= (R\omega R^t + R\dot{R}^t)(S^tR^t)(R\omega^tR^t + \dot{R}R^t) \\ &= \dot{R}\omega^tR^t + \dot{R}R^t\dot{R}R^t + R\dot{S}S^tR^tR\omega^tR^t + R\dot{S}R^t\dot{R}R^t\end{aligned} \tag{106}$$

We only have to consider the last term of the Lagrangian, since the other two have already been treated. The terms which require the evaluation of $\langle \dot{c}, \hat{E}_k\rangle$ can be written as

$$v^t = \left[\langle \dot{c}, \hat{E}_1\rangle, \langle \dot{c}, \hat{E}_2\rangle, \cdots\right] \tag{107}$$

which implies that we can write the Lagrangian as

$$\begin{aligned}L \approx\ & m(1-\gamma_{ij}\dot{x}^i\dot{x}^j)^{\frac{1}{2}} - \frac{I}{2}\frac{1}{(1-\gamma_{ij}\dot{x}^i\dot{x}^j)^{\frac{1}{2}}}(\operatorname{tr}(\dot{S}\dot{S}^t) + 2\operatorname{tr}(\dot{S}S^t\omega^t) + \operatorname{tr}(\omega\omega^t)) \\ & - \frac{I}{2}\frac{1}{(1-\gamma_{ij}\dot{x}^i\dot{x}^j)^{\frac{3}{2}}}v^t((\dot{S}\dot{S}^t) + (\omega\omega^t) + 2(\dot{S}S^t\omega^t))v\end{aligned} \tag{108}$$

Under the change of frame, v transforms as

$$\tilde{v} = Rv \tag{109}$$

Therefore the last term transforms accordingly to the terms in brackets multiplied on the left by R^t and on the right by R. We only have to study the transformation of the expression in v^tXv, since the remaining terms we already know do not change under change of frame. Computing the transformation we get the transformed term

$$\begin{aligned}& R^t\dot{R}\dot{R}^tR + R^t\dot{R}S\dot{S}^t + \dot{S}S^t\dot{R}^tR + S\dot{S}^t + \omega\omega^t + \omega R^t\dot{R} \\ &\quad + \dot{R}^tR\omega^t + \dot{R}^t\dot{R} + 2(R^t\dot{R}\omega^t + R^t\dot{R}R^t\dot{R} + \dot{S}S^t\omega^t + \dot{S}S^tR^t\dot{R})\end{aligned} \tag{110}$$

The terms which are common to L are easily isolated. Then we note that

$$R^t\dot{R}\dot{R}^tR + \dot{R}^t\dot{R} + R^t\dot{R}R^t\dot{R} = 2B^2 - 2B^2 = 0 \tag{111}$$

where we used $\dot{R} = RB$.

$$R^t\dot{R}\dot{S}^t + \dot{S}S^t\dot{R}^tR + 2\dot{S}S^tR^t\dot{R} = R^t\dot{R}S\dot{S}^t - \dot{S}S^t\dot{R}^tR \\ = [A, B] \tag{112}$$

where we used $\dot{S} = SA$.

Finally we get

$$\omega R^t\dot{R} + \dot{R}^tR\omega^t + 2R^t\dot{R}\omega^t = \omega B + (\omega B)^t + 2B\omega^t \\ = [\omega, B] \tag{113}$$

Therefore there appears an additional term to the Lagrangian such that

$$\tilde{L} = L + v^t[A + \omega, B]v \tag{114}$$

and since the commutator of skew-symmetric matrices is skew-symmetric, the last term vanishes. So, as in the classical top, we conclude that the Lagrangian does not change under change of frame and the existence of a Killing vector field is enough to obtain integrability.

3. Relativistic tangent Euler top under the influence of a gravitic potential

Finally, we extend our results to a more general metric given by

$$ds^2 = -e^{2\phi(x,y)} + \gamma_{ij}\dot{x}^i\dot{x}^j \tag{115}$$

where ϕ is the gravitational potential. This generalises the previous result by allowing the existence of a gravitational field.

In this case the connection forms ω_i^0 do not vanish; in fact

$$\omega^0 = e^{-\phi}\frac{\partial}{\partial t} \ , \ \omega^0(\dot{c}) = e^{\phi} \ , \ \omega_i^0 = -G_iw^0 \tag{116}$$

Therefore we can simply take the previous Lagrangian and change the terms in $1 - \gamma_{ij}$ and ω. Doing this we obtain (for the symmetric top)

$$L \approx m(e^{2\phi} - \gamma_{ij}\dot{x}^i\dot{x}^j)^{\frac{1}{2}} - \frac{I}{2}\frac{1}{(e^{2\phi} - \gamma_{ij}\dot{x}^i\dot{x}^j)^{\frac{1}{2}}}(\mathrm{tr}(\dot{S}\dot{S}^t) + 2\,\mathrm{tr}(\dot{S}S^t\omega^t) + \mathrm{tr}(\omega\omega^t) - e^{2\phi}GG^t) \\ - \frac{I}{2}\frac{1}{(e^{2\phi} - \gamma_{ij}\dot{x}^i\dot{x}^j)^{\frac{3}{2}}}(v^t((\dot{S}\dot{S}^t) + ((\omega\omega^t) + 2(\dot{S}S^t\omega^t))v - 2e^{2\phi}\dot{S}S^tG^tv + e^{4\phi}GG^t) \tag{117}$$

Starting form this Lagrangian we can construct the classical limit with a gravitational potential. For that we expand the Lagrangian in v, remembering that ϕ and G will be of order v^2 (order ε^2, for small ε). Additionally, we consider that $\dot{\theta}$ is of the order of 1, i.e. the top is rotating at about the speed of light. Physically, we are considering the motion of a fast rotating top, that moves slowly in a weak field. We obtain

$$L \approx m\left(1-\frac{1}{2}v^2+\phi\right)-I\left(1-\phi+v^2\right)\dot{\theta}^2+I\dot{\theta}\left(G_y\dot{x}-G_x\dot{y}\right) \tag{118}$$

In this approximation the curvature of the manifold disappears and we retained terms up to order ε^3. The equations of motion for this approximation are given by (for J constant)

$$\begin{cases} -2\dot{\theta}I\left(1+v^2-\phi\right)+I(G_y\dot{x}-G_x\dot{y})=J \\ -m\ddot{x}-2I\left(\ddot{x}\dot{\theta}^2+2\dot{x}\dot{\theta}\ddot{\theta}\right)+I\left(\ddot{\theta}G_y+\dot{\theta}\left(\frac{\partial G_y}{\partial y}\dot{y}+\frac{\partial G_y}{\partial x}\dot{x}\right)\right)=\frac{\partial\phi}{\partial x}\left(m+I\dot{\theta}^2\right)+I\dot{\theta}\left(\frac{\partial G_y}{\partial x}\dot{x}-\frac{\partial G_x}{\partial x}\dot{y}\right) \\ -m\ddot{y}-2I\left(\ddot{y}\dot{\theta}^2+2\dot{y}\dot{\theta}\ddot{\theta}\right)-I\left(\ddot{\theta}G_x+\dot{\theta}\left(\frac{\partial G_x}{\partial y}\dot{y}+\frac{\partial G_x}{\partial x}\dot{x}\right)\right)=\frac{\partial\phi}{\partial y}\left(m+I\dot{\theta}^2\right)+I\dot{\theta}\left(\frac{\partial G_y}{\partial y}\dot{x}-\frac{\partial G_x}{\partial y}\dot{y}\right) \end{cases} \tag{119}$$

It is easy to verify that the terms in $\ddot{\theta}IG_i$ will be of order ε^4 (for that, one can take the equation of motion for θ, and diferentiating it we get the order of $\ddot{\theta}$). Since the Lagrangian is of order ε^3, we disregard these terms. Therefore the last two equations can be written as

$$\begin{cases} m\ddot{x}+2I(\ddot{x}\dot{\theta}^2+2\dot{x}\dot{\theta}\ddot{\theta})=-G_x(m+I\dot{\theta}^2)+I\dot{\theta}(\frac{\partial G_x}{\partial x}+\frac{\partial G_y}{\partial y})\dot{y} \\ m\ddot{y}+2I(\ddot{y}\dot{\theta}^2+2\dot{y}\dot{\theta}\ddot{\theta})=-G_y(m+I\dot{\theta}^2)-I\dot{\theta}(\frac{\partial G_x}{\partial x}+\frac{\partial G_y}{\partial y})\dot{x} \end{cases} \tag{120}$$

where the first term on the right is the force due to the presence of the gravitational potential and the term in $\dot{\theta}$ show the coupling between the gravitic field and the angular velocity of the particle.

V. GENERALISATION FOR A N-DIMENSIONAL MANIFOLD

The strategy we followed for the case where the top moved over a surface will not suffice for a higher dimensional system: firstly the complexity of the computation will increase dramatically as n increases and at the same time we will loose insight on the problem. In this section we will try to generalise the previous results for an arbitrary space.

It is evident from the discussion above that we can think about the integrability of the problem in two separate parts: some first integrals will come from the fact that there are

Killing vectors defined on the manifold and the other part will come from the rotation of the frame.

We will begin by studying the translational dynamics. For that we consider that there are k Killing vector fields on M which have the same properties we considered above. Having k fields we would expect to obtain k first integrals. As before, we only have to study the dependencies for the metric tensor and the connection forms. As before, if we define our coordinate system as in proposition III.1, the metric tensor will only depend on the coordinates not associated with the Killing vector fields.

For the connection forms we start by defining the fields E_i which define a frame at $c(t)$. At each point of c, we can define the frame the following way: firstly we define k fields aligned with the Killing vector fields (notice that this vector fields commute and are independent) and for the remaining we use the Gram-Schmidt orthonormalisation process. The connection forms can be expressed as

$$\hat{\omega}^i_j(X) = \langle \nabla_X \hat{E}_j, \hat{E}_i \rangle = g_{\alpha\beta}(X^\gamma \partial_\gamma \hat{E}^\alpha_j + X^\gamma \Gamma^\gamma_{\beta\alpha}) E^\beta_i \tag{121}$$

This proves that the connection forms do not depend on k coordinates, since the Christoffel symbols only depend on the metric and the remaining quantities also do not depend on the same coordinates. Thus, k commuting and independent Killing vector fields always generate k first integrals.

We are only interested in cases where $k < n$. As a consequence, we need m first integrals coming from the rotational dynamics, such that $k + m = n + dim(SO(n))$. This is in the fact that main problem, since for higher values of n this does not seem to hold. Therefore, one should look ffor under conserved quantities or change the strategy.

For the case $n = 3$ the system is still integrable if we have two Killing vector fields. The conserved quantities will be H, the two momenta coming from the Killing vectors, the total angular moment and one of its components. The sixth conserved quantity can be obtained from ([1]), from the general equation motion for a three dimensional classical tangent Euler top

$$m\frac{D\dot{x}^\mu}{dt} + \frac{1}{2}\sigma_{ij}\hat{\Omega}^u_{ij\nu}\dot{x}^\nu = 0 \tag{122}$$

where the quantities in the equation are defined in ([1]). If we take the inner product of this equation with $\dot{x}^\alpha$, the second term vanishes (due to the anti-symmetric curvature forms Ω) and the remaining term gives us the conservation of the translational energy T. Therefore, there is an additional conserved quantity $H - T$. To fully prove this assertion one should show that this first integrals commute. This is not obvious at first, but the main point should be to verify this for the commutators $\{X, H - T\}$, where X is any first integral.

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