Study of Static Spatially Compact Universes

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Abstract

We construct a universe consisting of two identical static patches of the de Sitter Universe connected by a thick matter shell. This is something Hermann Weyl did in the 20th century. However, Weyl's solution is not correct as he overlooked the continuity of the derivative of g_{00} in the transition zone. Taking this into account, we try to find correct solutions by solving the TOV equations numerically using various equations of state for a perfect fluid, namely constant density, affine equation of state, linear equation of state and polytropic equation of state. The linear equation of state does not give a solution and more analysis is needed to conclude about the others. Using the Einstein cluster model instead of a perfect fluid for the thick matter shell, we obtain a set of analytical solutions.

1 Introduction

1.1 Spherically Symmetric Static Space-Time

A space-time is static if the metric does not depend on time and the transformation $t \mapsto -t$ leaves it invariant, which implies that the components g_{0i} vanish.

Thus, the metric can be reduced to

$$ds^2 = -e^{2\phi(x^1, x^2, x^3)}dt^2 + g_{ij}(x^1, x^2, x^3)dx^i dx^j \,. \tag{1}$$

If the metric has spherical symmetry, then it c can be further reduced to

$$ds^2 = e^{-2\phi}dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \,, \tag{2}$$

where $\phi = \phi(r)$ and $m = m(r)$ can be thought of as the gravitational potential and the mass function, respectively [1].

1.2 Einstein Equations

The Einstein Equations for general relativity are given by [1]

$$\mathrm{G}_{\mu\nu} + \Lambda\, \mathrm{g}_{\mu\nu} = 8\pi\, \mathrm{T}_{\mu\nu} \,, \tag{3}$$

where $\mathrm{G}_{\mu\nu}$ is the Einstein tensor, Λ is the cosmological constant, $\mathrm{g}_{\mu\nu}$ is the metric tensor and $\mathrm{T}_{\mu\nu}$ is the stress-energy tensor. If we put the term with cosmological constant on the right-hand side, we obtain an effective stress-energy tensor, resulting in

$$\mathrm{G}_{\mu\nu} = 8\pi\, \mathrm{T}^{(\mathrm{eff})}_{\mu\nu} \,. \tag{4}$$

1.2.1 Perfect Fluid with Spherical Symmetry

The effective stress-energy tensor for a perfect fluid is given by

$$\mathrm{T}^{(\mathrm{eff})}_{\mu\nu} = (\rho + p)U_\mu U_\nu + p\, \mathrm{g}_{\mu\nu} \,, \tag{5}$$

where ρ and p are the effective density and pressure arising from the stress-energy tensor and the cosmological constant.

Taking the spherically symmetric metric in equation (2) and the stress-energy tensor for a perfect fluid (5) and inputting it in (4), we get the standard TOV equations [1]

$$\frac{dm}{dr} = 4\pi r^2 \rho \,, \tag{6}$$

$$\frac{d\phi}{dr} = \frac{m + 4\pi r^3 p}{r^2 \left(1 - \frac{2m}{r}\right)} \,, \tag{7}$$

$$\frac{dp}{dr} = -\left(\rho + p\right) \frac{d\phi}{dr} \,. \tag{8}$$

Additionally, to close this system of equations, we also need an equation of state for the fluid, which we are free to choose:

$$p = p(\rho) \,. \tag{9}$$

1.3 Example: Einstein Static Universe

The most simple example of a spherically symmetric static space-time is the Einstein Universe. It is the solution of equations (6), (7) and (8) with positive Λ and with a uniform distribution of matter. Its metric is

$$ds^2 = -dt^2 + d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\varphi^2) \,. \tag{10}$$

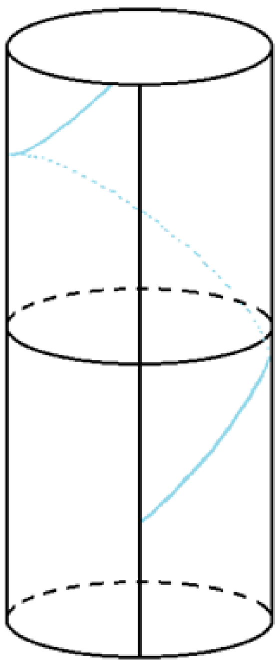

Figure 1: Diagram of the Einstein Universe. Time grows vertically and each circle slice represents a 3-sphere. Note that the radius stays constant throughout the time. In blue is represented the path of a photon.

2 De Sitter Universe

The de Sitter space-time is a maximally symmetric solution of the vacuum Einstein equations with positive cosmological constant [2]. We can set $\Lambda = 3$ since it is just a matter of choosing length units. This way, its metric is given by

$$ds^2 = -dt^2 + \cosh^2(t)\left(d\psi^2 + d\Omega^2\right) \tag{11}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

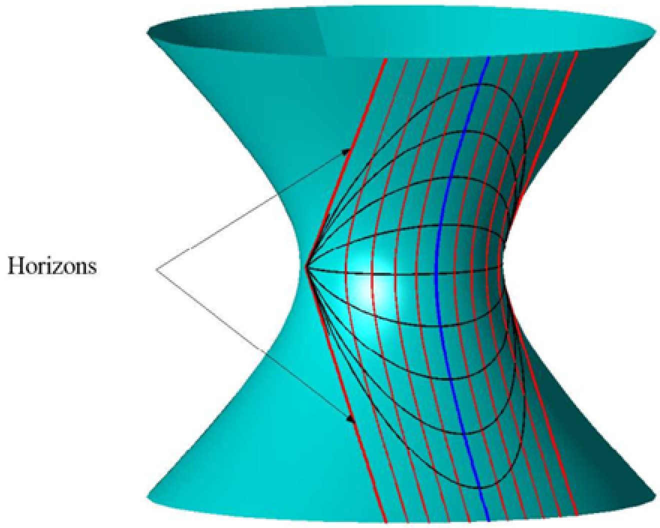

Figure 2: De Sitter Universe. Time grows vertically and each circle slice represents a 3-sphere. The radius (given by the term $\cosh(t)$) is initially decreasing until a certain minimum, where the cosmological constant starts dominating and makes the universe expand again. The delineated zone is the static patch of the de Sitter Universe. [3]

2.1 Static Patch

The metric in (11) is not static, since it depends explicitly on t. A static metric can be obtained by performing a coordinate transformation and the resulting metric is

$$ds^2 = -(1-r^2)dt^2 + (1-r^2)^{-1}dr^2 + r^2 d\Omega^2\,. \tag{12}$$

We could also obtain this by setting $\rho = \frac{\Lambda}{8\pi} = \frac{3}{8\pi}$ and $p = -\frac{\Lambda}{8\pi} = -\frac{3}{8\pi}$ and solving the TOV equations.

The static patch only describes a part of the de Sitter Universe, delineated in figure 2. We can see this in the Penrose diagram in figure 3 as well. Notice that there is a cosmological horizon for $r = 1$.

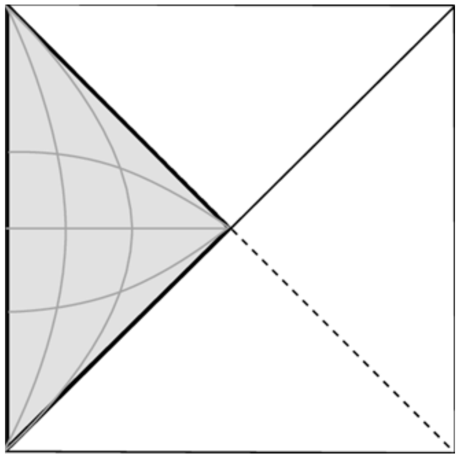

Figure 3: Penrose diagram of the de Sitter Universe. The left triangle is the static patch, the vertical curved lines represent lines of constant r and the horizontal ones represent lines of constant t. The left edge of the triangle is $r = 0$ and the other edges of the triangle are the cosmological horizon $r = 1$. [4]

3 Static Patch with Thick Matter Shell

3.1 Weyl's mistake

To avoid the cosmological horizon in the static patch, Hermann Weyl tried to add matter at some radius r_0 smaller than the horizon. That is, he tried to find a solution that consisted of the static patch for $0 < r < r_0$ and of a static and spherically symmetric solution for a perfect fluid with constant density for $r_0 < r$. He did this in his book "Space-Time-Matter" and he obtained the following solution [5]

$$g_{00} = -\left(1 - \frac{\lambda}{6}r^2\right) \qquad \text{in the de Sitter region,} \tag{13}$$

$$g_{00} = -\left(1 - \frac{2M}{r} - \frac{2\mu_0 + \lambda}{6}r^2\right) \qquad \text{in the matter region,} \tag{14}$$

with

$$\lambda, \mu_0 \qquad \text{constants,} \tag{15}$$

$$M = \frac{\mu_0}{6}r_0^3 \qquad \text{by the continuity of } g_{00} \text{ in } r_0. \tag{16}$$

Weyl correctly considered the continuity of g_{00} for $r = r_0$, which translates into the continuity of the function $\phi(r)$, the "gravitational potential". However, he did not take into account the continuity of the derivative, which can be easily verified from equations (13) and (14).

This means that there is a discontinuity in the "gravitational field". In analogy with electromagnetism, this implies the existence of a surface distribution of matter for $r = r_0$, which was not intended.

Notice that in this case the horizon in the static patch occurs for $r = \left(\frac{6}{\lambda}\right)^{1/2}$, so r_0 must be less than that. Nevertheless, this does not change any of the conclusions.

3.2 Thick Matter Shell

The objective is to construct a universe consisting of a static patch for $r < r_0$, followed by some matter and then a static patch again, as illustrated in figure 4.

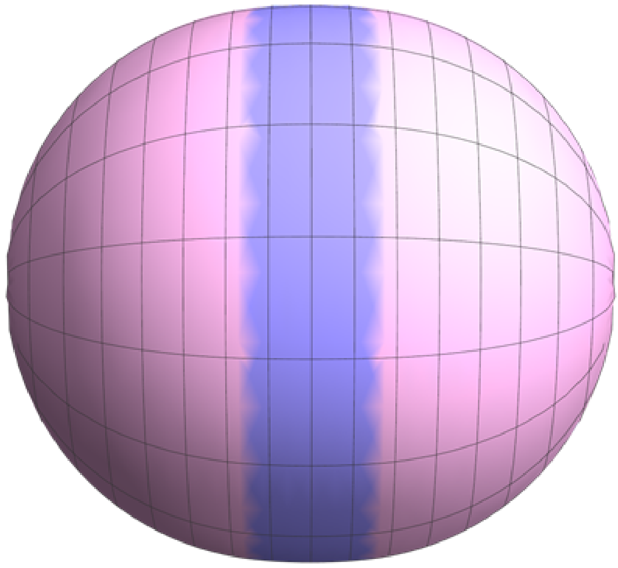

Figure 4: Diagram of two identical de Sitter Universes joined by a thick matter shell

This has been done before with a thin matter shell instead of a thick matter shell, that is, with a surface distribution of matter in [6]. It is different from Weyl's solutions as it joins two de Sitter regions directly.

Let us now rewrite the equations with quantities that relate only to the matter. We can set

$$\bar{\rho} = \rho - \frac{\Lambda}{8\pi} = \rho - \frac{3}{8\pi}\,, \tag{17}$$

$$\bar{p} = p + \frac{\Lambda}{8\pi} = p + \frac{3}{8\pi}\,, \tag{18}$$

$$\bar{m} = m - \frac{\Lambda r^3}{6} = m - \frac{r^3}{2}\,, \tag{19}$$

which is just subtracting from ρ, p and m the terms arising from the cosmological constant. Then the TOV equations become

$$\frac{d\bar{m}}{dr} = 4\pi r^2 \bar{\rho}\,, \tag{20}$$

$$\frac{d\phi}{dr} = \frac{\bar{m} + 4\pi r^3 \bar{p} - r^3}{r^2\left(1 - r^2 - \frac{2\bar{m}}{r}\right)}\,, \tag{21}$$

$$\frac{d\bar{p}}{dr} = -\left(\bar{\rho} + \bar{p}\right)\frac{d\phi}{dr}\,. \tag{22}$$

It is only necessary to solve the equations for the matter region until we reach the "equator" satisfying $r = r_1 > r_0$, and then the solution can be mirrored to the other side. The equator is defined as the largest radius of the universe, which is given by $\mathrm{g}^{rr} = 0$:

$$1 - r_1^2 - \frac{2\bar{m}}{r_1} = 0\,. \tag{23}$$

For a viable thick matter shell, there are several boundary conditions that need to be satisfied:

1. Continuity of ϕ at r_0;
2. Continuity of $\frac{d\phi}{dr}$ at r_0;

3. Continuity of $\bar{m}$ at r_0;

4. Derivative of ϕ must be 0 at r_1;

5. Derivative of $\bar{m}$ must be 0 at r_1.

The first two conditions are motivated by the continuity of the potential and of the gravitational field. The third one comes from the continuity of the metric and implies $\bar{m}(r_0) = 0$. The last two come from an argument of symmetry: we do not want the gravitational field to pull to one half or the other, nor the mass to grow in one direction or the other at the equator.

Since $\phi(r) = \frac{1}{2}\log(1-r^2)$ in the de Sitter region, the first condition implies $\phi(r_0) = \frac{1}{2}\log(1-r_0^2)$.

The second condition actually implies that $\bar{p}(r_0) = 0$ and this can be easily calculated. The derivative of ϕ in the matter region is given by equation (21). Then, in order for $\frac{d\phi}{dr}$ to be continuous, we have

$$\frac{d\phi}{dr}\left(r_0^-\right) = \frac{d\phi}{dr}\left(r_0^+\right) \Leftrightarrow -\frac{r_0}{1-r_0^2} = \frac{\bar{m}(r_0) + 4\pi r^3 \bar{p}(r_0) - r_0^3}{r^2\left(1 - r_0^2 - \frac{2\bar{m}(r_0)}{r_0}\right)} \,. \tag{24}$$

Since $\bar{m}(r_0) = 0$, the equation becomes

$$\bar{p}(r_0) = 0 \,. \tag{25}$$

The last two conditions must be treated more carefully, since r is not a good coordinate choice when $r \to r_1$, because $g_{rr} \to \infty$. We can choose another coordinate, namely the arc length l, given by

$$dl^2 = \left(1 - r^2 - \frac{2\bar{m}}{r}\right)^{-1} dr^2 \,. \tag{26}$$

The derivative of ϕ becomes

$$\frac{d\phi}{dl} = \sqrt{1 - r^2 - \frac{2\bar{m}}{r}}\frac{d\phi}{dr} \,. \tag{27}$$

When $r = r_1$, the term inside the square root is zero, by the definition of the equator in equation (23). So in order to have $\frac{d\phi}{dl}(r_1) = 0$, we just need $\frac{d\phi}{dr}$ to be finite as $r \to r_1$.

The same argument can be made for $\bar{m}$, and we reach the same conclusion, namely that $\frac{d\bar{m}}{dr}$ needs to be finite as $r \to r_1$; however, this condition is automatically satisfied due to equation (20), since $\bar{\rho}$ is assumed to be always finite.

4 Perfect Fluid with Constant Density

If we choose (as Weyl did)

$$\bar{\rho} = \frac{3k}{4\pi} \,, \tag{28}$$

where k is a positive constant, we obtain

$$\bar{m} = kr^3 - kr_0^3 \,, \tag{29}$$

$$\frac{d\phi}{dr} = \frac{(k-1)r^3 - kr_0^3 + 4\pi r^3 \bar{p}}{r^2\left(1 - r^2 - \frac{2\bar{m}}{r}\right)} \,, \tag{30}$$

$$\frac{d\bar{p}}{dr} = -\left(\bar{\rho} + \bar{p}\right)\frac{d\phi}{dr} \,. \tag{31}$$

By setting $k = 0.5$ and $r_0 = 0.3$, for example, we can solve the equations numerically.

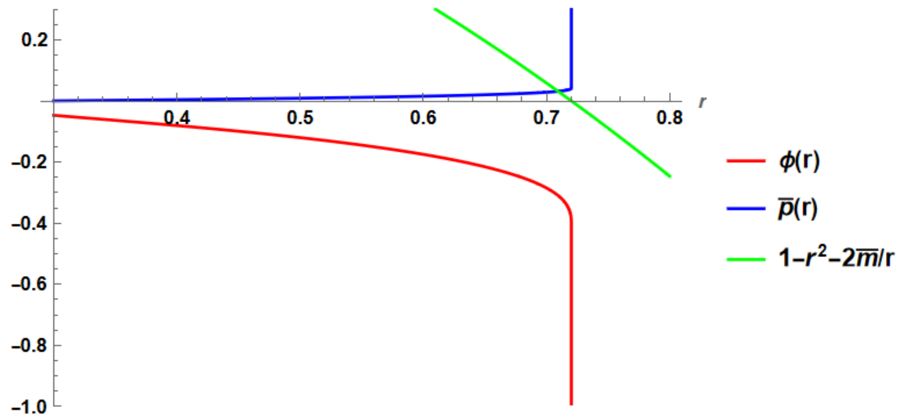

Figure 5: Plot of the numerical solutions obtained in Mathematica for ϕ and for $\bar{p}$ with constant density. The green function defines r_1 where it intersects the r axis.

We can see that the solutions for both ϕ and $\bar{p}$ blow up when $r \to r_1$, which means that the derivative of ϕ is not finite. Using other values for k we achieve similar results.

In order for $\frac{d\phi}{dr}$ to be finite, we need the numerator in (30) to be zero at r_1, that is,

$$(k-1)r_1^3 - kr_0^3 + 4\pi r_1^3 \bar{p}(r_1) = 0\,, \tag{32}$$

because the denominator vanishes by definition of the equator (23). This is a very strict relation between the r1 and the pressure at that point, which will be very hard to obtain numerically.

5 Pefect Fluid with Affine Equation of State

Instead of constant density, we can choose an affine equation of state, that is,

$$\bar{\rho} = \frac{\bar{p}}{c_s^2} + \bar{\rho}_0\,, \tag{33}$$

where c_s is the speed of sound and $\bar{\rho}_0$ is the density at zero pressure.

The TOV equations can be reduced to two equations by substituting the equation (21) for ϕ into equation (22) for $\bar{p}$, yielding

$$\frac{d\bar{m}}{dr} = 4\pi r^2 \bar{\rho}\,, \tag{34}$$

$$\frac{d\bar{p}}{dr} = -\left(\bar{\rho} + \bar{p}\right) \frac{\bar{m} + 4\pi r^3 \bar{p} - r^3}{r^2 \left(1 - r^2 - \frac{2\bar{m}}{r}\right)}\,. \tag{35}$$

Because of the relation between ϕ and $\bar{p}$ in equation (22), the condition for $\frac{d\phi}{dr}$ to be finite as $r \to r_1$ can be translated into $\frac{d\bar{p}}{dr}$ being finite as $r \to r_1$, as long as the term $(\bar{\rho} + \bar{p})$ is non-vanishing.

We can solve the equations numerically by setting for example $c_s^2 = 0.0004$, $\rho_0 = 75$ and $r_0 = 0.2$.

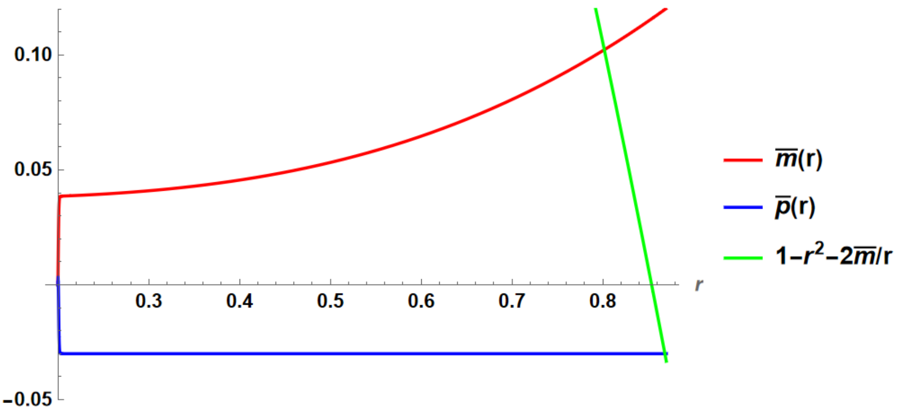

Figure 6: Plot of the numerical solutions obtained in Mathematica for $\bar{m}$ and for $\bar{p}$ with an affine equation of state.

It can be observed that the pressure initially drops to a negative value and then remains nearly constant, even when $r \to r_1$. This means that $\frac{d\bar{p}}{dr}$ remains finite. The term $(\bar{\rho} + \bar{p})$ turns out to be vanish as $r \to r_1$ in this case. If we plot $\frac{d\phi}{dr}$ given by equation (21), it is visible that $\frac{d\phi}{dr}$ blows up when $r \to r_1$, as shown in figure 7. So this solution is not correct.

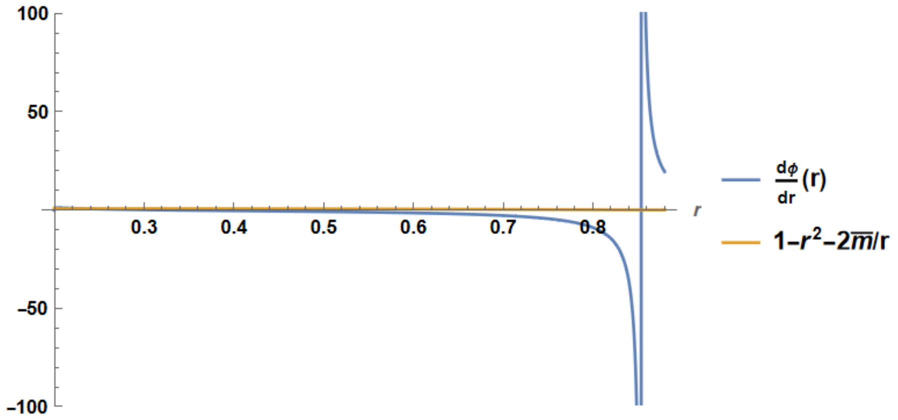

Figure 7: Plot of the numerical solution obtained in Mathematica for the derivative of ϕ

For $\frac{d\phi}{dr}$ in equation (21) to be finite, we need

$$\bar{m} + 4\pi r^3 \bar{p} - r^3 = 0 \tag{36}$$

when $r = r_1$, because the denominator vanishes by definition of the equator (23). We can write $\bar{m}(r_1)$ in terms of r_1 from that definition and substitute in (36), resulting in the following relation:

$$4\pi r_1^2 \bar{p}(r_1) + \frac{1}{2} r_1 - \frac{3}{2} r_1^3 = 0\,. \tag{37}$$

This is a very strict relation between the r_1 and the pressure at that point, which will be very hard to obtain numerically.

6 Perfect Fluid with Linear Equation of State

The linear equation of state is given by

$$\bar{\rho} = \frac{\bar{p}}{c_s^2} \tag{38}$$

where c_s is the speed of sound. The equations are the same as before:

$$\frac{d\bar{m}}{dr} = 4\pi r^2 \bar{\rho}\,, \tag{39}$$

$$\frac{d\bar{p}}{dr} = -\left(\bar{\rho} + \bar{p}\right) \frac{\bar{m} + 4\pi r^3 \bar{p} - r^3}{r^2 \left(1 - r^2 - \frac{2\bar{m}}{r}\right)}\,. \tag{40}$$

Setting $c_s^2 = 0.0004$ and $r_0 = 0.2$ and solving numerically, we obtain:

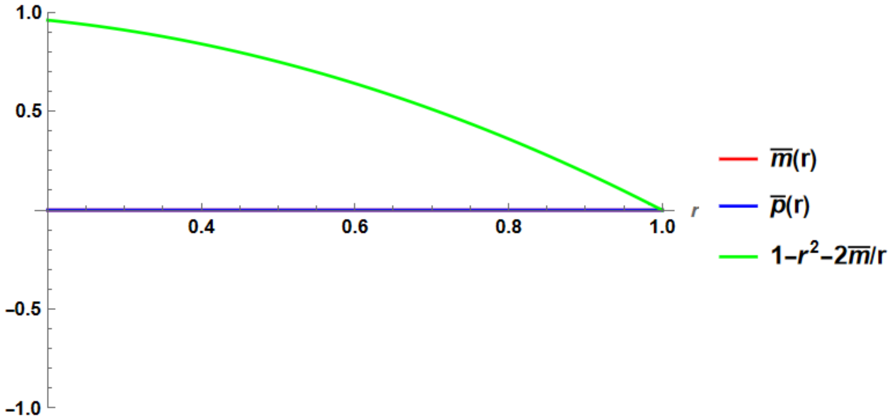

Figure 8: Plot of the numerical solutions obtained in Mathematica for $\bar{m}$ and for $\bar{p}$ with a linear equation of state.

The solutions for $\bar{m}$ and for $\bar{p}$ are identically zero, which means there is actually no matter and the solution is just the de Sitter static patch. What happens is that because the initial conditions are $\bar{m}(r_0) = \bar{p}(r_0) = 0$, the density is initially zero and it remains so because it satisfies the equations. This means that the linear equation of state is inadequate for our problem.

7 Perfect Fluid with Polytropic Equation of State

The polytropic equation of state is given by

$$\bar{\rho} = \left(\frac{\bar{p}}{k}\right)^{\frac{n}{n+1}}\,, \tag{41}$$

where k and n are constants.

If we solve this numerically, the solutions for $\bar{m}$ and $\bar{p}$ vanish identically, since it suffers from the same problem as the linear equation of state. However, the polytropic equation of state is not Lipschitz, and so there is more than one solution for the TOV equations, which cannot be found numerically and will require analytical techniques.

8 Einstein Cluster Model

We can try other matter models instead of the perfect fluid, such as the anisotropic fluid, which is similar to a spherically symmetric perfect fluid but has a radial pressure p_{rad} and a tangential pressure $p_{\tan}$ not necessarily equal [7]. The TOV equations are in this case

$$\frac{dm}{dr} = 4\pi r^2 \rho\,, \tag{42}$$

$$\frac{d\phi}{dr} = \frac{m + 4\pi r^3 p_{\rm rad}}{r^2\left(1 - \frac{2m}{r}\right)}\,, \tag{43}$$

$$\frac{dp_{\rm rad}}{dr} = \frac{2}{r}\left(p_{\rm tan} - p_{\rm rad}\right) - \left(\rho + p_{\rm rad}\right)\frac{d\phi}{dr}\,, \tag{44}$$

where both pressures include the terms from the cosmological constant.

If we keep $p_{\rm rad}$ constant and equal to $-\frac{\Lambda}{8\pi} = -\frac{3}{8\pi}$ (Einstein cluster condition [7]), for simplicity, we obtain from (42)–(44)

$$\frac{d\phi}{dr} = \frac{m - \frac{3}{2}r^3}{r^2\left(1 - \frac{2m}{r}\right)}\,, \tag{45}$$

$$p_{\rm tan} = -\frac{3}{8\pi} + \frac{r}{2}\left(\rho + p_{\rm rad}\right)\frac{d\phi}{dr}\,. \tag{46}$$

Just like before, if we define the barred quantities by subtracting the terms arising from the cosmological constant,

$$\bar{\rho} = \rho - \frac{\Lambda}{8\pi} = \rho - \frac{3}{8\pi}\,, \tag{47}$$

$$\bar{p}_{\rm tan} = p_{\rm tan} + \frac{\Lambda}{8\pi} = p_{\rm tan} + \frac{3}{8\pi}\,, \tag{48}$$

$$\bar{m} = m - \frac{r^3}{2}\,, \tag{49}$$

$$\bar{p}_{\rm rad} = p_{\rm rad} + \frac{\Lambda}{8\pi} = 0\,, \tag{50}$$

then we can rewrite the TOV equations as

$$\frac{d\bar{m}}{dr} = 4\pi r^2 \bar{\rho}\,, \tag{51}$$

$$\frac{d\phi}{dr} = \frac{\bar{m} - r^3}{r^2\left(1 - r^2 - \frac{2\bar{m}}{r}\right)}\,, \tag{52}$$

$$\bar{p}_{\rm tan} = \frac{r\bar{\rho}}{2}\frac{d\phi}{dr}\,. \tag{53}$$

In order to build a viable thick matter shell model, the same conditions discussed in section 3.2 must apply. However, the continuity of the derivative of ϕ in r_0 is automatically satisfied since $\bar{p}_{\rm rad} = 0$. So we just need $\phi(r_0) = \frac{1}{2}\log\left(1 - r_0^2\right)$, $\bar{m}(r_0) = 0$, and $\frac{d\phi}{dr}(r_1)$ and $\frac{d\bar{m}}{dr}(r_1)$ to be finite.

In order for (52) to yield a finite result in the equator defined as in equation (23), we must have

$$\bar{m} - r_1^3 = 0\,. \tag{54}$$

Therefore, we obtain

$$\begin{cases} r_1 = \frac{\sqrt{3}}{3} \\ \bar{m}(r_1) = \frac{\sqrt{3}}{9} \end{cases}\,. \tag{55}$$

A viable thick shell model can then be constructed by selecting a smooth function $\bar{m} = \bar{m}(r)$ which vanishes for $r < r_0 < \frac{\sqrt{3}}{3}$ and satisfies (55).

Additionally, in order for the matter density to be non-negative we must have, for any r,

$$\bar{m}' \geq 0\,. \tag{56}$$

Furthermore, in order for matter to satisfy the dominant energy condition, which basically stipulates that energy does not flow faster than the speed of light [2], we must have, for $r_0 < r < \frac{\sqrt{3}}{3}$,

$$\bar{\rho} \geq |\bar{p}_{\tan}| \Rightarrow \left|\frac{d\phi}{dr}\right| \leq \frac{2}{r}\,. \tag{57}$$

In this way, we obtain a family of exact solutions by choosing a smooth function for $\bar{m}(r)$ that obeys equations (55), (56) and (57) for $r_0 < r < \frac{\sqrt{3}}{3}$.

8.1 Example

For a simple example, we can choose $r_0 = \frac{2\sqrt{3}}{9}$ and

$$\bar{m}(r) = r - r_0 \tag{58}$$

for $r_0 < r < \frac{\sqrt{3}}{3}$, leading to

$$\bar{\rho} = \frac{1}{4\pi r^2}\,, \tag{59}$$

$$\frac{d\phi}{dr} = \frac{\left(r - \frac{\sqrt{3}}{3}\right)\left(r + \frac{2\sqrt{3}}{3}\right)}{r\left(r^2 + \frac{\sqrt{3}}{3}r + \frac{4}{3}\right)}\,, \tag{60}$$

$$\bar{p}_{\tan} = \frac{r\bar{\rho}}{2}\frac{d\phi}{dr}\,. \tag{61}$$

Note that

$$\frac{d\phi}{dr} \leq 0 \Rightarrow \bar{p}_{\tan} \leq 0\,, \tag{62}$$

that is, the matter is always under tension. This matter model resembles a membrane that is stretched along its tangent direction and that can be deformed without resistance perpendicularly (since $p_{\text{rad}} = 0$). Moreover,

$$\left|\frac{d\phi}{dr}\right| \leq \frac{\left(\frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{9}\right)\left(\frac{\sqrt{3}}{3} + \frac{2\sqrt{3}}{3}\right)}{r\left(\frac{4}{27} + \frac{2}{9} + \frac{4}{3}\right)} = \frac{9}{46r} \tag{63}$$

for $r_0 < r < \frac{\sqrt{3}}{3}$, and so the matter satisfies the dominant energy condition.

9 Conclusions

Our objective was to construct a universe made of two identical static patches of the de Sitter's Universe joined by a thick matter shell.

Assuming a perfect fluid as the matter matter model, we solved the TOV equations to obtain a solution, using various equations of state. The conditions for a viable thick matter shell lead to $\phi(r_0) = \frac{1}{2}\log(1 - r_0^2)$, $\bar{p}(r_0) = 0 = \bar{m}(r_0)$ and finite values for the derivatives of ϕ and $\bar{m}$ in r_1. The perfect fluid with constant density and with an affine equation of state show to potential to lead to a solution, but under very strict conditions, and we were not able to find such a solution numerically. The linear equation of state gives a vanishing solution for pressure and mass functions, meaning that it is inadequate for our problem. The polytropic equation of state also gives an identically zero solution. However, the solution is not unique since the polytropic equation of state is not Lipschitz. This means that analytical techniques must be used to draw further conclusions.

By using a different matter model, namely the Eistein cluster model, we obtained a viable thick matter shell by choosing a smooth function that vanishes for $r_0 < r < r_1$ and that obeys $r_1 = \frac{\sqrt{3}}{3}$ and $\bar{m}(r_1) = \frac{\sqrt{3}}{9}$ for $r_0 < r < r_1$.

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