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Relevant Aspects of Riemannian Geometry

Novos Talentos em Matemática

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I. Abstract

This report is intended to be a quick review over some of the most important tools and results of Riemannian Geometry. A selection of propositions and theorems will be illustrated with relevant examples in order to display its validity and applicability. The key concepts of Riemannian Geometry will also be introduced. Most of the theorems are presented without proof, even though its study was part of the work.

II. Contents

1. Tangent Vectors

In Riemannian geometry, one must look at tangent vectors as directional derivative operators acting on real-valued differentiable functions.

Definition 1.1. ([1]) Let $c : (-\varepsilon, \varepsilon) \to M$ be a differentiable curve on a smooth manifold M. Consider the set $C^\infty(p)$ of all functions $f : M \to \mathbb{R}$ that are differentiable at $c(0) = p$. The tangent vector to the curve c at p is the operator $\dot{c}(0) : C^\infty(p) \to \mathbb{R}$ given by

$$\dot{c}(0)(f) = \frac{d(f \circ c)}{dt}(0).$$

A tangent vector to M at p is a tangent vector to some differentiable curve $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$.

In order to reveal the relevance of tangent vectors, let's consider the following example.

Example 1.2. Let $\mathbb{R}P^1$ (real projective line) be the one-dimensional smooth manifold corresponding to the topological space of lines passing through the origin in $\mathbb{R}^2$, let φ, ψ be two different parameterizations of $\mathbb{R}P^1$ given by

$$\begin{aligned} \varphi : \mathbb{R} &\longrightarrow \mathbb{R}P^1 \\ x &\longmapsto \varphi(x) = \mathcal{L}\{(x, 2)\} \end{aligned}$$

and

$$\begin{aligned} \psi : \mathbb{R} &\longrightarrow \mathbb{R}P^1 \\ y &\longmapsto \psi(y) = \mathcal{L}\{(1, y)\}, \end{aligned}$$

and let f be a real function on $\mathbb{R}P^1$ given by

$$\begin{aligned} f : \mathbb{R}P^1 &\longrightarrow \mathbb{R} \\ \mathcal{L}\{(a, b)\} &\longmapsto f(\mathcal{L}\{(a, b)\}) = \frac{ab}{a^2 + b^2}. \end{aligned}$$

Notice that f is well-defined. Since $\mathbb{R}P^1$ is formed by taking the quotient of $\mathbb{R}^2 \backslash \{0\}$ under the equivalence relation $x \sim \lambda x$, f must verify $f(\mathcal{L}\{(\lambda a, \lambda b)\}) = f(\mathcal{L}\{(a, b)\})$, $\forall\, \lambda \in \mathbb{R} \backslash \{0\}$. In fact,

$$f(\mathcal{L}\{(\lambda a, \lambda b)\}) = \frac{\lambda^2}{\lambda^2} \frac{ab}{a^2 + b^2} = \frac{ab}{a^2 + b^2} = f(\mathcal{L}\{(a, b)\}).$$

In order to compute the directional derivative of f along x and y at $p = \mathcal{L}\{(1,2)\}$, one must consider the differentiable curves c_x, c_y on $\mathbb{R}P^1$ given by

$$\begin{aligned} c_x : \mathbb{R} &\longrightarrow \mathbb{R}P^1 \\ t &\longmapsto c_x(t) = \varphi(1 + t) = \mathcal{L}\{(1 + t, 2)\} \end{aligned}$$

and

$$\begin{aligned} c_y : \mathbb{R} &\longrightarrow \mathbb{R}P^1 \\ t &\longmapsto c_y(t) = \psi(2 + t) = \mathcal{L}\{(1, 2 + t)\}, \end{aligned}$$

and use them to compute the values of $\dot{c}_x(0)(f)$ and $\dot{c}_y(0)(f)$. One has

$$\dot{c}_x(0)(f) = \left.\frac{\partial}{\partial x}\right|_{p=\mathcal{L}\{(1,2)\}} f = \left.\frac{d}{dt}\right|_{t=0} f(\mathcal{L}\{(1+t,2)\}) = \left.\frac{d}{dt}\right|_{t=0} \frac{2\,(1+t)}{(1+t)^2+2} = \frac{6}{25}$$

and

$$\dot{c}_y(0)(f) = \left.\frac{\partial}{\partial y}\right|_{p=\mathcal{L}\{(1,2)\}} f = \left.\frac{d}{dt}\right|_{t=0} f(\mathcal{L}\{(1,2+t)\}) = \left.\frac{d}{dt}\right|_{t=0} \frac{2+t}{1+(2+t)^2} = -\frac{3}{25}.$$

In reality, there is a well-defined relation between the directional derivatives along x and y for $p \in \mathbb{R}P^1 \setminus \{\mathcal{L}\{(1,0)\};\ \mathcal{L}\{(0,1)\}\}$. One may write $y(x)$ as

$$y(x) = \psi^{-1}(\varphi(x)) = \psi^{-1}(\mathcal{L}\{(x,2)\}) = \psi^{-1}\left(\mathcal{L}\left\{\left(1,\frac{2}{x}\right)\right\}\right) = \frac{2}{x},$$

which is defined on $\mathbb{R}\setminus\{0\}$. Therefore, one has

$$\frac{\partial g}{\partial x} = \frac{\partial y}{\partial x}\frac{\partial g}{\partial y} = -\frac{2}{x^2}\frac{\partial g}{\partial y},$$

for any function $g \in C^\infty(\mathbb{R}P^1)$. In fact, one may see that f satisfies this relation at $p = \mathcal{L}\{(1,2)\}$:

$$\frac{6}{25} = \frac{\partial f}{\partial x} = -2\frac{\partial f}{\partial y} = -2\left(-\frac{3}{25}\right) = \frac{6}{25}.$$

2. Stokes Theorem

Riemannian geometry provides a way of expressing the Stokes Theorem in terms of differential forms.

Theorem 2.1. (Stokes Theorem) ([1]) Let M be an n-dimensional oriented smooth manifold with boundary, let ω be an $(n-1)$-differential form on M with compact support, and let $i:\ \partial M \to M$ be the inclusion of the boundary ∂M in M. Then

$$\int_{\partial M} i^*\omega = \int_M d\omega,$$

where we consider ∂M with the induced orientation.

Example 2.2. Consider the manifold

$$M = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\ \wedge\ z \geq 0\},$$

and let ω be the 2-form

$$\omega = 2\,dx \wedge dy.$$

In order to compute the two possible values of $\int_M \omega$, one will first consider the parameterization $\varphi : U \to M$ defined by

$$\varphi(\alpha, \theta) = (\sin\alpha\cos\theta\,, \sin\alpha\sin\theta\,, \cos\alpha)$$

on $U := (0, \pi/2) \times (0, 2\pi)$. One may notice that

$$\varphi^* dx = d(x \circ \varphi) = d(\sin\alpha\cos\theta) = \cos\alpha\cos\theta\, d\alpha - \sin\alpha\sin\theta\, d\theta;$$

$$\varphi^* dy = d(y \circ \varphi) = d(\sin\alpha\sin\theta) = \cos\alpha\sin\theta\, d\alpha + \sin\alpha\cos\theta\, d\theta,$$

and so

$$\varphi^*\omega = 2\,\varphi^* dx \wedge \varphi^* dy = \sin 2\alpha\, d\alpha \wedge d\theta.$$

Since the set $M \setminus \varphi(U)$ has measure zero, one may then conclude that

$$\int_M \omega = \pm \int_U \varphi^*\omega = \pm \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin 2\alpha\, d\alpha \wedge d\theta = \pm 2\pi,$$

where the $\pm$ sign depends on the choice of orientation. It is now possible to use the Stokes Theorem to confirm the previous result. Firstly, one may note that the form ω is exact. Indeed, one has, for instance, $\omega = d\beta$ with $\beta = xdy - ydx$. Secondly, it is clear that the manifold

$$N = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \,\wedge\, z = 0\}$$

corresponds to the boundary ∂M of M. Therefore, by the Stokes Theorem,

$$\int_M \omega = \int_M d\beta = \int_{\partial M} i^*\beta = \int_N i^*\beta,$$

where $i : N \to M$ is the inclusion map and N is equipped with the induced orientation. In order to compute this integral, one will consider the parameterization $\psi : V \to N$ defined by

$$\psi(\theta) = (\cos\theta\,, \sin\theta\,, 0)$$

on $V := (0, 2\pi)$. One may notice that

$$\psi^*(i^* dx) = d\big(x \circ (i \circ \psi)\big) = d(\cos\theta) = -\sin\theta\, d\theta;$$

$$\psi^*(i^* dy) = d\big(y \circ (i \circ \psi)\big) = d(\sin\theta) = \cos\theta\, d\theta,$$

and so

$$\psi^*(i^*\beta) = \big(x \circ (i \circ \psi)\big)\, \psi^*(i^* dy) - \big(y \circ (i \circ \psi)\big)\, \psi^*(i^* dx) = d\theta.$$

Since the set $N \setminus \psi(V)$ has measure zero, one may then conclude that

$$\int_N i^*\beta = \pm \int_V \psi^*(i^*\beta) = \pm \int_0^{2\pi} d\theta = \pm 2\pi,$$

where the $\pm$ sign depends on the choice of orientation. The Stokes Theorem hence corroborates the previously obtained result.

3. Riemannian Metrics

In order to define distances and angles in a general differentiable manifold, one must add structure to it by choosing a special 2-tensor field, called a Riemannian metric.

Definition 3.1. ([1]) A Riemannian metric on a smooth manifold M is a symmetric positive definite smooth covariant 2-tensor field g. A smooth manifold M equipped with a Riemannian metric g is called a Riemannian manifold, and is denoted by (M, g).

Basically, a Riemannian metric is a smooth assignment of an inner product to each tangent space. This leads to a number of concepts, such as the length of curves.

Definition 3.2. ([1]) If $(M, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold and $c : [a, b] \to M$ is a differentiable curve, the length of c is

$$l(c) = \int_a^b \|\dot{c}(t)\| \, dt = \int_a^b \sqrt{\langle \dot{c}(t), \dot{c}(t) \rangle} \, dt.$$

Example 3.3. Consider the manifold

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \ \wedge \ z > 0\},$$

which is represented in Figure 1.

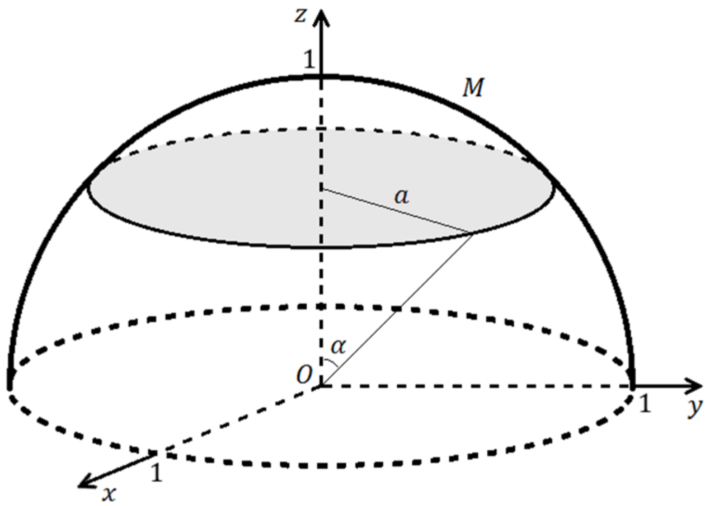

Figure 1 – Geometric Representation of M;

One will now use two different representations of the standard Riemannian metric on M to compute the perimeter of the circle of radius a represented in Figure 1, whose value is naturally expected to be $2\pi a$. One possible parameterization of M is, for instance, $\varphi : U \to M$ defined by

$$\varphi(x, y) = \left(x, y, \sqrt{1 - x^2 - y^2}\right)$$

on $U := \{(x, y) \in \mathbb{R}^2 | \; x^2 + y^2 < 1\}$. The metric induced on M by the Euclidean metric on $\mathbb{R}^3$ is

$$g = g_{xx} \, dx \otimes dx + g_{xy} \, dx \otimes dy + g_{yx} \, dy \otimes dx + g_{yy} \, dy \otimes dy,$$

where the coefficients of the metric tensor are given by

$$g_{xx} = \langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial x} \rangle = 1 + \frac{x^2}{1 - x^2 - y^2};$$

$$g_{xy} = g_{yx} = \langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \rangle = \frac{xy}{1 - x^2 - y^2};$$

$$g_{yy} = \langle \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial y} \rangle = 1 + \frac{y^2}{1 - x^2 - y^2}.$$

Indeed, one can easily note that the circle of radius a in Figure 1 may be parameterized by the differentiable curve $r : [0, 2\pi] \to M$ given in local coordinates by

$$\hat{r}(t) = (a \cos t \, , a \sin t).$$

Hence, one has

$$\dot{r}(t) = -a \sin t \frac{\partial}{\partial x} + a \cos t \frac{\partial}{\partial y},$$

and the perimeter of the circle of radius a in Figure 1 is therefore found to be

$$l(r) = \int_0^{2\pi} \|\dot{r}(t)\| \, dt = \int_0^{2\pi} \sqrt{a^2} \, dt = \int_0^{2\pi} a \, dt = 2\pi a.$$

Another possible parameterization of M is, for instance, $\psi : V \to M$ defined by

$$\psi(\beta, \theta) = (\sin \beta \cos \theta \, , \sin \beta \sin \theta \, , \cos \beta)$$

on $V := (0, \pi/2) \times (0,2\pi)$. The metric induced on M by the Euclidean metric on $\mathbb{R}^3$ will now be

$$h = d\beta \otimes d\beta + \sin^2 \beta \, d\theta \otimes d\theta,$$

as $h_{\beta\beta} = 1$, $h_{\beta\theta} = h_{\theta\beta} = 0$ and $h_{\theta\theta} = \sin^2 \beta$. Indeed, one can easily notice that the circle of radius a in Figure 1 may also be parameterized by the differentiable curve $s : [0, 2\pi] \to M$ given in the new local coordinates by

$$\hat{s}(t) = (\alpha, t).$$

Hence, one has

$$\dot{s}(t) = \frac{\partial}{\partial \theta},$$

and the perimeter of the circle of radius a in Figure 1 is therefore found to be

$$l(s) = \int_0^{2\pi} \|\dot{s}(t)\| \, dt = \int_0^{2\pi} \sqrt{\sin^2 \alpha} \, dt = \int_0^{2\pi} \sin \alpha \, dt = 2\pi \sin \alpha.$$

Keeping in mind that $\sin \alpha = a$, one can easily note that $l(s) = l(r) = 2\pi a$, that is, the two previously obtained results are in agreement with each other and with the expected value.

4. Curvature

No open set of the 2-sphere S^2 with the standard metric is isometric to an open set of the Euclidean plane. Indeed, the Euclidean plane has no curvature while the 2-sphere S^2 has. An immediate consequence is that no one can roll a piece of paper to obtain a sphere. However, one is still able to obtain a cylinder or a cone, implying that these two surfaces have no curvature. One will use the Cartan structure equations to confirm this statement in two different examples.

Example 4.1. Consider the smooth manifold M corresponding to the surface of a cylinder

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\},$$

and let $\varphi : U \to M$ be a parameterization of M defined by

$$\varphi(\theta, z) = (\cos\theta, \sin\theta, z)$$

on $U := (0, 2\pi) \times \mathbb{R}$. The metric induced on M by the Euclidean metric on $\mathbb{R}^3$ is

$$g = d\theta \otimes d\theta + dz \otimes dz.$$

One may now consider the field of frames $\{X_\theta, X_z\}$ on M, where

$$X_\theta := \frac{\partial}{\partial\theta},$$

and

$$X_z := \frac{\partial}{\partial z}.$$

Hence, $\langle X_\theta, X_\theta\rangle = 1$, $\langle X_\theta, X_z\rangle = 0$ and $\langle X_z, X_z\rangle = 1$, and so a field of orthonormal frames $\{E_\theta, E_z\}$ is given by $E_\theta := X_\theta$ and $E_z := X_z$, and $\{\omega^\theta, \omega^z\}$, with $\omega^\theta := d\theta$ and $\omega^z := dz$, is its associated field of dual co-frames. Since M is a 2-dimensional Riemannian manifold, one has $\omega_\theta^\theta = 0 = \omega_z^z$ and $\omega_\theta^z = -\omega_z^\theta$. Hence, the first Cartan Structure Equations ([1]) hold

$$d\omega^\theta = -\,\omega^z \wedge \omega_\theta^z \Leftrightarrow 0 = -dz \wedge \omega_\theta^z;$$

$$d\omega^z = \omega^\theta \wedge \omega_\theta^z \Leftrightarrow 0 = d\theta \wedge \omega_\theta^z,$$

implying $\omega_\theta^z = 0$. Finally, the third Cartan Structure Equation ([1]) yield $\Omega_\theta^z = -\Omega_z^\theta = d\omega_\theta^z = 0$ and $\Omega_\theta^\theta = 0 = \Omega_z^z$. As all curvature forms vanish, the manifold M corresponding to the surface of a cylinder has no curvature.

Example 4.2. Consider the smooth manifold M corresponding to the surface of a cone

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\},$$

and let $\varphi : U \to M$ be a parameterization of M defined by

$$\varphi(r, \theta) = (r\cos\theta, r\sin\theta, r)$$

on $U := \mathbb{R}_0^+ \times (0, 2\pi)$. The metric induced on M by the Euclidean metric on $\mathbb{R}^3$ is

$$g = 2\, dr \otimes dr + r^2\, d\theta \otimes d\theta.$$

One may now consider the field of frames $\{X_r, X_\theta\}$ on M, where

$$X_r := \frac{\partial}{\partial r},$$

and

$$X_\theta := \frac{\partial}{\partial \theta}.$$

Hence, $\langle X_r, X_r \rangle = 2$, $\langle X_r, X_\theta \rangle = 0$ and $\langle X_\theta, X_\theta \rangle = r^2$, and so a field of orthonormal frames $\{E_r, E_\theta\}$ is given by $E_r := \frac{1}{\sqrt{2}} X_r$ and $E_\theta := \frac{1}{r} X_\theta$, and $\{\omega^r, \omega^\theta\}$, with $\omega^r := \sqrt{2} dr$ and $\omega^\theta := r d\theta$, is its associated field of dual co-frames. Since M is a 2-dimensional Riemannian manifold, one has $\omega_r^r = 0 = \omega_\theta^\theta$ and $\omega_r^\theta = -\omega_\theta^r$. Hence, the first Cartan Structure Equations ([1]) hold

$$d\omega^r = -\,\omega^\theta \wedge \omega_r^\theta \Leftrightarrow 0 = -\,r d\theta \wedge \omega_r^\theta;$$

$$d\omega^\theta = \omega^r \wedge \omega_r^\theta \Leftrightarrow dr \wedge d\theta = \sqrt{2}\, dr \wedge \omega_r^\theta,$$

implying $\omega_r^\theta = \frac{1}{\sqrt{2}} d\theta$. Finally, the third Cartan Structure Equation ([1]) yields $\Omega_r^r = 0 = \Omega_\theta^\theta$ and $\Omega_r^\theta = -\Omega_\theta^r = d\omega_r^\theta = 0$. As all curvature forms vanish, the manifold M corresponding to the surface of a cone has no curvature.

5. Morse Theorem

One of the most interesting results provided by Riemannian geometry is unquestionably the Morse Theorem.

Theorem 5.1. (Morse Theorem) ([1]) Let M be a compact 2-dimensional differentiable manifold and let $f : M \to \mathbb{R}$ be a smooth Morse function, that is, a smooth function whose critical points are non-degenerate. Then

$$\chi(M) = m - s + n,$$

where $\chi(M)$ is the Euler characteristic of M, and m, n and s are the numbers of maxima, minima and saddle points respectively.

Example 5.2. Consider the compact 2-dimensional differentiable manifold represented in Figure 2, which corresponds to the torus $\mathbb{T}^2$, and let f be the smooth Morse function

$$\begin{aligned} f : \mathbb{T}^2 &\longrightarrow \mathbb{R} \\ p &\longmapsto f(p) = z. \end{aligned}$$

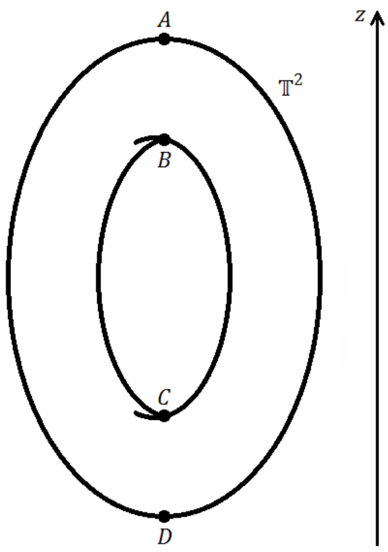

Figure 2 – Torus $\mathbb{T}^2$;

Indeed, the Euler characteristic of the torus $\mathbb{T}^2$ is 0, and one may easily notice that f has

- $m = 1$ maximum at A;
- $s = 2$ saddle points at B and C;
- $n = 1$ minimum at D.

Therefore, this example corroborates the Morse Theorem, since

$$m - s + n = 1 - 2 + 1 = 0 = \chi(\mathbb{T}^2).$$

Example 5.3. Consider the compact 2-dimensional differentiable manifold represented in Figure 3, which corresponds to the 2-sphere S^2, and let f be the smooth Morse function

$$\begin{aligned} f : S^2 &\longrightarrow \mathbb{R} \\ p &\longmapsto f(p) = x^2 - y^2. \end{aligned}$$

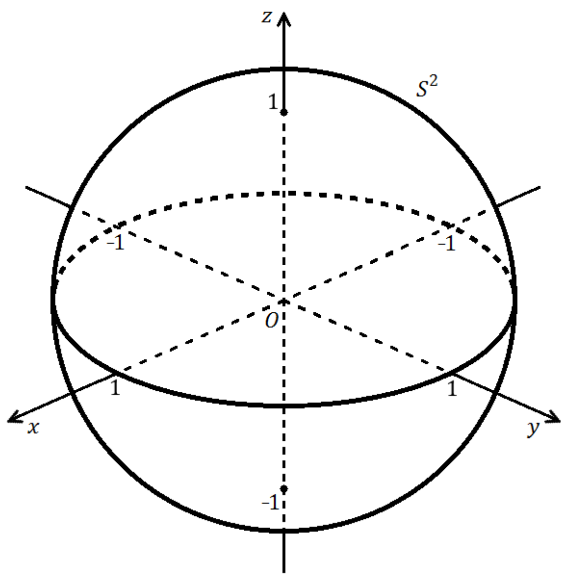

Figure 3 – 2-sphere S^2.

Indeed, the Euler characteristic of the 2-sphere S^2 is 2, and one may notice that f has

- $m = 2$ maxima at $(1, 0, 0)$ and $(-1, 0, 0)$;
- $s = 2$ saddle points at $(0, 0, 1)$ and $(0, 0, -1)$;
- $n = 2$ minima at $(0, 1, 0)$ and $(0, -1, 0)$.

Therefore, this example corroborates the Morse Theorem, since

$$m - s + n = 2 - 2 + 2 = 2 = \chi(S^2).$$

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7. References

[1] Godinho L, Natário J. *An Introduction to Riemannian Geometry with Applications to Mechanics and Relativity*. Lisbon: Springer International Publishing (2014)