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Decay of solutions to the Klein-Gordon equation on some expanding cosmological spacetimes

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Abstract

The decay of solutions to the Klein-Gordon equation is studied in two expanding cosmological spacetimes, namely

- the de Sitter universe in flat Friedmann-Lemaître-Robertson-Walker (FLRW) form
- the cosmological region of the Reissner-Nordström-de Sitter (RNdS) model.

Using energy methods, for initial data with finite higher order energies, bounds on the decay rates of the solution are obtained. Also, a previously established bound on the decay rate of the time derivative of the solution to the wave equation, in an expanding de Sitter universe in flat FLRW form, is improved, proving Rendall's conjecture. A similar improvement is also given for the wave equation in the cosmological region of the RNdS spacetime.

Popular science description

One of the most profound realisations of physics of the twentieth century is that space and time should be considered as a single whole, a four-dimensional geometric object called spacetime, rather than two separate entities. Gravitation arising from matter manifests itself as curvature of spacetime, and the motion of matter is described by the straightest possible curves on this curved spacetime. Theoretically there are many possible spacetimes, all of which arise as solutions to the so-called Einstein field equations, relating matter content on the one hand with the curvature of spacetime on the other. While these are highly nonlinear difficult equations, important insights are achievable by simplifications obtained by linearising them. These simplifications take the form of the familiar wave equations one meets in everyday life, for example those used in describing ripples of water on the surface of a pond, albeit on a curved spacetime. A more general wave equation, which is satisfied by matter fields on spacetime is the so-called Klein-Gordon equation. It is a natural question to ask what the qualitative long-term behaviour is, of solutions to these wave equations. Are they bounded, do they grow, or do they decay? If they decay, at what rate do they decay? The answers to these questions have played an important role in recent theoretical developments in general relativity, for example in progress towards the resolution of the cosmic no-hair conjecture (roughly speaking saying that all solutions to the Einstein equations with a positive cosmological constant 'eventually' look alike, namely they resemble the so-called de Sitter spacetime). This thesis contributes answers to the questions of obtaining exact decay rates in two expanding spacetimes for solutions to the Klein-Gordon equation and to the wave equation.

Contents

List of acronyms

Acronym	Phrase
FLRW	Friedman-Lemaître-Robertson-Walker
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
RNdS	Reissner-Nordström-de Sitter

1 Introduction

The aim of this thesis is to obtain exact decay rates for solutions to the Klein-Gordon equation in a fixed background of some expanding cosmological spacetimes. The two spacetimes we will consider are the de Sitter universe in flat Friedmann-Lemaître-Robertson-Walker (FLRW) form, and the cosmological region of the Reissner-Nordström-de Sitter (RNdS) model. We give some background and motivation below.

In the theory of general relativity, a spacetime (M, g) is a differentiable Lorentzian manifold M with a Lorentzian metric g, satisfying the Einstein field equations, given by

$$\mathrm{Ric} - \frac{S}{2} g + \Lambda g = 8\pi T,$$

where Ric denotes the Ricci tensor, S is the scalar curvature, Λ is the cosmological constant, and T is the energy-momentum tensor of the matter content of spacetime. Gravitation manifests itself as curvature of spacetime, and the equation of motion for matter is given by the geodesic equation on (M, g). The Einstein field equations, which are the core of general relativity, form a hyperbolic system of partial differential equations (PDEs). The simplest hyperbolic equations are the familiar wave equations. It is natural to first try understanding, via PDE theory, the linearised wave equations obtained from the Einstein equations. There are several works on the study of the Einstein field equations from a PDE perspective; see for example [7], [30], [31], [32]. In particular, it is now established that the Einstein field equations allow a formulation as an initial value problem, that is, a Cauchy problem. In fact, Einstein himself viewed them as a system of evolution equations when he gave an argument justifying that gravitational waves propagate at the speed of light [12], [14]. There he essentially studied a linearised problem by considering a metric close to that of the Minkowski space, and with a special choice of coordinates, derived a wave equation for the perturbation, which he used to conclude that gravitation waves propagate at the speed of light.

One can study linear wave equations, such as the Klein-Gordon equation $\partial_\mu \partial^\mu \phi - m^2 \phi = 0$, on curved spacetimes, by replacing the usual partial derivative ∂_μ used in flat Minkowski spacetime, with the covariant derivative ∇_μ, derived from the Levi-Civita connection on M induced by the metric g. The study of wave equations on Lorentzian manifolds is interesting from several points of view, ranging from the realm of partial differential equations and differential geometry in pure mathematics, to theoretical physics. Within general relativity, the Klein-Gordon equation is of interest since it is a proxy for the Einstein equations, and it also arises in a geometric model for dark matter [5].

From the pure mathematical perspective, analysis of linear wave equations on Lorentzian manifolds is a natural topic of study within the domain of hyperbolic partial differential equations and differential geometry. The texts such as [2], [36, §2.7, Chap.6], [6] discuss the global theory in contrast to [18] and [23], where the emphasis was to present the classical work of Hadamard and M. Riesz (Lund University) in modern language, using the theory of distributions and differential geometry.

The problem we consider is made linear, in that the background spacetime is fixed. This constitutes a first step towards understanding the more complicated nonlinear coupled problem, where one also considers the effect of the energy-momentum tensor of the solution to the Klein-Gordon equation on the Einstein equation. This nonlinear coupled problem is much more complicated, and usually requires, as a first step, a detailed understanding of our simpler linear problem.

One may consider the linear wave equations as a proxy for the Einstein equations, with the ultimate goal of understanding the qualitative behaviour of solutions to the Einstein equations by slicing spacetime into spacelike hypersurfaces. After this first step, one may then proceed to consider linearised Einstein equations (which can be reduced to tensor wave-like linear equations), and finally, the full nonlinear Einstein equations.

In the modern qualitative theory of differential equations, pioneered by Poincaré and Lyapunov, one studies the behaviour of solutions to differential equations without explicitly solving them (as the solutions may not be available in the first place, and even if analytic expressions are available, for example in the form of unwieldy series expansions or integral expressions, they may be quite useless to answer fundamental questions such as whether they stay bounded, tend to zero, and what their behavior is for large values of the argument). Such a qualitative knowledge of the behaviour of solutions may also aid the development of more accurate numerical schemes for finding approximate solutions using the computer. In particular, viewing the simplified situation of the field equations, where one has a linear wave equation, such as the Klein-Gordon equation on a fixed background spacetime, one may be interested in the behaviour of sections of the solution for large values of one of the coordinates.

A cosmological constant Λ was first introduced by Einstein in 1917 in [13], to obtain a static cosmological solution, but he later dismissed it as his 'biggest blunder' in view of Hubble's galaxy redshift observations. However, it made a triumphant comeback into the standard model of cosmology in 1998 due to the observation of the accelerated expansion of the Universe consistent with a positive value of Λ. For late cosmological times, one expects $\Lambda > 0$ to completely dominate the dynamics, damping all inhomogeneities and anisotropies. This prompted the 'cosmic no-hair conjecture' of Gibbons and Hawking, from 1977 [19]: Generic expanding solutions of Einstein's field equations with a positive cosmological constant approach the de Sitter solution asymptotically.

In our case of the Klein-Gordon equation, the expectation is that the accelerated expansion from a positive cosmological constant has a dominating effect on the decay of solutions. Precise estimates on solutions may then prove useful in formulating and proving cosmic no-hair theorems; see e.g. [3], [8]. For example, the results of [9] (some special cases of which are improved in this thesis), had provided important insights for the recent analysis of the cosmic no-hair conjecture in spherically symmetric spacetimes in [8].

The wave equation $\Box_g \phi = 0$ in expanding cosmological spacetimes (M, g) has been amply studied in the literature, see for example [6], [9], [10], [34], and the references therein. It is a natural question to also study the Klein-Gordon equation $\Box_g \phi - m^2\phi = 0$, the degenerate version of which, when $m^2 = 0$, is the wave equation. For example, in [34, §6], also the case of the Klein-Gordon equation in the Schwarzschild-de Sitter spacetime is considered. In [33], the asymptotic behaviour of the solutions to the Klein-Gordon equation near the Big Bang singularity is studied, while we investigate the asymptotics of the Klein-Gordon equation in the far future in the case of the de Sitter universe in flat FLRW form, and in the cosmological region of the Reissner-Nordström-de Sitter solution. Recently, in [11], among other things, decay estimates for the solutions to the Klein-Gordon equation were obtained in de Sitter models (see in particular, Corollary 2.1 and the less obvious Proposition 3.1). However, these results are proved via Fourier transformation (reminiscent of our mode calculation in Appendix A), and are not as sharp[1] as our Theorem 3.2.

The wave equation in the de Sitter spacetime having flat 3-dimensional spatial sections was considered in Rendall [29]. There it was shown that the time derivative $\dot{\phi} := \partial_t \phi$ decays at least as $e^{-Ht} = (a(t))^{-1}$, where $H = \sqrt{\Lambda/3}$ is the Hubble constant, and $\Lambda > 0$ is the cosmological constant. Moreover, it was conjectured that the decay is of the order $e^{-2Ht} = (a(t))^{-2}$. The almost-exact conjectured decay rate of $|\dot{\phi}| \lesssim (a(t))^{-2+\delta}$ (where $\delta > 0$ can be chosen arbitrarily at the outset) follows as a corollary of a result shown recently in [9, Remark 1.1]. We improve this result, to obtaining full conformity with Rendall's conjecture, in our result Theorem 3.2 below.

A naive heuristic indication of the effect of the accelerated expansion on the decay of the solution, based on physical energy considerations, can be obtained as follows. Considering an expanding FLRW model with flat n-dimensional spatial sections of radius $a(t)$, we have on the one hand that the energy density of a solution ϕ of the Klein-Gordon equation is of the order of $m^2\phi^2$. On the one hand, the instantaneous energy is the energy density over the time slice times the volume $(a(t))^n$ of space at time t, and hence is given by $m^2\phi^2(a(t))^n$. On the other hand, if the wavelength of the particles associated with ϕ follows the expansion, then it is proportional to $a(t)$, and so the instantaneous energy, varies as

$$E^2 \sim m^2 + p^2 \sim A + \frac{B}{(a(t))^2},$$

where $A, B > 0$ are constants. Thus the instantaneous energy

$$m^2\phi^2(a(t))^n \propto \left(A + \frac{B}{(a(t))^2}\right),$$

giving

$$m^2\phi^2 \sim (a(t))^{-n}\left(A + \frac{B}{(a(t))^2}\right).$$

As $\dot{a} \geqslant 0$ (expanding FLRW spacetime), the term $A + \frac{B}{(a(t))^2}$ approaches a finite positive value, and so one may expect

$$\phi \sim (a(t))^{-\frac{n}{2}}.$$

[1] For example, in [11, Corollary 2.1], when $|m| < n/2$, our established decay rate from Theorem 3.2 is obtained only under the additional assumption that $\sqrt{n^2 - 4m^2} \in [1, n)$ while we make no such assumption.

We will find out that in fact things are much more complicated: this decay rate is valid only for[2] $|m| \geqslant \frac{n}{2}$. In order to obtain precise conjectures on the expected decay, we will consider Fourier modes for spatially-periodic solutions to the Klein-Gordon equation, or equivalently, consider the expanding de Sitter universe in flat FLRW form with toroidal spatial sections. This exercise already demonstrates that the underlying decay mechanism is the cosmological expansion, as opposed to dispersion. The Fourier mode analysis, which is peripheral to the rest of the thesis, is relegated to Appendix A. We will use 'energy methods' to prove our results, and we give elaborate on the general ideas behind this technique in Section 2.

In the cosmological region of the Reissner-Nordström-de Sitter spacetimes, the expanding region is foliated by spacelike hypersurfaces of 'constant r'. One expects the decay rate with respect to r, for the solution to the Klein-Gordon equation, in the cosmological region of the Reissner-Nordström-de Sitter spacetime, to be the same as the one for the de Sitter universe in flat FLRW form, when e^t is replaced by r, as is explained in Remark 5.2, 6.2. We show that this expectation is correct, and a suitable modification of the technique used in the case of the de Sitter universe in flat FLRW form does enable one to obtain the expected decay rates also for the case of the Reissner-Nordström-de Sitter spacetime.

Our main results are as follows:

- Theorem 3.2 considers the $m = 0$ case (wave equation), and we obtain a decay estimate on $\partial_t \phi$, improving a corollary of [9, Theorem 1], and proving the aforementioned Rendall's conjecture.
- Theorem 6.3 again considers the $m = 0$ case (wave equation), and improves [9, Theorem 2]. We obtain a decay estimate on $\partial_r \phi$, using a similar method to the one we use for proving Rendall's conjecture.
- Theorem 4.1 gives the decay rate of the solutions ϕ to the Klein-Gordon equation in the de Sitter universe in flat FLRW form.
- Theorem 5.3 gives the decay rate of the solutions ϕ to the Klein-Gordon equation in the cosmological region of the RNdS model.

The organisation of the thesis is as follows. Theorems 3.2, 4.1, 5.3, 6.3 are stated and proved in Sections 3, 4, 5, 6, respectively. The Fourier mode analysis for spatially-periodic

[2] We remark the seemingly odd comparison of m with the dimensionless $n/2$ done here arises as follows. Firstly, in geometrised units ($c = 1$, $G = 1$, and all quantities measured in meters), the cosmological constant has the dimensions L^{-1}. Then by setting the Hubble constant $H = 1$, or equivalently $\Lambda = 3$, we are choosing the length unit to be the length scale determined by the cosmological constant (roughly the radius of the cosmological horizon), thus rendering all our quantities dimensionless. In particular, the m^2 in the Klein-Gordon equation, which in geometrised units has the dimensions of L^{-2} (and in conventional units, one writes $\frac{m^2c^2}{h^2}$ instead of m^2), also becomes dimensionless, and can be compared to $\frac{n^2}{4}$. More physically, to say that $m > \frac{n}{2}$, for instance, means that the Compton wavelength λ associated with the mass m (in conventional units $\lambda = \frac{h}{mc}$) is smaller than $\frac{2}{n}$ times the cosmological radius.

solutions to the Klein-Gordon equation is given in Appendix A, while Appendix B contains a technical lemma which is needed in the proof of Theorem 4.1. Finally, in Appendix C, we establish the sharpness of the bound of one of the key estimates in the proof of the $|m| = \frac{n}{2}$ case of Theorem 4.1.

2 Methods and background

In this section we give a bird's-eye view of the methods used in order to prove our four theorems, and also fix some preliminary notation, which will used throughout the thesis.

We will use the standard notation from differential geometry (e.g. [21]), and from the theory of partial differential equations (e.g. [36]). For a quick introduction to integration on oriented Lorentzian manifolds and Stokes' theorem (used in our proofs), we refer the reader to [37, Appendix B]. For general relativity background, we refer the reader to [17], [27], and for more mathematical treatments, to [24], [37].

In order to facilitate the smooth transition from the index notation preferred in the physicist's literature to the index-free notation prevalent in the mathematics world, we give the table below, to serve as a convenient notational dictionary.

Object	**Mathematicians**	**Physicists**
Vector field	X	X^μ
Metric	$g \equiv \langle \cdot, \cdot \rangle$	$g_{\mu\nu}$
Inner product	$g(X,Y) \equiv \langle X, Y \rangle$	$g_{\mu\nu} X^\mu Y^\nu$
Associated covector	$g(X, \cdot)$	$X_\nu := g_{\mu\nu} X^\mu$
Covariant derivative	$\nabla_X Y$	$X^\mu \nabla_\mu Y^\nu$
Covariant derivative tensor	∇X	$\nabla_\mu X^\nu := \partial_\mu X^\nu + \Gamma^\nu_{\mu\sigma} X^\sigma.$
Ricci tensor	Ric	$R_{\mu\nu}$
Scalar curvature	S	R
Energy momentum tensor	T	$T_{\mu\nu}$
Spatial dimensions	n	$d-1$

We will use energy methods for getting estimates on the decay rates. While these are covered in the context of linear PDEs in domains in $\mathbb{R}^n$ in most undergraduate books on the subject, we will work on Lorentzian manifolds. A quick introduction at the graduate level can be found in [36, Chap. 2 and 6], while a detailed treatment is covered in the monographs [31], [32]. We give a rough description below.

The basic idea behind the so-called energy methods in the mathematics literature (alternatively called Lyapunov methods in the engineering community) is as follows. Given a PDE and a solution ψ, one constructs a suitable scalar function $E(t)$, which is nonnegative for all t, and is typically given by an integral over the spatial dimensions of ψ and its partial derivatives. One then studies the evolution in time of E along a solution to the PDE by considering $E'(t)$ obtained by differentiating with respect to t under the integral sign, and using the PDE, typically in combination with a version of Stokes theorem, one obtains a differential inequality for the energy function. A suitable integration with respect to the t variable then gives a bound on the energy function, which the results in estimates for the norms of the partial derivatives of the solution in suitable function spaces.

For an interval $I \subset \mathbb{R}$, we define $C^1(I) := \{a : I \to \mathbb{R} : a \text{ is continuously differentiable on } I\}$. Also,

$$L^\infty(\mathbb{R})^n := \left\{ \varphi : \mathbb{R}^n \to \mathbb{R} : \|\varphi\|_{L^\infty(\mathbb{R}^n)} := \sup_{\mathbf{x} \in \mathbb{R}^n} |\varphi(\mathbf{x})| < +\infty \right\}.$$

For preliminaries on Sobolev spaces, we refer the reader to [36, Chap. 4] or [16, Chap. 5], although we recall the definitions and the results we use in the text of the thesis.

3 Decay in the de Sitter universe in flat FLRW form; $m = 0$

In this section, we will prove Rendall's Conjecture in Theorem 3.2 below. But before we explain the statement of Rendall's Conjecture, we recall [9, Theorem 1], since it gives the hitherto best known estimate in connection with Rendall's Conjecture.

Theorem 3.1.

Suppose that

- $\delta > 0$,
- $I \subset \mathbb{R}$ *is an open interval of the form* $(t_*, +\infty)$, $t_0 \in I$,
- $a(\cdot) \in C^1(I)$ *with* $\dot{a}(t) \geqslant 0$ *for* $t \geqslant t_0$, *and* $\epsilon > 0$ *is such that* $\displaystyle\int_{t_0}^{\infty} \frac{1}{(a(t))^\epsilon} dt < +\infty$,
- $n \geqslant 2$,
- (M, g) *is an expanding* FLRW *spacetime with flat* n*-dimensional sections, given by* $I \times \mathbb{R}^n$, *with the metric* $g = -dt^2 + (a(t))^2 \left((dx^1)^2 + \cdots + (dx^n)^2\right)$,
- $k > \frac{n}{2} + 2$, $\phi_0 \in H^k(\mathbb{R}^n)$, $\phi_1 \in H^{k-1}(\mathbb{R}^n)$, *and*
- ϕ *is a smooth solution to the Cauchy problem*

$$\begin{cases} \Box_g \phi = 0, & (t \geqslant t_0,\ \mathbf{x} \in \mathbb{R}^n), \\ \phi(t_0, \mathbf{x}) = \phi_0(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^n), \\ \partial_t \phi(t_0, \mathbf{x}) = \phi_1(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^n). \end{cases}$$

Then for all $t \geqslant t_0$, $\|\partial_t \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2+\epsilon+\delta}$.

Here, the symbol $\lesssim$ is used to mean that there exists a constant $C(\delta)$, independent of ϵ, such that

$$\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leqslant C(\delta)\,(a(t))^{-2+\epsilon+\delta}.$$

We also use the standard notation $H^k(\mathbb{R}^n)$ for the Sobolev space,

$$\|\phi\|^2_{H^k(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \sum_{|\boldsymbol{\alpha}|\leqslant k} (\partial_{\boldsymbol{\alpha}}\phi)^2 d^n\mathbf{x} < +\infty \quad \text{for } \phi \in H^k(\mathbb{R}^n),$$

where $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$, $\mathbb{N}_0$ is the set of nonnegative integers, $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$, and $\partial_{\boldsymbol{\alpha}} := (\partial_{x_1})^{\alpha_1}\cdots(\partial_{x_n})^{\alpha_n}$; see for example [37, p.249] or [36, Chap. 4].

In Theorem 3.1, in particular, if $a(t) = e^{Ht}$, where H is the Hubble constant, then as $\epsilon > 0$ can be taken to be arbitrarily small, we obtain $\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2+\delta} = e^{-(2-\delta)Ht}$, and this is in agreement with Rendall's conjecture up to the small quantity $\delta > 0$. We will show below that in fact one gets the exact rate $(a(t))^{-2}$ when $n > 2$. There is no loss of generality in assuming that $H = 1$. Our result is the following.

Theorem 3.2.

Suppose that

- $I \subset \mathbb{R}$ *is an open interval of the form* $(t_*, +\infty)$, $t_0 \in I$,
- $n > 2$,
- (M, g) *is the expanding de Sitter universe in flat FLRW form, with flat n-dimensional sections, given by* $I \times \mathbb{R}^n$, *with the metric* $g = -dt^2 + e^{2t}\left((dx^1)^2 + \cdots + (dx^n)^2\right)$,
- $k > \frac{n}{2} + 2$, $\quad \phi_0 \in H^k(\mathbb{R}^n)$, $\quad \phi_1 \in H^{k-1}(\mathbb{R}^n)$, *and*
- ϕ *is a smooth solution to the Cauchy problem*

$$\begin{cases} \Box_g\phi &= 0, & (t \geqslant t_0,\ \mathbf{x} \in \mathbb{R}^n), \\ \phi(t_0,\mathbf{x}) &= \phi_0(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^n), \\ \partial_t\phi(t_0,\mathbf{x}) &= \phi_1(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^n). \end{cases}$$

Then for all $t \geqslant t_0$, $\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2} = e^{-2t}$.

Proof. We proceed in several steps.

Step 1: Bound on $\Delta\phi$.

We will follow the preliminary steps of the proof of [9, Theorem 1] in order to obtain a bound on $\Delta\phi$, which will be needed in the proof of our Theorem 3.2. We repeat this preliminary step here from [9, §2.2] for the sake of completeness and for the convenience of the reader. Also, we divide this somewhat long step further into subparts (a)-(d).

(a) In this part rewrite the wave equation using the $(t,\mathbf{x})$-coordinates. For a vector field $X = X^\mu\partial_\mu$, it can be shown that

$$\nabla_\mu X^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,X^\mu),$$

where $g := \det[g_{\mu\nu}]$ is the determinant of the matrix $[g_{\mu\nu}]$ describing the metric in the coordinate system/chart. Then it follows that

$$\Box_g \phi = \nabla_\mu(\partial^\mu \phi) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,\partial^\mu \phi).$$

Thus $\Box_g \phi = 0$ can be rewritten as $\partial_\mu(\sqrt{-g}\,\partial^\mu \phi) = 0$. With the metric for the de Sitter universe in flat FLRW form given by $g = -dt^2 + (a(t))^2\left((dx^1)^2 + \cdots + (dx^n)^2\right)$, the wave equation can be rewritten as $\partial_\mu(a^n \partial^\mu \phi) = 0$, that is,

$$-\ddot{\phi} - \frac{n\dot{a}}{a}\dot{\phi} + \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi = 0.$$

(b) In this part, we will construct an appropriate current J (for use in the application of the divergence theorem later in part (d) below) by using the energy-momentum tensor T and a suitable multiplier X. We recall (see e.g. [37, Appendix E]) that the energy-momentum tensor for the wave equation is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{g_{\mu\nu}}{2}\partial_\alpha\phi\partial^\alpha\phi. \tag{1}$$

Then it can be shown that $\nabla_\mu T^{\mu\nu} = 0$. From (1), we have in particular that

$$T_{00} = \frac{1}{2}\left(\dot{\phi}^2 + a^{-2}\delta^{ij}\partial_i\phi\partial_j\phi\right).$$

Define the vector field[3]

$$X = a^{2-n}\frac{\partial}{\partial t}.$$

Then X is future-pointing ($g(X, \partial_t) < 0$) and causal[4] (X is time-like since $g(X, X) < 0$). We form the current J, given by $J_\mu = T_{\mu\nu}X^\nu$. Then it can be shown that

$$J = (X \cdot \phi)\text{grad } \phi - \frac{1}{2}g(\text{grad } \phi, \text{grad } \phi)X.$$

(Here $X \cdot \phi$ means the application of the vector field X on ϕ, and grad ϕ is the vector field obtained from the one-form $d\phi$ by 'raising indices'[5].) It follows that $g(J, J) \leqslant 0$, so that

[3]Let $C^\infty(M)$ denote the real vector space (with pointwise operations of addition and scalar multiplication) of all smooth maps $f : M \to \mathbb{R}$. We recall that a *tangent vector* v at a point $p \in M$ is a map $v : C^\infty(M) \to \mathbb{R}$ such that it is linear and obeys the Leibniz product rule, that is, $v(f \cdot g) = f(p)v(g) + g(p)v(f)$ for all $f, g \in C^\infty(M)$. The vector space of all tangent vectors at p is denoted by T_pM. Having constructed a differentiable structure for the tangent bundle $TM = \bigcup_{p \in M} T_pM$ of M (see e.g. [21, Exercise 4.8]), with the natural projection map $\pi : TM \to M$ defined by $\pi(v) = p$ for $v \in T_pM$, a *vector field* $\chi : M \to TM$ is a smooth map such that $\pi \circ \chi = \text{id}_M$, the identity map on M.

[4]We call a vector field *causal* if it is timelike or null.

[5]In geometric parlance, the operation of 'raising indices' corresponds to using the musical/canonical isomorphism between the tangent bundle TM and the cotangent bundle T^*M induced by the metric g.

J is causal. Also, J is past-pointing. To see this, we choose $E_1, \ldots, E_n$ orthogonal and spacelike such that $\{X, E_1, \ldots, E_n\}$ forms an orthogonal basis in each tangent space. Then expressing grad $\phi = c^0 X + c^1 E_1 + \cdots + c^n E_n$, we obtain

$$\begin{aligned} g(J, X) &= (g(X, \text{grad } \phi))^2 - \frac{1}{2} g(\text{grad } \phi, \text{grad } \phi) \cdot g(X, X) \\ &= \frac{(c^0)^2}{2} (g(X, X))^2 - \frac{1}{2} \left((c^1)^2 g(E_1, E_1) + \cdots + (c^n)^2 g(E_n, E_n)\right) \cdot g(X, X) \geqslant 0. \end{aligned}$$

(c) In this part, we define an associated energy $E(t)$ with the time slice at time t, and also show that $\nabla_\mu J^\mu \geqslant 0$ everywhere.

Set $N = \dfrac{\partial}{\partial t}$, the future unit normal vector field. We define the energy E by

$$E(t) = \int_{\{t\} \times \mathbb{R}^n} J_\mu N^\mu = \int_{\mathbb{R}^n} a^2 T_{00} d^n \mathbf{x} = \int_{\mathbb{R}^n} \frac{1}{2} \left(a^2 \dot{\phi}^2 + \delta^{ij} \partial_i \partial_j \phi \right) d^n \mathbf{x}.$$

The deformation tensor Π associated with the multiplier X is

$$\Pi := \frac{1}{2} \mathcal{L}_X g = -dt \mathcal{L}_X dt + \dot{a} a^{3-n} \delta_{ij} dx^i dx^j.$$

(Here $\mathcal{L}_X$ denotes the Lie derivative in the direction of the vector field X. We refer the reader to [37, Appendix C.2, p.439-441] or [21, p.33, 71] for the definition of the Lie derivative and its properties.) It can be shown that $\mathcal{L}_X dt = (2-n) \dot{a} a^{1-n} dt$. Thus $\Pi = (n-2) \dot{a} a^{1-n} dt^2 + \dot{a} a^{3-n} \delta_{ij} dx^i dx^j$.

Claim: $\nabla_\mu J^\mu = T^{\mu\nu} \Pi_{\mu\nu}$.

We have

$$\begin{aligned} \Pi_{\mu\nu} &= \frac{1}{2} (\mathcal{L}_X g)(\partial_\mu, \partial_\nu) = \frac{1}{2} \left(\mathcal{L}_X (g_{\mu\nu}) - g(\mathcal{L}_X \partial_\mu, \partial_\nu) - g(\partial_\mu, \mathcal{L}_X \partial_\nu) \right) \\ &= \frac{1}{2} \left(X(g_{\mu\nu}) - g([X, \partial_\mu], \partial_\nu) - g(\partial_\mu, [X, \partial_\nu]) \right). \end{aligned}$$

(Here, by the *commutator* $[Y, Z]$ of two vector fields Y, Z, we mean the vector field defined by $[Y, Z] f := Y \cdot (Z \cdot f) - Z \cdot (Y \cdot f)$, for $f \in C^\infty(M)$, the space of all smooth functions $f : M \to \mathbb{R}$ on the manifold M.) But by the definition of the Levi-Civita connection ∇,

$$\begin{aligned} g(\nabla_{\partial_\mu} \partial_\nu, X) &= \frac{1}{2} \left(\partial_\mu (g(\partial_\nu, X)) + \partial_\nu (g(\partial_\mu, X)) - X(g(\partial_\mu, \partial_\nu)) \right. \\ &\quad \left. + g([X, \partial_\mu], \partial_\nu) + g([X, \partial_\nu], \partial_\mu) + g(X, [\partial_\mu, \partial_\nu]) \right) \\ &= \frac{1}{2} \left(\partial_\mu (g(\partial_\nu, X)) + \partial_\nu (g(\partial_\mu, X)) \right) - \frac{1}{2} \left(X(g_{\mu\nu}) - g([X, \partial_\mu], \partial_\nu) - g(\partial_\mu, [X, \partial_\nu]) \right). \end{aligned}$$

So

$$\begin{aligned}
\Pi_{\mu\nu} &= \frac{1}{2}\left(X(g_{\mu\nu}) - g([X,\partial_\mu],\partial_\nu) - g(\partial_\mu,[X,\partial_\nu])\right) \\
&= \frac{1}{2}\left(\partial_\mu(g(\partial_\nu,X)) + \partial_\nu(g(\partial_\mu,X))\right) - g(\nabla_\mu\partial_\nu,X).
\end{aligned}$$

Writing $X = k\partial_0$, where $k(t) := (a(t))^{2-n}$, we have

$$\Pi_{\mu\nu} = \frac{1}{2}\left(\partial_\mu(kg_{\nu 0}) + \partial_\nu(kg_{\mu 0})\right) - g(\nabla_\mu\partial_\nu, k\partial_0) = -\dot{k}\delta_{\mu 0}\delta_{\nu 0} + k\Gamma^0_{\mu\nu}.$$

We have $\nabla_\mu X_\nu = \nabla_\mu(-k\delta_{\nu 0}) = -\dot{k}\delta_{\mu 0}\delta_{\nu 0} + k\Gamma^0_{\mu\nu}$. So $\Pi_{\mu\nu} = \nabla_\mu X_\nu$, and consequently,

$$\nabla_\mu J^\mu = \nabla_\mu(T^{\mu\nu}X_\nu) = 0 + T^{\mu\nu}\nabla_\mu X_\nu = T^{\mu\nu}\Pi_{\mu\nu}.$$

This completes the proof of our claim that $\nabla_\mu J^\mu = T^{\mu\nu}\Pi_{\mu\nu}$.

So the 'bulk term' is

$$\begin{aligned}
\nabla_\mu J^\mu &= T^{\mu\nu}\Pi_{\mu\nu} \\
&= (n-2)\dot{a}a^{1-n}\dot{\phi}^2 + \frac{n-2}{2}\dot{a}a^{1-n}\partial_\alpha\phi\partial^\alpha\phi + \dot{a}a^{-1-n}\delta^{ij}\partial_i\phi\partial_j\phi - \frac{n}{2}\dot{a}a^{1-n}\partial_\alpha\phi\partial^\alpha\phi \\
&= (n-1)\dot{a}a^{1-n}\dot{\phi}^2 \geqslant 0.
\end{aligned}$$

(d) In this part, we use the divergence theorem on a suitable domain with the current J in order to get our uniform-in-time bound on $\|\Delta\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}$. For each $R > 0$, define the set $B_0 := \{(t_0,\mathbf{x}) \in I \times \mathbb{R}^n : \langle \mathbf{x},\mathbf{x}\rangle_{\mathbb{R}^n} \leqslant R^2\}$. The future domain of dependence of B_0 is the set

$$D^+(B_0) := \left\{ p \in M \;\middle|\; \text{Every past inextendible causal curve through } p \text{ intersects } B_0. \right\}.$$

Here by a *causal curve*, we mean one whose tangent vector at each point is a causal vector. A curve $c : (a,b) \to M$ which is smooth and future directed (that is, $\dot{c}$ is future-pointing) is called *past inextendible* if $\lim_{t\to a} c(t)$ does not exist.

Let $t_1 > t_0$. We will now apply the divergence theorem to the region

$$\mathcal{R} := D^+(B_0) \cap \{(t,\mathbf{x}) \in M : t \leqslant t_1\}.$$

See Figure 3. For preliminaries on the divergence theorem in the context of a time-oriented Lorentzian manifold, we refer the reader to [37, Appendix B]. We have

$$\int_{\mathcal{R}} (\nabla_\mu J^\mu)\epsilon = \int_{\partial\mathcal{R}} J \lrcorner \epsilon,$$

where $\partial\mathcal{R}$ denotes the boundary of $\mathcal{R}$, ϵ is the volume form on M induced by g, and $\lrcorner$ denotes contraction in the first index.

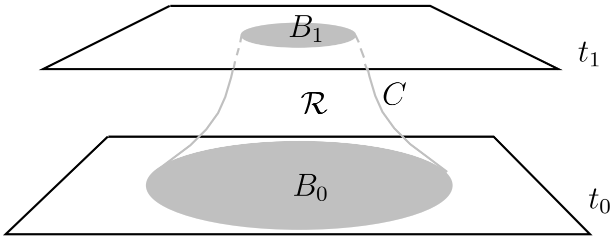

Figure 1: The region $\mathcal{R}$ with its boundary $\partial\mathcal{R}$.

Since J is past-pointing, the boundary integral over the null portion C of the boundary $\partial\mathcal{R}$ is nonpositive. Also, because $\nabla_\mu J^\mu$ is nonnegative, we have that the volume integral over $\mathcal{R}$ is nonnegative. This gives an inequality on the two boundary integrals, one over B_0, and the other over $B_1 := D^+(B_0) \cap \{t = t_1\}$, as follows:

$$\int_{B_0} \frac{1}{2}(a^2\dot{\phi}^2 + \delta^{ij}\partial_i\phi\partial_j\phi)d^n\mathbf{x} \geqslant \int_{B_1} \frac{1}{2}(a^2\dot{\phi}^2 + \delta^{ij}\partial_i\phi\partial_j\phi)d^n\mathbf{x}.$$

Passing the limit $R \to \infty$ yields $E(t_0) \geqslant E(t_1)$. As the choice of $t_1 > t_0$ was arbitrary, we have for all $t \geqslant t_0$, $E(t) \leqslant E(t_0) < \infty$. The finiteness of $E(t_0)$ follows from our assumption that $\phi_0 \in H^k(\mathbb{R}^n)$ and $\phi_1 \in H^{k-1}(\mathbb{R}^n)$ for a k satisfying $k > \frac{n}{2} + 2 \geqslant 1$. From here, it follows that for all $t \geqslant t_0$,

$$\int_{\mathbb{R}^n} \dot{\phi}^2 d^n\mathbf{x} \lesssim \frac{1}{a^2}, \quad \text{and} \quad \int_{\mathbb{R}^n} \delta^{ij}\partial_i\phi\partial_j\phi d^n\mathbf{x} \lesssim 1.$$

But since each partial derivative $\partial_i\phi$ is also a solution of the wave equation, and as $k \geqslant 2$, we obtain, by applying the above to the partial derivatives $\partial_i\phi$, that also $\int_{\mathbb{R}^n}(\Delta\phi)^2 d^n\mathbf{x} \lesssim 1$. In fact, since $k > \frac{n}{2} + 2$, we also obtain that for a $k' > \frac{n}{2}$, $\|\Delta\phi\|_{H^{k'}(\mathbb{R}^n)} \lesssim 1$. Finally, by the Sobolev inequality (see e.g. [20, (7.30), p.158]), we obtain

$$\|\Delta\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim 1. \tag{2}$$

This completes Step 1 of the proof of Theorem 3.2.

Step 2: The wave equation in conformal coordinates.

The key point of departure from the earlier derivation of the estimates from [9] is the usage of 'conformal coordinates', which renders the wave equation in a form where it becomes possible to integrate, leaving essentially just the time derivative of ϕ with other terms (e.g. $\Delta\phi$) for which we have a known bound. An application of the triangle inequality will then deliver the desired bound.

Define $\tau = \displaystyle\int_{t_0}^t \frac{1}{a(s)}ds$. Then $\dfrac{d\tau}{dt} = \dfrac{1}{a(t)}$ and $a(t)\dfrac{d}{dt} = \dfrac{d}{d\tau}$.

With a slight abuse of notation, we write $a(\tau) := a(t(\tau))$. Then $dt = a(\tau)d\tau$. So

$$g = -dt^2 + (a(t))^2\left((dx^1)^2 + \cdots + (dx^n)^2\right) = (a(\tau))^2\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right).$$

The wave equation $\Box_g \phi = 0$ can be rewritten as $\partial_\mu(\sqrt{-g}\,\partial^\mu \phi) = 0$, which becomes

$$\partial_\mu(a^{n+1}\partial^\mu \phi) = 0.$$

Separating the partial derivative operators with respect to the τ and $\mathbf{x}$ coordinates, we obtain the wave equation in conformal coordinates $\partial_\tau(a^{n-1}\partial_\tau \phi) = a^{n-1}\Delta\phi$, where Δ is the usual Laplacian on $\mathbb{R}^n$. This completes Step 2 of the proof of Theorem 3.2.

Step 3: $n > 2$ **and** $a(t) = e^t$**.**

We have

$$\tau = \int_{t_0}^{t} \frac{1}{e^s} ds = e^{-t_0} - \frac{1}{e^t} = e^{-t_0} - \frac{1}{a}, \tag{3}$$

and so $a(\tau) = \dfrac{1}{e^{-t_0} - \tau}$. We note that $\tau \in [0, e^{-t_0})$. Also, $a(\tau = 0) = \dfrac{1}{e^{-t_0}} = e^{t_0} = a(t = t_0)$. Integrating $\partial_\tau(a^{n-1}\partial_\tau \phi) = a^{n-1}\Delta\phi$ from $\tau = 0$ to τ, we obtain

$$a^{n-1}\partial_\tau \phi - a(t_0)^{n-1}\ \partial_\tau \phi|_{\tau=0} = \int_0^\tau \Delta\phi \frac{1}{(e^{-t_0} - \tau)^{n-1}} d\tau,$$

and so $a^{n-1} a \partial_t \phi = a(t_0)^{n-1} a(t_0)\ \partial_t \phi|_{t=t_0} + \displaystyle\int_0^\tau \Delta\phi \frac{1}{(e^{-t_0} - \tau)^{n-1}} d\tau$, that is,

$$\partial_t \phi = (a(t))^{-n} \left(a(t_0)^n \phi_1 + \int_0^\tau \Delta\phi \frac{1}{(e^{-t_0} - \tau)^{n-1}} d\tau \right).$$

Hence, using the bound from (2), namely $\|\Delta\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leqslant C$ for all $t \geqslant t_0$, we obtain

$$\begin{aligned}
\|\partial_t \phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} &\leqslant (a(t))^{-n}\left(a(t_0)^n \|\phi_1\|_{L^\infty(\mathbb{R}^n)} + \int_0^\tau \|\Delta\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \frac{1}{(e^{-t_0} - \tau)^{n-1}} d\tau \right) \\
&\leqslant (a(t))^{-n}\left(a(t_0)^n \|\phi_1\|_{L^\infty(\mathbb{R}^n)} + \frac{C}{n-2}\left((e^{-t_0} - \tau)^{2-n} - (e^{-t_0})^{2-n}\right)\right) \\
&\leqslant (a(t))^{-n}\left(a(t_0)^n \|\phi_1\|_{L^\infty(\mathbb{R}^n)} + \frac{C}{n-2}\left((a(t))^{n-2} - (a(t_0))^{n-2}\right)\right) \\
&\leqslant (a(t))^{-n}(a(t))^{n-2}\left(\frac{a(t_0)^n \|\phi_1\|_{L^\infty(\mathbb{R}^n)}}{(a(t))^{n-2}} + \frac{C}{n-2}\left(1 - \left(\frac{a(t_0)}{a(t)}\right)^{n-2}\right)\right) \\
&\leqslant \frac{1}{(a(t))^2}\left(\frac{a(t_0)^n \|\phi_1\|_{L^\infty(\mathbb{R}^n)}}{(a(t_0))^{n-2}} + \frac{C}{n-2}(1-0)\right).
\end{aligned}$$

Hence $\|\partial_t \phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leqslant \dfrac{1}{(a(t))^2}\left((a(t_0))^2 \|\phi_1\|_{L^\infty(\mathbb{R}^n)} + \dfrac{C}{n-2}\right)$, and so

$$\|\partial_t \phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2}.$$

This completes the proof of Theorem 3.2. $\square$

Remark 3.3. The case when $n = 2$ and $a(t) = e^t$:

Integrating $\partial_\tau(a\partial_\tau\phi) = a\Delta\phi$ from $\tau = 0$ to τ, we obtain

$$a\partial_\tau\phi - a(t_0)\, \partial_\tau\phi|_{\tau=0} = \int_0^\tau \Delta\phi \frac{1}{e^{-t_0} - \tau} d\tau,$$

and so $\partial_t\phi = (a(t))^{-2}\left(a(t_0)^2\phi_1 + \int_0^\tau \Delta\phi \frac{1}{e^{-t_0} - \tau} d\tau\right)$. Hence

$$\begin{aligned}
\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} &\leqslant (a(t))^{-2}\left(a(t_0)^2\|\phi_1\|_{L^\infty(\mathbb{R}^2)} + \int_0^\tau \|\Delta\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \frac{1}{e^{-t_0} - \tau} d\tau\right) \\
&\leqslant (a(t))^{-2}\left(a(t_0)^2\|\phi_1\|_{L^\infty(\mathbb{R}^2)} + C\left(-\log(e^{-t_0} - \tau)\Big|_0^\tau\right)\right) \\
&\leqslant (a(t))^{-2}(\log a(t))\left(\frac{a(t_0)^2\|\phi_1\|_{L^\infty(\mathbb{R}^2)}}{\log a(t)} + C\left(1 - \frac{\log a(t_0)}{\log a(t)}\right)\right) \\
&\leqslant (a(t))^{-2}(\log a(t))\left(\frac{a(t_0)^2\|\phi_1\|_{L^\infty(\mathbb{R}^2)}}{t_0} + C\right),
\end{aligned}$$

and so $\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2}\log a(t)$.

This can be viewed as an improvement to [9, Theorem 1] in the special case when $a(t) = e^t$ and $n = 2$, since $\log a(t) = t \lesssim e^{\delta t} = 1 + \delta t + \cdots$.

Using a similar method, one can also obtain an improvement to [9, Theorem 2]. But we will postpone this discussion until after Section 5, and prove this result as Theorem 6.3 of Section 6, since we will need some preliminaries about the RNdS spacetime, which will be established in Section 5.

4 Decay in the de Sitter universe in flat FLRW form

In this section, we will obtain decay rates on $\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}$ for a solution the the Klein-Gordon equation in the de Sitter universe in flat FLRW form, that is, we will prove Theorem 4.1, stated below.

Recall that the Klein-Gordon equation is $\Box_g\phi - m^2\phi = 0$, that is,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, \partial^\mu\phi) - m^2\phi = 0.$$

In the case of the de Sitter universe in flat FLRW form, we obtain

$$-\ddot{\phi} - \frac{n\dot{a}}{a}\dot{\phi} + \frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi - m^2\phi = 0. \tag{4}$$

The result we will show in this section is the following.

Theorem 4.1.

Suppose that

- $I \subset \mathbb{R}$ *is an open interval of the form* $(t_*, +\infty)$, $t_0 \in I$,
- $m \in \mathbb{R}$,
- $n > 2$,
- (M, g) *is the expanding de Sitter universe in flat FLRW form, with flat* n*-dimensional sections, given by* $I \times \mathbb{R}^n$, *with the metric* $g = -dt^2 + e^{2t}\left((dx^1)^2 + \cdots + (dx^n)^2\right)$,
- $k > \frac{n}{2} + 2$, $\phi_0 \in H^k(\mathbb{R}^n)$, $\phi_1 \in H^{k-1}(\mathbb{R}^n)$, *and*
- ϕ *is a smooth solution to the Cauchy problem*

$$\begin{cases} \Box_g \phi - m^2 \phi &= 0, & (t \geqslant t_0,\ \mathbf{x} \in \mathbb{R}^n), \\ \phi(t_0, \mathbf{x}) &= \phi_0(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^n), \\ \partial_t \phi(t_0, \mathbf{x}) &= \phi_1(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^n). \end{cases}$$

Then for all $t \geqslant t_0$, *with* $a(t) := e^t$, *we have*

$$\|\phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim \begin{cases} a^{-\frac{n}{2}} & \text{if} \quad |m| > \frac{n}{2}, \\ a^{-\frac{n}{2}} \log a & \text{if} \quad |m| = \frac{n}{2}, \\ a^{-\frac{n}{2} + \sqrt{\frac{n^2}{4} - m^2}} & \text{if} \quad |m| < \frac{n}{2}. \end{cases}$$

We arrive at the guesses for the specific estimates given in Theorem 4.1 above, based on an analysis using Fourier modes, assuming spatially periodic solutions. This Fourier mode analysis is given in Appendix A.

4.1 Preliminary energy function and estimates

In this subsection, we will obtain preliminary bounds on the norms $\|\dot{\phi}(t, \cdot)\|_{H^{k-1}(\mathbb{R}^n)}$ and $\|\partial_i \phi(t, \cdot)\|_{H^{k-1}(\mathbb{R}^n)}$, which will be needed in the subsequent steps for proving Theorem 4.1.

Define the energy-momentum tensor T by $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \dfrac{1}{2} g_{\mu\nu}(\partial_\alpha \phi \partial^\alpha \phi + m^2 \phi^2)$.

Then $\nabla_\mu T^{\mu\nu} = 0$. Also, in particular, $T_{00} = \dfrac{1}{2}\left(\dot{\phi}^2 + \dfrac{1}{a^2}|\nabla \phi|^2 + m^2 \phi^2\right) = T^{00}$.

Set $X = a^{-n} \dfrac{\partial}{\partial t}$. Then X is time-like and hence causal, and X is future pointing.

Define J by $J^\mu = T^{\mu\nu} X_\nu$. Then J is causal and past-pointing.

Let $N = \dfrac{\partial}{\partial t}$. Define the energy E by

$$E(t) = \int_{\{t\}\times\mathbb{R}^n} J_\mu N^\mu = \int_{\mathbb{R}^n} \frac{1}{2}\left(\dot{\phi}^2 + \frac{1}{a^2}|\nabla\phi|^2 + m^2\phi^2\right) d^n\mathbf{x}.$$

Define $\Pi = \frac{1}{2}\mathcal{L}_X g = -dt\mathcal{L}_X dt + a^{-n+1}\dot{a}\left((dx^1)^2 + \cdots + (dx^n)^2\right)$. As $\mathcal{L}_X dt = -na^{-n-1}\dot{a}dt$, we have $\Pi = na^{-n-1}\dot{a}dt^2 + a^{-n+1}\dot{a}\left((dx^1)^2 + \cdots + (dx^n)^2\right)$. Hence

$$\nabla_\mu J^\mu = T^{\mu\nu}\Pi_{\mu\nu} = \frac{a^{-n-1}\dot{a}}{2}\left(2n\dot{\phi}^2 + \frac{2}{a^2}|\nabla\phi|^2\right) \geqslant 0.$$

For $R > 0$, define $B_0 := \{(t_0, \mathbf{x}) \in I \times \mathbb{R}^n : \langle \mathbf{x}, \mathbf{x}\rangle_{\mathbb{R}^n} \leqslant R^2\}$. The future domain of dependence of B_0 is denoted by $D^+(B_0)$.

Let $t_1 > t_0$. We will now apply the divergence theorem to the region

$$\mathcal{R} := D^+(B_0) \cap \{(t,\mathbf{x}) \in M : t \leqslant t_1\}.$$

We have $\int_{\mathcal{R}} (\nabla_\mu J^\mu)\epsilon = \int_{\partial\mathcal{R}} J \lrcorner \epsilon$. Using

- $\nabla_\mu J^\mu \geqslant 0$, and
- the fact that the boundary contribution on C, the null portion of $\partial\mathcal{R}$, is nonpositive (since J is causal and past-pointing),

we obtain the inequality

$$\int_{B_0} \frac{1}{2}\left(\dot{\phi}^2 + \frac{1}{a^2}|\nabla\phi|^2 + m^2\phi^2\right) d^n\mathbf{x} \geqslant \int_{B_1} \frac{1}{2}\left(\dot{\phi}^2 + \frac{1}{a^2}|\nabla\phi|^2 + m^2\phi^2\right) d^n\mathbf{x}.$$

Passing the limit $R \to \infty$ yields $E(t_1) \leqslant E(t_0) < +\infty$. As $t_1 > t_0$ was arbitrary, we obtain for all $t \geqslant t_0$,

$$E(t) = \int_{\mathbb{R}^n} \frac{1}{2}\left(\dot{\phi}^2 + \frac{1}{a^2}|\nabla\phi|^2 + m^2\phi^2\right) d^n\mathbf{x} \leqslant E(t_0) \leqslant \infty.$$

From here, it follows that for all $t \geqslant t_0$,

$$\int_{\mathbb{R}^n} \dot{\phi}^2 d^n\mathbf{x} \lesssim 1, \quad \int_{\mathbb{R}^n} |\nabla\phi|^2 d^n\mathbf{x} \lesssim a^2, \quad \text{and} \quad \int_{\mathbb{R}^n} \phi^2 d^n\mathbf{x} \lesssim 1 \quad (\text{if } m \neq 0).$$

But since each partial derivative $(\partial_{x^1})^{i_1}\cdots(\partial_{x^n})^{i_n}\phi$ is also a solution of the Klein-Gordon equation, it follows from $\phi_0 \in H^k(\mathbb{R}^n)$ and $\phi_1 \in H^{k-1}(R^n)$ for a $k > \frac{n}{2} + 2$, that also $\phi(t,\cdot) \in H^k(\mathbb{R}^n)$ and $\partial_t\phi(t,\cdot) \in H^{k-1}(\mathbb{R}^n)$, and moreover

$$\|\dot{\phi}\|_{H^{k'}(\mathbb{R}^n)} \lesssim 1, \quad \|\partial_i\phi\|_{H^{k'}(\mathbb{R}^n)} \lesssim a, \quad \text{and} \quad \|\phi\|_{H^{k'}(\mathbb{R}^n)} \lesssim 1 \ \ (\text{if } m \neq 0),$$

where $k' := k - 1$.

4.2 The auxiliary function ψ and its PDE

In this subsection, we introduce the auxiliary function ψ constructed from the solution ϕ to the Klein-Gordon equation, and also derive a PDE satisfied by ψ.

Motivated by the decay rate we anticipate for ϕ, we define the auxiliary function ψ by

$$\psi := a^\kappa \phi,$$

where

$$\kappa := \begin{cases} \frac{n}{2} & \text{if } |m| \geqslant \frac{n}{2}, \\ \frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2} & \text{if } |m| \leqslant \frac{n}{2}. \end{cases}$$

Then, using (4), it can be shown that ψ satisfies the equation

$$\ddot{\psi} + (\kappa^2 - n\kappa + m^2)\psi + (n - 2\kappa)\dot{\psi} - \frac{1}{a^2}\Delta\psi = 0. \tag{5}$$

4.3 The case $|m| > \frac{n}{2}$

In this subsection, we will give the proof of Theorem 4.1 in the case when $|m| > \frac{n}{2}$.

We have $\kappa = \frac{n}{2}$, so that $n - 2\kappa = 0$, while $\kappa^2 - n\kappa + m^2 = m^2 - \frac{n^2}{4}$, and thus (5) becomes

$$\ddot{\psi} - \frac{1}{a^2}\Delta\psi + \left(m^2 - \frac{n^2}{4}\right)\psi = 0.$$

We note that if $\phi \in H^\ell(\mathbb{R}^n)$ and $\dot{\phi} \in H^{\ell-1}(\mathbb{R}^n)$ for some ℓ, then $\psi \in H^\ell(\mathbb{R}^n)$ too, and also

$$\dot{\psi} = \frac{n}{2} a^{\frac{n}{2}-1}\dot{a}\phi + a^{\frac{n}{2}}\dot{\phi} \in H^{\ell-1}(\mathbb{R}^n).$$

Define the new energy $\mathcal{E}$, associated with the ψ-evolution, by

$$\mathcal{E}(t) := \frac{1}{2}\int_{\mathbb{R}^n} \left(\dot{\psi}^2 + \frac{1}{a^2}|\nabla\psi|^2 + \left(m^2 - \frac{n^2}{4}\right)\psi^2\right) d^n\mathbf{x} \geqslant 0.$$

Then using the fact that $a = e^t = \dot{a} > 0$, and also equation (5), we obtain

$$\begin{aligned} \dot{\mathcal{E}}(t) &= \int_{\mathbb{R}^n} \left(\dot{\psi}\ddot{\psi} - \frac{a\dot{a}}{a^4}|\nabla\psi|^2 + \frac{1}{a^2}\langle\nabla\psi, \nabla\dot{\psi}\rangle + \left(m^2 - \frac{n^2}{4}\right)\psi\dot{\psi}\right) d^n\mathbf{x} \\ &\leqslant \int_{\mathbb{R}^n} \left(\dot{\psi}\ddot{\psi} + \frac{1}{a^2}\langle\nabla\psi, \nabla\dot{\psi}\rangle + \left(m^2 - \frac{n^2}{4}\right)\psi\dot{\psi}\right) d^n\mathbf{x} \\ &\leqslant \int_{\mathbb{R}^n} \left(\dot{\psi}\Big(\frac{1}{a^2}\Delta\psi - \Big(m^2 - \frac{n^2}{4}\Big)\psi\Big) + \frac{1}{a^2}\langle\nabla\psi, \nabla\dot{\psi}\rangle + \left(m^2 - \frac{n^2}{4}\right)\psi\dot{\psi}\right) d^n\mathbf{x}. \end{aligned}$$

But

$$\int_{\mathbb{R}^n} \left(\dot{\psi}\Big(\frac{1}{a^2}\Delta\psi - \Big(m^2 - \frac{n^2}{4}\Big)\psi\Big) + \frac{1}{a^2}\langle\nabla\psi, \nabla\dot{\psi}\rangle + \Big(m^2 - \frac{n^2}{4}\Big)\psi\dot{\psi}\right) d^n\mathbf{x}.$$
$$= \frac{1}{a^2}\int_{\mathbb{R}^n} \left(\dot{\psi}\Delta\psi + \langle\nabla\psi, \nabla\dot{\psi}\rangle\right) d^n\mathbf{x} = \frac{1}{a^2}\int_{\mathbb{R}^n} \nabla\cdot(\dot{\psi}\nabla\psi) d^n\mathbf{x}.$$

For a fixed t, and for a ball $B(\mathbf{0}, r) \subset \mathbb{R}^n$, where $r > 0$, it follows from the divergence theorem (since $\dot{\psi}$ and $\nabla\psi$ are smooth), that

$$\int_{B(\mathbf{0},r)} \nabla\cdot(\dot{\psi}\nabla\psi)\, d^n\mathbf{x} = \int_{\partial B(\mathbf{0},r)} \dot{\psi}\langle\nabla\psi, \mathbf{n}\rangle\, d\sigma_r,$$

where $d\sigma_r$ is the surface area measure on the sphere $S_r = \partial B(\mathbf{0}, r)$, and $\mathbf{n}$ is the outward-pointing unit normal. The right hand side surface integral tends to 0 as $r \to +\infty$, by an application of Lemma B.1, given in Appendix B.

So for $t \geqslant t_0$, we have $\mathcal{E}'(t) \leqslant 0$, which yields $\mathcal{E}(t) \leqslant \mathcal{E}(t_0)$. In particular, for all $t \geqslant t_0$, $\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim C$, that is, $\|a^{\frac{n}{2}}\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim C$, and so[6] $\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim a^{-\frac{n}{2}}$. Then with enough regularity on ϕ_0, ϕ_1 at the outset, that is, if $\phi_0 \in H^k(\mathbb{R}^n)$ and $\phi_1 \in H^{k-1}(\mathbb{R}^n)$ for a $k > \frac{n}{2} + 2$, and by considering $(\partial_{x^1})^{i_1}\cdots(\partial_{x^n})^{i_n}\phi$ as a solution to the Klein-Gordon equation, we arrive at[7] $\|\phi(t,\cdot)\|_{H^{k'}(\mathbb{R}^n)} \lesssim a^{-\frac{n}{2}}$, where $k' := k - 2 > \frac{n}{2}$. Using the Sobolev inequality, we then obtain for all $t \geqslant t_0$, $\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim a^{-\frac{n}{2}}$. This completes the proof of Theorem 4.1 in the case when $|m| > \frac{n}{2}$.

4.4 The case $|m| < \frac{n}{2}$

In this subsection, we will give the proof of Theorem 4.1 in the case when $|m| < \frac{n}{2}$.

We have $\kappa = \frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2}$, $n - 2\kappa = 2\sqrt{\frac{n^2}{4} - m^2} > 0$, and $\kappa^2 - n\kappa + m^2 = 0$.

Equation (5) becomes $\ddot{\psi} + 2\Big(\sqrt{\frac{n^2}{4} - m^2}\Big)\,\dot{\psi} - \frac{1}{a^2}\Delta\psi = 0.$

Defining

$$\widetilde{\mathcal{E}}(t) := \frac{1}{2}\int_{\mathbb{R}^n} \left(\dot{\psi}^2 + \frac{1}{a^2}|\nabla\psi|^2\right) d^n\mathbf{x} \geqslant 0,$$

[6] We note that to reach this conclusion, we used Lemma B.1, and needed $\dot{\psi}(t,\cdot), \nabla\psi(t,\cdot) \in H^1(\mathbb{R}^n)$, which means that it is sufficient that the initial conditions for ϕ are such that $\varphi_0 \in H^2(\mathbb{R}^n)$ and $\varphi_1 \in H^1(\mathbb{R}^n)$.

[7] Note that in order to use the estimate $\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leqslant a^{-\frac{n}{2}}$, for $D\phi := (\partial_{x^1})^{i_1}\cdots(\partial_{x^n})^{i_n}\phi$ replacing ϕ, where $|(i_1,\ldots,i_n)| =: k'$, we must ensure that the initial conditions for $D\phi$, namely $(D\phi(t_0,\cdot), D\dot{\phi}(t_0,\cdot))$ is in $(H^2(\mathbb{R}^n), H^1(\mathbb{R}^n))$, which is guaranteed if the initial condition for ϕ, namely (ϕ_0, ϕ_1) is in $(H^k(\mathbb{R}^n), H^{k-1}(\mathbb{R}^n))$, with $k - k' = 2$.

we obtain

$$\begin{aligned}\dot{\widetilde{\mathcal{E}}}(t) &= \int_{\mathbb{R}^n}\left(\dot\psi\ddot\psi-\frac{\dot a}{a^3}|\nabla\psi|^2+\frac{1}{a^2}\langle\nabla\psi,\nabla\dot\psi\rangle\right)d^n\mathbf{x}\\ &= \int_{\mathbb{R}^n}\left(\dot\psi\Big(-2\Big(\sqrt{\frac{n^2}{4}-m^2}\Big)\,\dot\psi+\frac{1}{a^2}\Delta\psi\Big)-\frac{\dot a}{a^3}|\nabla\psi|^2+\frac{1}{a^2}\langle\nabla\psi,\nabla\dot\psi\rangle\right)d^n\mathbf{x}\\ &= -2\Big(\sqrt{\frac{n^2}{4}-m^2}\Big)\int_{\mathbb{R}^n}\dot\psi^2d^n\mathbf{x}-\frac{\dot a}{a^3}\int_{\mathbb{R}^n}|\nabla\psi|^2d^n\mathbf{x}.\end{aligned}$$

Using $a=e^t=\dot a$, we obtain

$$\begin{aligned}\dot{\widetilde{\mathcal{E}}}(t) &= -4\Big(\sqrt{\frac{n^2}{4}-m^2}\Big)\,\frac{1}{2}\int_{\mathbb{R}^n}\dot\psi^2d^n\mathbf{x}-2\frac{1}{2}\int_{\mathbb{R}^n}\frac{1}{a^2}|\nabla\psi|^2d^n\mathbf{x}\\ &\leqslant -\min\left\{4\Big(\sqrt{\frac{n^2}{4}-m^2}\Big),\ 2\right\}\cdot\frac{1}{2}\int_{\mathbb{R}^n}\left(\dot\psi^2+\frac{1}{a^2}|\nabla\psi|^2\right)d^n\mathbf{x}=-\,\theta\cdot\widetilde{\mathcal{E}}(t),\end{aligned}$$

where $\theta:=\min\left\{4\Big(\sqrt{\frac{n^2}{4}-m^2}\Big)\ ,\ 2\right\}>0$. So $\dot{\widetilde{\mathcal{E}}}(t)+\theta\cdot\widetilde{\mathcal{E}}(t)\leqslant 0$.

Multiplying throughout by $e^{\theta t}>0$, we obtain $\dfrac{d}{dt}\Big(e^{\theta t}\cdot\widetilde{\mathcal{E}}(t)\Big)\leqslant 0$.

Integrating from t_0 to t yields $e^{\theta t}\cdot\widetilde{\mathcal{E}}(t)\leqslant e^{\theta t_0}\cdot\widetilde{\mathcal{E}}(t_0)$, that is, $\widetilde{\mathcal{E}}(t)\lesssim e^{-\theta t}$.

In particular, $\|\dot\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)}\leqslant\sqrt{2\widetilde{\mathcal{E}}(t)}\lesssim e^{-\frac{\theta}{2}t}$.

We have $\psi(t,\mathbf{x})=\psi(t_0,\mathbf{x})+\displaystyle\int_{t_0}^t(\partial_t\psi)(s,\mathbf{x})ds$, and so

$$\begin{aligned}\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\leqslant \|\psi(t_0,\cdot)\|_{L^2(\mathbb{R}^n)}+\int_{t_0}^t\|(\partial_t\psi)(s,\cdot)\|_{L^2(\mathbb{R}^n)}ds,\\ &\lesssim A+\int_{t_0}^t Be^{-\frac{\theta}{2}s}ds=A+B\frac{e^{-\frac{\theta}{2}t_0}-e^{-\frac{\theta}{2}t}}{\theta/2}\lesssim C.\end{aligned}$$

Thus for all $t\geqslant t_0$, we have $\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)}=a^{-\kappa}\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)}\lesssim a^{-\kappa}$. By considering $(\partial_{x^1})^{i_1}\cdots(\partial_{x^n})^{i_n}\phi$, and using the Sobolev inequality, we have for all $t\geqslant t_0$,

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}\lesssim a^{-\kappa}=a^{-(\frac{n}{2}-\sqrt{\frac{n^2}{4}-m^2})}.$$

This completes the proof of Theorem 4.1 in the case when $|m|<\frac{n}{2}$.

4.5 The case $|m|=\frac{n}{2}$

Finally, in this section, we will give the proof of Theorem 4.1 in the remaining case, namely when $|m|=\frac{n}{2}$.

We have $\kappa = \frac{n}{2}$, and equation (5) becomes $\ddot{\psi} - \dfrac{1}{a^2}\Delta\psi = 0$.

Defining the same energy as we used earlier in the case when $|m| < \frac{n}{2}$,

$$\widetilde{\mathcal{E}}(t) := \frac{1}{2}\int_{\mathbb{R}^n} \left(\dot{\psi}^2 + \frac{1}{a^2}|\nabla\psi|^2\right) d^n\mathbf{x} \geqslant 0,$$

we obtain

$$\begin{aligned}\dot{\widetilde{\mathcal{E}}}(t) &= \int_{\mathbb{R}^n} \left(\dot{\psi}\ddot{\psi} - \frac{\dot{a}}{a^3}|\nabla\psi|^2 + \frac{1}{a^2}\langle\nabla\psi, \nabla\dot{\psi}\rangle\right) d^n\mathbf{x} \\ &= \int_{\mathbb{R}^n} \left(\dot{\psi}\frac{1}{a^2}\Delta\psi - \frac{\dot{a}}{a^3}|\nabla\psi|^2 + \frac{1}{a^2}\langle\nabla\psi, \nabla\dot{\psi}\rangle\right) d^n\mathbf{x} = -\frac{\dot{a}}{a^3}\int_{\mathbb{R}^n} |\nabla\psi|^2 d^n\mathbf{x} \leqslant 0.\end{aligned}$$

So $\widetilde{\mathcal{E}}(t) \leqslant \widetilde{\mathcal{E}}(t_0)$ for $t \geqslant t_0$. In particular, $\|\dot{\psi}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim B$ for $t \geqslant t_0$.

Again, $\psi(t,\mathbf{x}) = \psi(t_0,\mathbf{x}) + \displaystyle\int_{t_0}^{t} (\partial_t\psi)(s,\mathbf{x})ds$, gives

$$\begin{aligned}\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\leqslant \|\psi(t_0,\cdot)\|_{L^2(\mathbb{R}^n)} + \int_{t_0}^{t} \|(\partial_t\psi)(s,\cdot)\|_{L^2(\mathbb{R}^n)} ds, \\ &\lesssim A' + \int_{t_0}^{t} B ds \lesssim A + Bt \lesssim \log a.\end{aligned}$$

Thus for all $t \geqslant t_0$, we have[8] $\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)} = a^{-\kappa}\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim a^{-\kappa}\log a$. Hence (by considering $(\partial_{x^1})^{i_1}\cdots(\partial_{x^n})^{i_n}\phi$, and using the Sobolev inequality)

$$\forall t \geqslant t_0, \quad \|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim a^{-\frac{n}{2}}\log a. \tag{6}$$

This completes the proof of Theorem 4.1.

5 Decay in the cosmological region of the RNdS spacetime

In this section, we will obtain decay rates on $\|\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})}$ for a solution to the Klein-Gordon equation in the cosmological region of the Reissner-Nordström-de Sitter (RNdS) spacetime, that is, we will prove Theorem 5.3, stated below. We will begin with introducing the RNdS spacetime, and collecting some technical facts which will be used while proving Theorem 5.3.

[8] One can show that this bound is sharp; see Appendix C.

The Reissner-Nordström-de Sitter spacetime (M, g) is a solution to the Einstein-Maxwell equations with a positive cosmological constant, and it represents a pair[9] of antipodal charged black holes in a spherical[10] universe which is undergoing accelerated expansion. The Reissner-Nordström-de Sitter metric in $n+1$ dimensions is given by

$$g = -\frac{1}{V}dr^2 + V dt^2 + r^2 d\Omega^2,$$

where $V = r^2 + \dfrac{2M}{r^{n-2}} - \dfrac{e^2}{r^{n-1}} - 1$, and $d\Omega^2$ is the unit round metric on S^{n-1}.

The constants M and e are proportional to the mass and the charge, respectively, of the black holes, and the cosmological constant is chosen to be

$$\Lambda = \frac{n(n-1)}{2}$$

by an appropriate choice of units.

Consider the polynomial

$$p(r) := r^{n-1}V(r) = r^{n+1} - r^{n-1} + 2Mr - e^2.$$

As $p(0) = -e^2 < 0$ and as $p(r) \overset{r\to\infty}{\longrightarrow} \infty$, it follows that p will have a real root in $(0, +\infty)$, and the largest real root of p, which we denote by r_c, must be positive. If $r > r_c$, then clearly $p(r) > 0$, and so also $V(r) > 0$.

It can also be seen that p has at most three distinct positive roots. Suppose, on the contrary, that p has more than three distinct positive roots: $r_1 < r_2 < r_3 < r_4$. Applying Rolle's theorem to p on $[r_i, r_{i+1}]$ $(i = 1, 2, 3)$, we conclude that p' must have three distinct roots $r'_i \in (r_i, r_{i+1})$ $(i = 1, 2, 3)$. Applying Rolle's theorem to p' on $[r'_i, r'_{i+1}]$ $(i = 1, 2)$, we conclude that p'' must have two distinct roots $r''_i \in (r'_i, r'_{i+1})$ $(i = 1, 2)$. But

$$p'' = r^{n-3}n(n+1)\Big(r^2 - \frac{(n-1)(n-2)}{n(n+1)}\Big),$$

which has only one positive root, a contradiction.

The 'subextremality' assumption on the RNdS spacetime made in Theorem 5.3, refers to a nondegeneracy of the positive roots of p: we assume that there are exactly three positive roots, r_-, r_+ and r_c, and

$$0 < r_- < r_+ < r_c.$$

[9] We note that there is no solution analogous to RNdS but with only one black hole. This is analogous to (but much more complicated than, and still not fully understood) the fact that one cannot have a single electric charge on a spherical universe (Gauss's law requires that the total charge must be zero). In fact, the fundamental solution of the Laplace equation on the sphere gives a unit positive charge at some point and a unit negative charge at the antipodal point. One can have more than two black holes, for instance the so-called Kastor-Traschen solution [26].

[10] "Spherical" here means that the Cauchy hypersurface (that is, "space") is an n-sphere.

These describe the event horizon $r = r_+$, and the Cauchy 'inner' horizon $r = r_-$. It can be seen that the subextremality condition then implies $p'(r_c) > 0$. (Indeed, $p'(r_c)$ cannot be negative, as otherwise p would acquire a root larger than r_c since $p(r) \stackrel{r\to\infty}{\longrightarrow} \infty$. Also, if $p'(r_c) = 0$, then Rolle's theorem implies again that p' would have three positive roots, ones in (r_-, r_+) and (r_+, r_c), and one at r_c, which is impossible, as we had seen above.) $p'(r_c) > 0$ implies that $V'(r_c) > 0$. We will also assume that

$$V''(r_c) > 0.$$

Our assumptions give following, which will be used in our proof of Theorem 5.3.

Lemma 5.1 (Global redshift)**.** *$V'(r) > 0$ for all $r \geqslant r_c$.*

Proof. We have $V'(r) = \dfrac{rp'(r) - (n-1)p(r)}{r^n} = \dfrac{2r^{n+1} + 2(3-n)Mr + (n-1)e^2}{r^n} =: \dfrac{q(r)}{r^n}$. As $V'(r_c) > 0$, we have $q(r_c) > 0$. Also $V''(r_c) > 0$ and so V' is increasing near r_c. But then $q(r) = r^n V'(r)$ is also increasing near r_c, and in particular, $q'(r_c) \geqslant 0$. Let us suppose that there exists an $r_* > r_c$ such that $V'(r_*) = 0$, and let r_* be the smallest such root. Then $q(r_*) = 0$ too. We note that $q' = 2(n+1)r^n + 2(3-n)M$, and so q' can have only one nonnegative root, namely $\left(\frac{(n-3)}{n+1}M\right)^{\frac{1}{n}} \geqslant 0$.

1° r_* is a repeated root of q. Then $q'(r_*) = 0$.

   If in addition $q'(r_c) = 0$, then we arrive at a contradiction, since q' then has two positive roots (at r_c and at r_*), which is impossible.

   If $q'(r_c) > 0$, then we arrive at a contradiction as follows. As q is increasing near r_c, and since $q(r_c) > 0 = q(r_*)$, it follows by the intermediate value theorem that there is some $r_c' \in (r_c, r_*)$ such that $q(r_c') = q(r_c)$. But by Rolle's theorem applied to q on $[r_c, r_c']$, there must exist an $r_*' \in (r_c, r_c')$ such that $q'(r_*') = 0$. Again q' acquires two zeros (at r_* and at r_*'), which is impossible.

2° r_* is a simple root of q. But as $q(r) \stackrel{r\to\infty}{\longrightarrow} \infty$, it follows that there must be at least one more root $r_{**} > r_*$ of q. By Rolle's theorem applied to q on $[r_*, r_{**}]$, it follows that $q'(r_{**}') = 0$ for some $r_{**}' \in (r_*, r_{**})$.

   If in addition $q'(r_c) = 0$, then we arrive at a contradiction, since q' then has two positive roots (at r_c and at r_{**}'), which is impossible.

   If $q'(r_c) > 0$, then, as in the last paragraph of 1°, there exists an $r_*' \in (r_c, r_c') \subset (r_c, r_*)$ such that $q'(r_*') = 0$. Thus q' again gets two positive roots (at r_*' and at r_{**}'), which is impossible.

This shows that our assumption the V' is zero beyond r_c is incorrect. □

In the cosmological region, where $r > r_c$, the hypersurfaces of constant r are spacelike cylinders with a future-pointing unit normal vector field $N = V^{\frac{1}{2}} \frac{\partial}{\partial r}$, and volume element $dV_n = V^{\frac{1}{2}} r^{n-1} dt d\Omega$.

The global structure of a maximal spherically symmetric extension of this metric can be depicted by a conformal Penrose diagram shown below, repeated periodically; see for example [8].

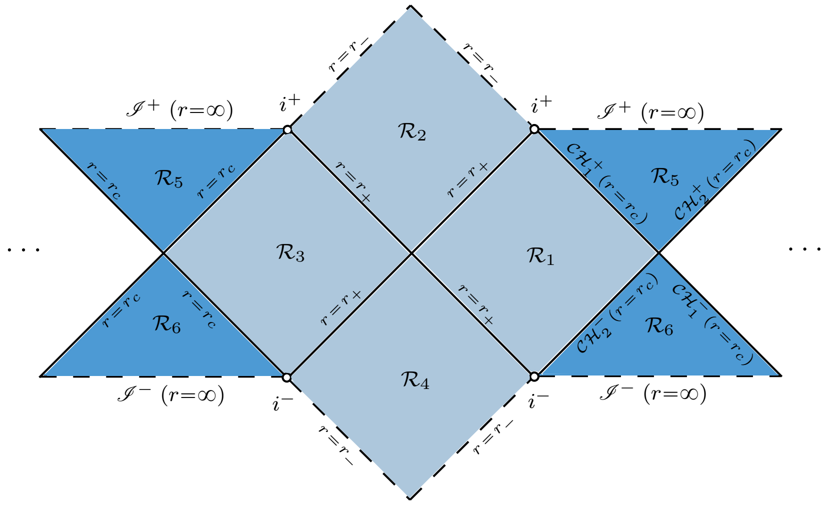

Figure 2: Conformal diagram of the Reissner-Nordström-de Sitter spacetime.

We are interested in the behaviour of the solution to the Klein-Gordon equation in the cosmological region $\mathcal{R}_5$ of this spacetime (see Figure 2), bounded by the cosmological horizon branches $\mathcal{CH}_1^+$, $\mathcal{CH}_2^+$, the future null infinity $\mathscr{I}^+$, and the point i^+. In particular, we want to obtain estimates for the decay rate of ϕ as $r \to \infty$. We guess the decay rates simply by substituting r instead of e^t in the estimates we had obtained for the decay rate of ϕ with respect to t in the case of the de Sitter universe in flat FLRW form from the previous Section 4, and the rationale behind this expectation is elaborated in the remark below.

Remark 5.2. We note that for large r the metric of the RNdS spacetimes looks like

$$-\frac{1}{r^2 - 1} dr^2 + (r^2 - 1) dt^2 + r^2 d\Omega^2. \tag{7}$$

Defining τ by $\tau = t + \frac{1}{2}\log(r^2 - 1)$, we have $d\tau = dt - \frac{r}{1 - r^2} dr$.

The metric from (7) in the $(r, \tau, \ldots)$-coordinates is given by

$$-d\tau^2 + dr^2 - 2r dr d\tau + r^2 d\tau^2 + r^2 d\Omega^2.$$

Now define x via the relation $r = e^\tau x$. Then $dr = e^\tau dx + e^\tau x d\tau$, and this yields

$$dx = \frac{1}{e^\tau} dr - \frac{r}{e^\tau} d\tau.$$

So the metric from (7) in the $(x, \tau, \ldots)$-coordinates takes the form

$$-d\tau^2 + (e^\tau dx)^2 + (e^\tau x)^2 d\Omega^2 = -d\tau^2 + e^{2\tau}(dx^2 + x^2 d\Omega^2),$$

which we recognise as the de Sitter metric in flat FLRW form. A free-falling 'galaxy' in these coordinates corresponds to an observer of constant x and constant Ω, with proper time $\tau = t$. For observers with fixed x one then has $r \sim e^\tau = e^t$. This justifies our guess that the same estimates from Theorem 4.1 for de Sitter spacetime in flat FLRW form ought to work in the cosmological region of RNdS, if we replace e^t by r. See also Remark 6.2.

We will prove the following result.

Theorem 5.3.

Suppose that

- $\epsilon > 0$,
- $m \in \mathbb{R}$,
- $M > 0$,
- $e > 0$,
- $n > 2$,
- (M, g) *is the* $(n+1)$*-dimensional subextremal Reissner-Nordström-de Sitter solution given by the metric* $g = -\frac{1}{V}dr^2 + V dt^2 + r^2 d\Omega^2$, *where* $V = r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1$, *and* $d\Omega^2$ *is the metric of the unit* $(n-1)$*-dimensional sphere* S^{n-1},
- $k > \frac{n}{2} + 2$, *and*
- ϕ *is a smooth solution to* $\Box_g \phi - m^2 \phi = 0$ *such that*

$$\|\phi\|_{H^k(\mathcal{CH}_1^+)} < +\infty \quad \text{and} \quad \|\phi\|_{H^k(\mathcal{CH}_2^+)} < +\infty,$$

where $\mathcal{CH}_1^+ \simeq \mathcal{CH}_2^+ \simeq \mathbb{R} \times S^{n-1}$ *are the two components of the future cosmological horizon, parameterised by the flow parameter*[11] *of the global Killing vector field* $\frac{\partial}{\partial t}$.

Then there exists a r_0 *large enough so that for all* $r \geqslant r_0$,

$$\|\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim \begin{cases} r^{-\frac{n}{2}+\epsilon} & \text{if} \quad |m| > \frac{n}{2}, \\ r^{-\frac{n}{2}+\sqrt{\frac{n^2}{4}-m^2}+\epsilon} & \text{if} \quad |m| \leqslant \frac{n}{2}. \end{cases}$$

[11] An *integral curve* $\gamma = (\lambda \mapsto \gamma(\lambda))$ of a vector field X is a curve, which for all parameter values λ, satisfies that the velocity vector $v_{\gamma(\lambda)}$ at the point $\gamma(\lambda)$ of the curve is such that $v_{\gamma(\lambda)} = X_{\gamma(\lambda)}$. The parameter ('time') of the integral curve is determined up to an additive constant; see for instance [15, Box 3.1, p.49]. We remark that the flow parameter along the cosmological horizon is replacing the time coordinate t, which is not defined on the cosmological horizon.

5.1 Preliminary energy function

In this section, we introduce some convenient notation, and also introduce a preliminary energy function $r \mapsto E(r)$, which will play an important role in the Subsection 5.5 on the red-shift estimates.

For a ϕ defined in the cosmological region $\mathcal{R}_5$, we define $\phi' := \frac{\partial \phi}{\partial r}$ and $\dot{\phi} := \frac{\partial \phi}{\partial t}$.

We will also use the following notation:

$\mathring{\nabla}\phi$	gradient of ϕ on S^{n-1} with respect to the unit round metric,
$\lvert\mathring{\nabla}\phi\rvert$	norm with respect to the unit round metric,
$\mathring{\Delta}\phi$	Laplacian of ϕ on S^{n-1} with respect to the unit round metric,
$\mathring{g\!\!\!/}$	determinant of the unit round metric.

Suppose that ϕ satisfies the Klein-Gordon equation $\Box_g \phi - m^2\phi = 0$.

The energy-momentum tensor associated with ϕ is $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu}(\partial_\alpha \partial^\alpha \phi + m^2\phi^2)$.

Recall that $N = V^{\frac{1}{2}} \frac{\partial}{\partial r}$. Thus

$$\begin{aligned} T(N,N) &= \left(\phi'^2 - \frac{1}{2}\frac{(-1)}{V}\left(\phi'^2(-V) + \dot{\phi}^2 \frac{1}{V} + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + m^2\phi^2\right)\right) V \\ &= \frac{1}{2}\left(V\phi'^2 + \frac{1}{V}\dot{\phi}^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + m^2\phi^2\right). \end{aligned}$$

Define $X := \frac{V^{\frac{1}{2}}}{r^{n-1}} N = \frac{V^{\frac{1}{2}}}{r^{n-1}} V^{\frac{1}{2}} \frac{\partial}{\partial r} = \frac{V}{r^{n-1}} \frac{\partial}{\partial r}$. We define the energy

$$E(r) := \int_{\mathbb{R}\times S^{n-1}} T(X,N) dV_n = \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \left(V^2\phi'^2 + \dot{\phi}^2 + \frac{V}{r^2}|\mathring{\nabla}\phi|^2 + m^2 V\phi^2\right) dt d\Omega.$$

5.2 The auxiliary function ψ and its PDE

In this subsection, we introduce the auxiliary function ψ constructed from the solution ϕ to the Klein-Gordon equation, and also derive a PDE satisfied by ψ.

The Klein-Gordon equation $\Box_g \phi - m^2\phi = 0$ can be rewritten as:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,\partial^\mu\phi) - m^2\phi = 0, \quad \Leftrightarrow \quad \frac{1}{r^{n-1}\sqrt{\mathring{g\!\!\!/}}}\partial_\mu\left(r^{n-1}\sqrt{\mathring{g\!\!\!/}}\, g^{\mu\nu}\partial_\nu\phi\right) - m^2\phi = 0.$$

This becomes

$$\frac{1}{r^{n-1}\sqrt{\mathring{g\!\!\!/}}}\left(\partial_r\left(r^{n-1}\sqrt{\mathring{g\!\!\!/}}\,(-V)\partial_r\phi\right) + \partial_t\left(r^{n-1}\sqrt{\mathring{g\!\!\!/}}\,\frac{1}{V}\partial_t\phi\right) + \frac{r^{n-1}\sqrt{\mathring{g\!\!\!/}}}{r^2}\,\mathring{\Delta}\phi\right) - m^2\phi = 0$$

that is,

$$-(V\phi')' - \frac{(n-1)}{r}V\phi' + \frac{\ddot{\phi}}{V} + \frac{1}{r^2}\mathring{\Delta}\phi - m^2\phi = 0,$$
$$\Leftrightarrow \quad \phi'' + \frac{(n-1)}{r}\phi' + \frac{V'}{V}\phi' - \frac{\ddot{\phi}}{V^2} - \frac{1}{r^2V}\mathring{\Delta}\phi + \frac{m^2}{V}\phi = 0.$$

Define

$$\psi := r^\kappa\phi,$$

where

$$\kappa = \begin{cases} \dfrac{n}{2} & \text{if } |m| \geqslant \dfrac{n}{2}, \\ \dfrac{n}{2} - \sqrt{\dfrac{n^2}{4} - m^2} & \text{if } |m| \leqslant \dfrac{n}{2}. \end{cases}$$

Then, using the PDE for ϕ, it can be shown that

$$\psi'' + \left(\frac{V'}{V} + \frac{n-1}{r} - \frac{2\kappa}{r}\right)\psi' - \frac{\ddot{\psi}}{V^2} - \frac{1}{r^2V}\mathring{\Delta}\psi + \theta\psi = 0, \tag{8}$$

where

$$\theta := \frac{m^2}{V} + \frac{\kappa}{r}\left(\frac{1}{r} - \frac{V'}{V}\right) - \frac{\kappa}{r^2}(n-1-\kappa).$$

5.3 The case $|m| > \frac{n}{2}$

In this subsection, we will consider the case $|m| > \frac{n}{2}$ of Theorem 5.3.

Then $\kappa = \frac{n}{2}$, and (8) becomes

$$\psi'' + \left(\frac{V'}{V} - \frac{1}{r}\right)\psi' - \frac{\ddot{\psi}}{V^2} - \frac{1}{r^2V}\mathring{\Delta}\psi + \theta\psi = 0, \tag{9}$$

where $\theta := -\dfrac{n}{2}\left(\dfrac{n}{2} - 1\right)\dfrac{1}{r^2} + \dfrac{m^2}{V} + \dfrac{n}{2r}\left(\dfrac{1}{r} - \dfrac{V'}{V}\right)$.

We will use an energy function to obtain the required decay of ψ for large r, and in order to do so, we will need to keep careful track of the limiting behaviour of the various functions appearing in the expression for θ and the coefficients of the PDE (9). We will do this step-by-step in a sequence of lemmas.

Lemma 5.4. *Given any $\epsilon > 0$, there exists an r_0 large enough so that for all $r \geqslant r_0$,*

$$\frac{2+\epsilon}{r} \geqslant \frac{V'}{V} \geqslant \frac{2-\epsilon}{r}.$$

Proof. This follows immediately from

$$\lim_{r\to\infty} r\frac{V'}{V} = \lim_{r\to\infty} r \cdot \frac{2r - \frac{2(n-2)M}{r^{n-1}} + \frac{e^2(n-1)}{r^n}}{r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1} = \lim_{r\to\infty} \frac{2 - \frac{2(n-2)M}{r^n} + \frac{e^2(n-1)}{r^{n+1}}}{1 + \frac{2M}{r^n} - \frac{e^2}{r^{n+1}} - \frac{1}{r^2}} = 2.$$

□

Lemma 5.5. *There exists r_0 large enough so that for $r \geqslant r_0$, we have $\theta > 0$.*

(We note that the proof uses the fact that $|m| > \frac{n}{2}$, and so this result is specific to this subsection.)

Proof. We have $\lim_{r\to\infty} \frac{r^2}{V} = \lim_{r\to\infty} \frac{1}{1 + \frac{2M}{r^n} - \frac{e^2}{r^{n+1}} - \frac{1}{r^2}} = 1$, and so there exists a r_0' such that

$$\frac{r^2}{V} \geqslant 1 - \epsilon$$

for $r \geqslant r_0'$. Also, by the previous lemma, there exists a $r_0 > r_0'$ such that

$$\frac{V'}{V} \leqslant \frac{2+\epsilon}{r}$$

for all $r \geqslant r_0$. Then we have for $r > r_0$ that

$$\begin{aligned}
\theta &= \frac{1}{r^2}\left(-\frac{n}{2}\left(\frac{n}{2} - 1\right) + m^2\frac{r^2}{V} + \frac{n}{2}\left(1 - \frac{V'}{V}r\right)\right) \\
&\geqslant \frac{1}{r^2}\left(-\frac{n^2}{4} + \frac{n}{2} + m^2(1-\epsilon) + \frac{n}{2}\left(1 - \frac{(2+\epsilon)}{r}r\right)\right) \\
&= \frac{1}{r^2}\left(\delta - \epsilon\left(\delta + \frac{n}{2} + \frac{n^2}{4}\right)\right),
\end{aligned}$$

where

$$\delta := m^2 - \frac{n^2}{4} > 0.$$

Taking ϵ at the outset small enough so as to satisfy

$$0 < \epsilon < \frac{\delta}{\delta + \frac{n}{2} + \frac{n^2}{4}},$$

we see that $\theta > 0$ for $r \geqslant r_0$. □

Define the energy

$$\mathcal{E}(r) := \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \left(\psi'^2 + \frac{1}{V^2}\dot{\psi}^2 + \frac{1}{r^2V}|\mathring{\nabla}\psi|^2 + \theta\psi^2\right) dtd\Omega.$$

(We assume for the moment that this is finite for a sufficiently large r_0. Later on, in the subsection on redshift estimates, we will see how our initial finiteness of Sobolev norms of ϕ on the two branches $\mathcal{CH}_1^+$, $\mathcal{CH}_2^+$ of the cosmological horizon guarantees this.)

We now proceed to find an expression for $\mathcal{E}'(r)$, and to simplify it, we will use (9), and the divergence theorem, to get rid of the terms involving $\ddot{\psi}$ and $\mathring{\Delta}\psi$, the spherical Laplacian of ψ:

$$\begin{aligned}
\mathcal{E}'(r) &= \int_{\mathbb{R}\times S^{n-1}} \Big(\psi'\psi'' + \frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2 + \frac{1}{V^2}\dot{\psi}\dot{\psi}' + \frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2 \\
&\qquad + \frac{1}{r^2V}\langle\mathring{\nabla}\psi,(\mathring{\nabla}\psi)'\rangle + \frac{\theta'}{2}\psi^2 + \theta\psi\psi'\Big)\, dtd\Omega \\
&= \int_{\mathbb{R}\times S^{n-1}} \Big(\psi'\Big(-\Big(\frac{V'}{V} - \frac{1}{r}\Big)\psi' + \frac{1}{V^2}\ddot{\psi} + \frac{1}{r^2V}\mathring{\Delta}\psi - \cancel{\theta\psi}\Big) \\
&\qquad + \frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2 + \frac{1}{V^2}\dot{\psi}\dot{\psi}' + \frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2 \\
&\qquad + \frac{1}{r^2V}\langle\mathring{\nabla}\psi,(\mathring{\nabla}\psi)'\rangle + \frac{\theta'}{2}\psi^2 + \cancel{\theta\psi\psi'}\Big)\, dtd\Omega \\
&= \int_{\mathbb{R}\times S^{n-1}} \Big(-\Big(\frac{V'}{V} - \frac{1}{r}\Big)\psi'^2 + \cancel{\frac{1}{V^2}\ddot{\psi}\psi'} + \cancel{\frac{1}{V^2}\dot{\psi}\dot{\psi}'} + \frac{1}{r^2V}\Big(\cancel{\psi'\mathring{\Delta}\psi + \langle\mathring{\nabla}\psi,(\mathring{\nabla}\psi)'\rangle}\Big) \\
&\qquad + \frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2 + \frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2 + \frac{\theta'}{2}\psi^2\Big)\, dtd\Omega.
\end{aligned}$$

We note that in the above, getting rid of the spherical Laplacian by using the divergence theorem is allowed because the compact sphere S^{n-1} has no boundary. For the second time derivative, however, there is a boundary at infinity (with two connected components), namely

$$\lim_{t\to+\infty}\int_{S^{n-1}} \dot{\psi}\psi' d\Omega - \lim_{t\to-\infty}\int_{S^{n-1}} \dot{\psi}\psi' d\Omega,$$

which can be seen to be equal to 0, by Lemma B.3 from Appendix B. Thus

$$\mathcal{E}'(r) = \int_{\mathbb{R}\times S^{n-1}} \Big(-\Big(\frac{V'}{V} - \frac{1}{r}\Big)\psi'^2 + \frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2 + \frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2 + \frac{\theta'}{2}\psi^2\Big)\, dtd\Omega.$$

Let $\epsilon > 0$ be given. Then there exists an r_0 large enough such that:

(a) $\displaystyle \frac{V'}{V} - \frac{1}{r} \geqslant \frac{2-\epsilon}{r} - \frac{1}{r} = \frac{1-\epsilon}{r}$,

(b) $\displaystyle \Big(\frac{1}{V^2}\Big)' = -2\frac{V'}{V}\frac{1}{V^2} \leqslant -2\frac{(2-\epsilon)}{r}\frac{1}{V^2}$,

(c) $\displaystyle \Big(\frac{1}{r^2V}\Big)' = -\frac{1}{r^2V}\Big(\frac{2}{r} + \frac{V'}{V}\Big) \leqslant -\frac{1}{r^2V}\Big(\frac{2}{r} + \frac{2-\epsilon}{r}\Big) = -\frac{1}{r^2V}\frac{(4-\epsilon)}{r}$,

(d) $\displaystyle \frac{\theta'}{\theta} = \frac{1}{r}\Big(\frac{-\frac{n}{2}(\frac{n}{2}-1)(-2) - \frac{m^2V'}{r^3V^2} - \frac{n}{2r^5}(\frac{1}{r} - \frac{V'}{V}) + \frac{n}{2r^4}(-\frac{1}{r^2} - \frac{V''V - V'^2}{V^2})}{-\frac{n}{2}(\frac{n}{2}-1) + \frac{m^2}{r^2V} + \frac{n}{2r^3}(\frac{1}{r} - \frac{V'}{V})}\Big) \leqslant \frac{1}{r}(-2+\epsilon)$.

Hence, using (a)-(d) above, we obtain

$$\begin{aligned}
\mathcal{E}'(r) &= \int_{\mathbb{R}\times S^{n-1}} \Big(-\Big(\frac{V'}{V} - \frac{1}{r}\Big)\psi'^2 + \frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2 + \frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2 + \frac{\theta'}{2}\psi^2\Big)\, dtd\Omega \\
&\leqslant \int_{\mathbb{R}\times S^{n-1}} \Big(-\frac{(1-\epsilon)}{r}\psi'^2 + \frac{1}{2}\frac{(-2)(2-\epsilon)}{rV^2}\dot{\psi}^2 + \frac{1}{2}\frac{(-1)}{r^2V}\frac{(4-\epsilon)}{r}|\mathring{\nabla}\psi|^2 \\
&\qquad + \frac{1}{2}\frac{1}{r}(-2+\epsilon)\theta\psi^2\Big)\, dtd\Omega \\
&= -\frac{1}{r}\frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \Big(2(1-\epsilon)\psi'^2 + 2(2-\epsilon)\frac{1}{V^2}\dot{\psi}^2 + (4-\epsilon)\frac{1}{r^2V}|\mathring{\nabla}\psi|^2 \\
&\qquad + (2-\epsilon)\theta\psi^2\Big)\, dtd\Omega \\
&\leqslant -\frac{2(1-\epsilon)}{r}\mathcal{E}(r).
\end{aligned}$$

Using Grönwall's inequality (see e.g. [16, Appendix B(j)]), we obtain

$$\mathcal{E}(r) \leqslant \mathcal{E}(r_0)e^{\int_{r_0}^{r} -\frac{2(1-\epsilon)}{r}dr} = \mathcal{E}(r_0)\Big(\frac{r}{r_0}\Big)^{-2(1-\epsilon)} \lesssim r^{-2+2\epsilon}.$$

Thus $\int_{\mathbb{R}\times S^{n-1}} \theta\psi^2 dtd\Omega \leqslant 2\mathcal{E}(r) \lesssim r^{-2+2\epsilon}$, and so $\int_{\mathbb{R}\times S^{n-1}} \psi^2 dtd\Omega \lesssim \frac{r^{2\epsilon}}{r^2\theta} \lesssim \frac{r^{2\epsilon}}{r^2\frac{1}{r^2}} = r^{2\epsilon}$.

Hence $\|\psi(r,\cdot)\|_{L^2(\mathbb{R}\times S^{n-1})} \lesssim r^{\epsilon}$. Consequently, $\|\phi(r,\cdot)\|_{L^2(\mathbb{R}\times S^{n-1})} \lesssim r^{-\frac{n}{2}+\epsilon}$.

Recall that S^{n-1} admits $\frac{n(n-1)}{2}$ independent Killing vectors, given by $L_{ij} = x^i\frac{\partial}{\partial x^j} - x^j\frac{\partial}{\partial x^i}$, for $i < j$ (under the usual embedding $S^{n-1} \subset \mathbb{R}^n$). As $\frac{\partial}{\partial t}$ and L_{ij} are Killing vector fields, it follows that $\dot{\phi}$ and $L_{ij}\cdot\phi$ are also solutions to $\Box_g\phi - m^2\phi = 0$. Commuting with the Killing vector fields $\frac{\partial}{\partial t}$ and L_{ij}, if we assume for now[12] that at r_0 we have $\|\phi(r_0,\cdot)\|_{H^k(\{r=r_0\})} < +\infty$, then we also obtain for all $r \geqslant r_0$ that $\|\phi(r,\cdot)\|_{H^{k'}(\mathbb{R}\times S^{n-1})} \lesssim r^{-\frac{n}{2}+\epsilon}$, where $k' = k-2 > \frac{n}{2}$. By the Sobolev inequality[13], $\|\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim r^{-\frac{n}{2}+\epsilon}$.

This completes the proof of Theorem 5.3 in the case when $|m| > \frac{n}{2}$ (provided we show the aforementioned finiteness of energy, which will be carried out in Subsection 5.5 on redshift estimates).

5.4 The case $|m| \leqslant \frac{n}{2}$

In this subsection, we will consider the remaining case of Theorem 5.3, namely the case when $|m| \leqslant \frac{n}{2}$.

[12] This will be proved later in the subsection on redshift estimates.

[13] The part of the Sobolev embedding theorem concerning inclusion in Hölder spaces holds for a complete Riemannian manifold with a positive injectivity radius and a bounded sectional curvature; see for example [25, §3.3, Thm.3.4] or [4, Ch.2].

Let $\epsilon' > 0$ be given. Define $\widetilde{\mathcal{E}}(r) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}\times S^{n-1}} \left(\psi'^2 + \frac{1}{V^2}\dot{\psi}^2 + \frac{1}{r^2 V}|\mathring{\nabla}\psi|^2 + \frac{\epsilon'}{r^2}\psi^2\right) dtd\Omega.$

We now proceed to find an expression for $\widetilde{\mathcal{E}}'(r)$, and we will simplify it using (8) and the divergence theorem, in order to get rid of the terms involving $\ddot{\psi}$ and the spherical Laplacian of ψ:

$$
\begin{aligned}
\widetilde{\mathcal{E}}'(r) &= \int_{\mathbb{R}\times S^{n-1}} \Bigg(\psi'\psi'' + \frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2 + \frac{1}{V^2}\dot{\psi}\dot{\psi}' + \frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2 \\
&\qquad + \frac{1}{r^2V}\langle\mathring{\nabla}\psi, (\mathring{\nabla}\psi)'\rangle - \frac{\epsilon'}{r^3}\psi^2 + \frac{\epsilon'}{r^2}\psi\psi'\Bigg) dtd\Omega \\
&= \int_{\mathbb{R}\times S^{n-1}} \Bigg(\psi'\Big(- \Big(\frac{V'}{V} + \frac{n-1}{r} - \frac{2\kappa}{r}\Big)\psi' + \cancel{\frac{\ddot{\psi}}{V^2}} + \cancel{\frac{1}{r^2V}\mathring{\Delta}\psi} - \theta\psi\Big) \\
&\qquad + \frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2 + \cancel{\frac{1}{V^2}\dot{\psi}\dot{\psi}'} + \frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2 + \cancel{\frac{1}{r^2V}\langle\mathring{\nabla}\psi, (\mathring{\nabla}\psi)'\rangle} \\
&\qquad - \frac{\epsilon'}{r^3}\psi^2 + \frac{\epsilon'}{r^2}\psi\psi'\Bigg) dtd\Omega \\
&= \int_{\mathbb{R}\times S^{n-1}} \Bigg(-\Big(\frac{V'}{V} + \frac{n-1}{r} - \frac{2\kappa}{r}\Big)\psi'^2 + \frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2 + \frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2 - \frac{\epsilon'}{r^3}\psi^2\Bigg) dtd\Omega \\
&\quad + \Big(\frac{\epsilon'}{r^2} - \theta\Big)\int_{\mathbb{R}\times S^{n-1}} \psi\psi' dtd\Omega.
\end{aligned}
$$

Again, for getting rid of the spherical Laplacian, we use the divergence theorem, noting that the sphere S^{n-1} has no boundary. For handling the second time derivative, as before, we note that there is a boundary at infinity (with two connected components), which can be seen to be equal to 0, by Lemma B.3 from Appendix B. Thus

$$
\begin{aligned}
\widetilde{\mathcal{E}}'(r) &= \int_{\mathbb{R}\times S^{n-1}} \Bigg(-\Big(\frac{V'}{V} + \frac{n-1}{r} - \frac{2\kappa}{r}\Big)\psi'^2 + \frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2 + \frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2 - \frac{\epsilon'}{r^3}\psi^2\Bigg) dtd\Omega \\
&\quad + \Big(\frac{\epsilon'}{r^2} - \theta\Big)\int_{\mathbb{R}\times S^{n-1}} \psi\psi' dtd\Omega.
\end{aligned}
$$

Now there exists an r_0 large enough such that for all $r \geqslant r_0$, we have:

(i) $\dfrac{V'}{V} + \dfrac{n-1}{r} - \dfrac{2\kappa}{r} \geqslant \dfrac{2-\epsilon'}{r} + \dfrac{n-1}{r} - \dfrac{2\kappa}{r} = \dfrac{1-\epsilon' + (n-2\kappa)}{r} \geqslant \dfrac{1-\epsilon'}{r}$, using $n - 2\kappa \geqslant 0$.

(ii) $\Big(\dfrac{1}{V^2}\Big)' \leqslant -\dfrac{2(2-\epsilon')}{r}\cdot\dfrac{1}{V^2}$.

(iii) $\Big(\dfrac{1}{r^2V}\Big)' \leqslant -\dfrac{1}{r^2V}\Big(\dfrac{2}{r} + \dfrac{2-\epsilon'}{r}\Big)$.

Using (i), (ii) and (iii), it can be seen that

$$\begin{aligned}\widetilde{\mathcal{E}}'(r) &\leqslant \int_{\mathbb{R}\times S^{n-1}}\Big(-\frac{(1-\epsilon')}{r}\psi'^2-\frac{1}{2}\frac{2(2-\epsilon')}{rV^2}\dot{\psi}^2-\frac{1}{2}\frac{1}{r^2V}\Big(\frac{2}{r}+\frac{2-\epsilon'}{r}\Big)|\mathring{\nabla}\psi|^2-\frac{\epsilon'}{r^3}\psi^2\Big)\,dtd\Omega \\ &\quad+\Big(\frac{\epsilon'}{r^2}-\theta\Big)\int_{\mathbb{R}\times S^{n-1}}\psi\psi'dtd\Omega \\ &\leqslant -\frac{1}{r}\frac{1}{2}\int_{\mathbb{R}\times S^{n-1}}\Big(2(1-\epsilon')\psi'^2+2(2-\epsilon')\frac{1}{V^2}\dot{\psi}^2+(4-\epsilon')\frac{1}{r^2V}|\mathring{\nabla}\psi|^2+\frac{2\epsilon'}{r^2}\psi^2\Big)\,dtd\Omega \\ &\quad+\Big(\frac{\epsilon'}{r^2}-\theta\Big)\int_{\mathbb{R}\times S^{n-1}}\psi\psi'dtd\Omega.\end{aligned}$$

Hence $\widetilde{\mathcal{E}}'(r)\leqslant -\frac{2(1-\epsilon')}{r}\widetilde{\mathcal{E}}(r)+\Big(\frac{\epsilon'}{r^2}-\theta\Big)\int_{\mathbb{R}\times S^{n-1}}\psi\psi'dtd\Omega.$

We have $\theta=\frac{m^2}{V}+\frac{\kappa}{r}\Big(\frac{1}{r}-\frac{V'}{V}\Big)-\frac{\kappa}{r^2}(n-1-\kappa)=\frac{1}{r^2}\left(\frac{m^2}{\frac{V}{r^2}}+\kappa\Big(1-\frac{V'}{V}r\Big)-\kappa(n-1-\kappa)\right).$

As $\frac{V}{r^2}\overset{r\to\infty}{\longrightarrow}1$ and $\frac{V'}{V}r\overset{r\to\infty}{\longrightarrow}2$, $r^2\theta\overset{r\to\infty}{\longrightarrow}\frac{m^2}{1}+\kappa(1-2)-\kappa(n-1-\kappa)=m^2-\kappa n+\kappa^2=0.$

Thus, given $\epsilon'>0$, there exists an r_0 such that for $r\geqslant r_0$, $|r^2\theta|<\epsilon'$, that is, $|\theta|<\frac{\epsilon'}{r^2}$. So

$$\begin{aligned}\widetilde{\mathcal{E}}'(r) &\leqslant -\frac{2(1-\epsilon')}{r}\widetilde{\mathcal{E}}(r)+\Big(\frac{\epsilon'}{r^2}-\theta\Big)\int_{\mathbb{R}\times S^{n-1}}\psi\psi'dtd\Omega \\ &\leqslant -\frac{2(1-\epsilon')}{r}\widetilde{\mathcal{E}}(r)+\Big(\frac{\epsilon'}{r^2}+\frac{\epsilon'}{r^2}\Big)\left|\int_{\mathbb{R}\times S^{n-1}}\psi\psi'dtd\Omega\right|.\end{aligned}$$

The Cauchy-Schwarz inequality applied to the last integral gives

$$\begin{aligned}\left|\int_{\mathbb{R}\times S^{n-1}}\psi\psi'dtd\Omega\right| &\leqslant \sqrt{\int_{\mathbb{R}\times S^{n-1}}\psi^2dtd\Omega}\cdot\sqrt{\int_{\mathbb{R}\times S^{n-1}}\psi'^2dtd\Omega} \\ &\leqslant \sqrt{\frac{2r^2}{\epsilon'}\widetilde{\mathcal{E}}(r)}\cdot\sqrt{2\widetilde{\mathcal{E}}(r)}=\frac{2r}{\sqrt{\epsilon'}}\widetilde{\mathcal{E}}(r).\end{aligned}$$

So we obtain $\widetilde{\mathcal{E}}'(r)\leqslant -\frac{2(1-\epsilon')}{r}\widetilde{\mathcal{E}}(r)+\frac{2\epsilon'}{r^2}\frac{2r}{\sqrt{\epsilon'}}\widetilde{\mathcal{E}}(r)=(-2+2\epsilon'+4\sqrt{\epsilon'})\frac{1}{r}\widetilde{\mathcal{E}}(r).$

By Grönwall's inequality, $\widetilde{\mathcal{E}}(r)\leqslant\widetilde{\mathcal{E}}(r_0)e^{\int_{r_0}^{r}(-2+2\epsilon'+4\sqrt{\epsilon'})\frac{1}{r}dr}=\frac{\widetilde{\mathcal{E}}(r_0)}{r_0^{-2+2\epsilon'+4\sqrt{\epsilon'}}}r^{-2+2\epsilon'+4\sqrt{\epsilon'}}$. So

$$\begin{aligned}\int_{\mathbb{R}\times S^{n-1}}\psi^2dtd\Omega &= \frac{2r^2}{\epsilon'}\frac{1}{2}\int_{\mathbb{R}\times S^{n-1}}\frac{\epsilon'}{r^2}\psi^2dtd\Omega\leqslant\frac{2r^2}{\epsilon'}\widetilde{\mathcal{E}}(r) \\ &\leqslant \frac{2r^2}{\epsilon'}\frac{\widetilde{\mathcal{E}}(r_0)}{r_0^{-2+2\epsilon'+4\sqrt{\epsilon'}}}r^{-2+2\epsilon'+4\sqrt{\epsilon'}}=\frac{2\widetilde{\mathcal{E}}(r_0)}{\epsilon' r_0^{-2+2\epsilon'+4\sqrt{\epsilon'}}}r^{2\epsilon'+4\sqrt{\epsilon'}}.\end{aligned}$$

Thus $\|\psi(r,\cdot)\|_{L^2(\mathbb{R}\times S^{n-1})} \leqslant \sqrt{\frac{2\widetilde{\mathcal{E}}(r_0)}{\epsilon'}}\frac{1}{r_0^{-1+\epsilon'+2\sqrt{\epsilon'}}} r^{\epsilon'+2\sqrt{\epsilon'}}$, and so

$$\|\phi(r,\cdot)\|_{L^2(\mathbb{R}\times S^{n-1})} \leqslant \sqrt{\frac{2\widetilde{\mathcal{E}}(r_0)}{\epsilon'}}\frac{1}{r_0^{-1+\epsilon'+2\sqrt{\epsilon'}}} r^{-\kappa+\epsilon'+2\sqrt{\epsilon'}}.$$

Given $\epsilon > 0$, arbitrarily small, we take $\epsilon' = \epsilon'(\epsilon) > 0$ small enough so that $\epsilon' + 2\sqrt{\epsilon'} < \epsilon$ at the outset, so that $\|\phi(r,\cdot)\|_{L^2(\mathbb{R}\times S^{n-1})} \lesssim r^{-\kappa+\epsilon}$. Again assuming at the moment that at r_0 we have $\|\phi(r_0,\cdot)\|_{H^k(\{r=r_0\})} < +\infty$, and by commuting with the Killing vector fields $\frac{\partial}{\partial t}$ and L_{ij}, then we also obtain for all $r \geqslant r_0$ that

$$\|\phi(r,\cdot)\|_{H^{k'}(\mathbb{R}\times S^{n-1})} \lesssim r^{-(\frac{n}{2}-\sqrt{\frac{n^2}{4}-m^2})+\epsilon},$$

where $k' = k - 2 > \frac{n}{2}$. By the Sobolev inequality, this yields

$$\|\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim r^{-(\frac{n}{2}-\sqrt{\frac{n^2}{4}-m^2})+\epsilon}.$$

This completes the proof of Theorem 5.3 in the case when $|m| \leqslant \frac{n}{2}$ (provided we show the finiteness of energy, which will be carried out in the subsection on redshift estimates below).

5.5 Redshift estimates

In this subsection, we complete the last remaining step, namely to use redshift estimates to transfer finiteness of the energies along the branches $\mathcal{CH}_1^+$ and $\mathcal{CH}_2^+$ of the cosmological horizon to finiteness at $r = r_0$, justifying the finiteness of the energies $\mathcal{E}(r_0)$ and $\widetilde{\mathcal{E}}(r_0)$ assumed in the previous two subsections. We divide this rather long subsection into parts (a)-(e) for ease of readability.

(a) In this first step we introduce two new coordinates u, v. Moreover, we define some vector fields (K, Y), and also express (the earlier defined) vector fields X, N using the new coordinates.

Define the new coordinate u by $u = t + \displaystyle\int_{r_*}^{r} \frac{1}{V} dr$, where $r_* > r_c$ is arbitrary, but fixed. Then $du = dt + \dfrac{1}{V}dr$. The Reissner-Nordström-de Sitter metric can be rewritten using the coordinates $(u, r, \dots)$, instead of the old $(t, r, \dots)$-coordinates, as follows

$$\begin{aligned} g &= -\frac{1}{V}dr^2 + Vdt^2 + r^2 d\Omega^2 = V\Big(-\frac{1}{V^2}dr^2 + dt^2\Big) + r^2 d\Omega^2 \\ &= -V\Big(\frac{1}{V}dr + dt\Big)\Big(\frac{1}{V}dr - dt\Big) + r^2 d\Omega^2 \\ &= -Vdu\Big(-du + \frac{2}{V}dr\Big) + r^2 d\Omega^2 = Vdu^2 - 2dudr + r^2 d\Omega^2. \end{aligned}$$

The matrix of the metric in the $(u, r, \ldots)$-coordinate system is

$$[g_{\mu\nu}] = \left[\begin{array}{cc|c} V & -1 & \\ -1 & 0 & \\ \hline & & * \end{array}\right].$$

Since

$$\det \begin{bmatrix} V & -1 \\ -1 & 0 \end{bmatrix} = -1,$$

this coordinate system extends across the cosmological horizon $r = r_c$ (where $V = 0$). The hypersurfaces of constant u are null and transverse to the cosmological horizon. Thus only one of the branches of the cosmological horizon, namely $\mathcal{CH}_1^+$, is covered by the $(u, r, \ldots)$-coordinates. (In order to cover the other branch $\mathcal{CH}_2^+$, where $u = -\infty$, we can introduce

$$v := -t + \int_{r_*}^{r} \frac{1}{V} dr,$$

and use the $(v, r, \ldots)$-coordinate chart. See Figure 5.5.)

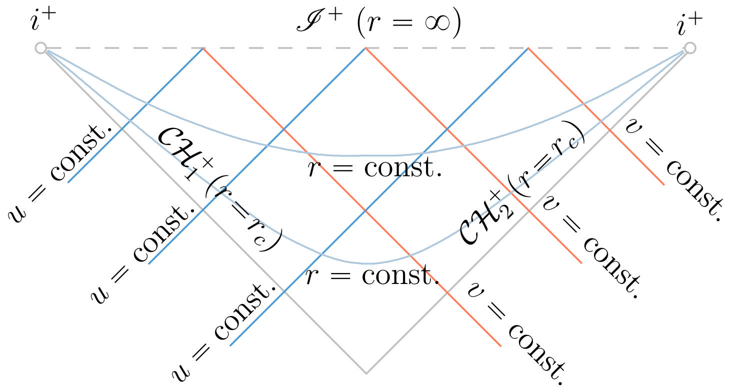

Figure 3: Coordinates u and v.

We will only consider $\mathcal{CH}_1^+$ in the remainder of this subsection, since $\mathcal{CH}_2^+$ can be treated in an analogous manner.

The Killing vector field

$$K = \frac{\partial}{\partial u} = \frac{\partial}{\partial t}$$

is well-defined across $\mathcal{CH}_1^+$, and is null on the cosmological horizon $\mathcal{CH}_1^+$, even though the t-coordinate is not defined there. Consider the vector field in the $(u, r, \ldots)$-coordinate chart given by

$$Y = \left(\frac{\partial}{\partial r}\right)_u.$$

The subscript u means that the integral curves of Y in the $(u, r, \ldots)$-coordinate chart have a constant u-coordinate. Then we have $du(Y) = 0$ and $dr(Y) = 1$, and so in the old $(t, r, \ldots)$-coordinate chart, the vector field Y can be expressed as

$$Y = \frac{\partial}{\partial r} - \frac{1}{V}\frac{\partial}{\partial t}.$$

Let the vector field X be defined by

$$X = \frac{V^{\frac{1}{2}}}{r^{n-1}} N = \frac{V}{r^{n-1}}\frac{\partial}{\partial r}$$

in the old $(t, r, \ldots)$-coordinate chart. To find the expression for X in the $(u, r, \ldots)$-coordinate chart induced basis vectors, we first find

$$N = -\frac{\text{grad } r}{|\text{grad } r|}$$

in the $(u, r, \ldots)$-coordinate chart induced basis vectors. Since

$$\begin{bmatrix} V & -1 \\ -1 & 0 \end{bmatrix}^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & 1 \\ 1 & V \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & -V \end{bmatrix},$$

we have $\langle \text{grad } r, \text{grad } r\rangle = \langle dr, dr\rangle = -V$.

If $\omega := -\frac{1}{\sqrt{V}}dr$, then $N = g^{\mu\nu}\omega_\nu = \dfrac{1}{\sqrt{V}}\Big(\dfrac{\partial}{\partial u} + V\dfrac{\partial}{\partial r}\Big)$. So

$$X = \frac{\sqrt{V}}{r^{n-1}} N = \frac{1}{r^{n-1}}\Big(\frac{\partial}{\partial u} + V\frac{\partial}{\partial r}\Big).$$

(b) In this step we will define the energy $\widetilde{E}$.

Recall that in Subsection 5.1, we had defined the preliminary energy function E. We have

$$\begin{aligned} E(r) = \int_{\mathbb{R}\times S^{n-1}} T(X, N) dV_n \quad &= \quad \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \Big(V^2\phi'^2 + \dot{\phi}^2 + \frac{V}{r^2}|\mathring{\nabla}\phi|^2 + m^2 V\phi^2\Big) dt d\Omega \\ &\xrightarrow{r\to r_c} \frac{1}{2}\int\limits_{\substack{\mathbb{R}\times S^{n-1} \\ (\mathcal{CH}_1^+)}} (K\cdot\phi)^2 du d\Omega + \frac{1}{2}\int\limits_{\substack{\mathbb{R}\times S^{n-1} \\ (\mathcal{CH}_2^+)}} (K\cdot\phi)^2 dv d\Omega \end{aligned}$$

(since $V(r_c) = 0$). So $E(r)$ 'loses control' of the transverse and angular derivatives as $r \to r_c$. To remedy this problem, we define a new energy $\widetilde{E}$, by adding Y to X, obtaining

$$\widetilde{E}(r) := E(r) + \int_{\mathbb{R}\times S^{n-1}} T(Y, N) dV_n.$$

In the old $(t, r, \ldots)$-coordinates, $N = \sqrt{V}\,\partial_r$, and so

$$T(Y,N) = T(\partial_r, N) - \frac{1}{V}T(\partial_t, N) = \frac{1}{\sqrt{V}}T(N,N) - \frac{1}{\sqrt{V}}T(\partial_t, \partial_r) = \frac{1}{\sqrt{V}}\left(T(N,N) - T(\partial_t, \partial_r)\right).$$

We have $T(\partial_t, \partial_r) = \dot{\phi}\phi'$. So

$$\begin{aligned}
\widetilde{E}(r) &= E(r) + \int_{\mathbb{R}\times S^{n-1}} \left(\frac{1}{\sqrt{V}}\frac{1}{2}\left(V\phi'^2 + \frac{\dot{\phi}^2}{V} + \frac{|\mathring{\nabla}\phi|^2}{r^2} + m^2\phi^2\right) - \frac{1}{\sqrt{V}}\dot{\phi}\phi' \right) \sqrt{V} r^{n-1} dt d\Omega \\
&= E(r) + \int_{\mathbb{R}\times S^{n-1}} \frac{1}{2}\left(V\left(\phi' - \frac{1}{V}\dot{\phi}\right)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + m^2\phi^2 \right) r^{n-1} dt d\Omega \\
&= E(r) + \int_{\mathbb{R}\times S^{n-1}} \frac{1}{2}\left(V(Y\cdot\phi)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + m^2\phi^2 \right) r^{n-1} dt d\Omega.
\end{aligned}$$

We now have

$$\begin{aligned}
\widetilde{E}(r_c) &= E(r_c) + \frac{r_c^{n-3}}{2} \int_{\substack{\mathbb{R}\times S^{n-1} \\ (\mathcal{CH}_1^+)}} |\mathring{\nabla}\phi|^2 du d\Omega + \frac{r_c^{n-3}}{2} \int_{\substack{\mathbb{R}\times S^{n-1} \\ (\mathcal{CH}_2^+)}} |\mathring{\nabla}\phi|^2 dv d\Omega \\
&\quad + \frac{m^2 r_c^{n-1}}{2} \int_{\substack{\mathbb{R}\times S^{n-1} \\ (\mathcal{CH}_1^+)}} \phi^2 du d\Omega + \frac{m^2 r_c^{n-1}}{2} \int_{\substack{\mathbb{R}\times S^{n-1} \\ (\mathcal{CH}_2^+)}} \phi^2 dv d\Omega,
\end{aligned}$$

so that using $\widetilde{E}$ instead of E allows us to regain some control of the angular derivatives as $r \to r_c$. We note that $\widetilde{E}(r_c)$ is equivalent to $\|\phi\|^2_{H^1(\mathcal{CH}_1^+)} + \|\phi\|^2_{H^1(\mathcal{CH}_2^+)}$.

(c) In this step, we will compute the deformation tensor Ξ corresponding to Y.

We have

- $\left[-\frac{1}{V}\partial_t, \partial_r\right] = -\frac{V'}{V}\partial_t$,
- $\mathcal{L}_{\partial_r} g = \frac{V'}{V}dr^2 - \frac{2}{V}dr\mathcal{L}_{\partial_r}dr + V'dt^2 + 2rd\Omega^2 = \frac{V'}{V}dr^2 + V'dt^2 + 2rd\Omega^2$,
- $\mathcal{L}_{-\frac{1}{V}\partial_t} g = -\frac{2}{V}dr\mathcal{L}_{-\frac{1}{V}\partial_t}dr + 2V dt\mathcal{L}_{-\frac{1}{V}\partial_t}dt = 2V dt\left(-\frac{V'}{V^2}\right)dr.$

Hence

$$\begin{aligned}
\Xi &= \frac{1}{2}\mathcal{L}_Y g = \frac{1}{2}\mathcal{L}_{\partial_r - \frac{1}{V}\partial_t} g = \frac{1}{2}\mathcal{L}_{\partial_r} g + \frac{1}{2}\mathcal{L}_{-\frac{1}{V}\partial_t} g = \frac{1}{2}\frac{V'}{V^2}dr^2 + \frac{V'}{2}dt^2 + rd\Omega^2 + \frac{V'}{V}dtdr \\
&= \frac{1}{2}V'\left(dt + \frac{1}{V}dr\right)^2 + rd\Omega^2 = \frac{1}{2}V'du^2 + rd\Omega^2.
\end{aligned}$$

(d) In this step, we will prove an important inequality, namely the estimate from below for $T^{\mu\nu}\Xi_\mu$ given in (12) below.

We have

$$du(\partial_r) = \Big(dt + \frac{1}{V}\Big)\partial_r = \frac{1}{V}, \quad \text{and} \quad du(\partial_t) = 1,$$

and on the other hand,

$$-g(Y, \partial_r) = -g\Big(\partial_r - \frac{1}{V}\partial_t, \partial_r\Big) = \frac{1}{V}, \text{ and } -g(Y, \partial_t) = -g\Big(\partial_r - \frac{1}{V}\partial_t, \partial_t\Big) = 1,$$

showing that

$$du = -g(Y, \cdot).$$

Also, we recall that

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{g_{\mu\nu}}{2}\left(\partial_\alpha\phi\partial^\alpha\phi + m^2\phi^2\right) = \Big(d\phi\otimes d\phi - \frac{1}{2}\langle d\phi, d\phi\rangle g - \frac{1}{2}m^2\phi^2 g\Big)_{\mu\nu}.$$

It follows that

$$T^{\mu\nu}\Xi_{\mu\nu} \;=\; \Xi^{\mu\nu}T_{\mu\nu} = \frac{1}{2}V'(Y\cdot\phi)^2 + \frac{1}{r^3}|\mathring{\nabla}\phi|^2 - \frac{n-1}{2r}\langle d\phi, d\phi\rangle - \frac{n-1}{2r}m^2\phi^2. \quad (10)$$

We have $d\phi(\partial_\mu) = \partial_\mu\phi$. So

$$\langle d\phi, d\phi\rangle = g^{\alpha\beta}(d\phi)_\alpha(d\phi)_\beta = g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) = (\partial_\alpha\phi)(\partial^\alpha\phi) = \phi'^2(-V) + \dot{\phi}^2\frac{1}{V} + \frac{1}{r^2}|\mathring{\nabla}\phi|^2.$$

Also, $-2(K\cdot\phi)(Y\cdot\phi) - V(Y\cdot\phi)^2 = -2\dot{\phi}\Big(\phi' - \frac{1}{V}\dot{\phi}\Big) - V\Big(\phi' - \frac{1}{V}\dot{\phi}\Big)^2 = \frac{1}{V}\dot{\phi}^2 - V\phi'^2$. So

$$\langle d\phi, d\phi\rangle = -2(K\cdot\phi)(Y\cdot\phi) - V(Y\cdot\phi)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2. \quad (11)$$

Combining (10) and (11), we obtain

$$\begin{aligned} T^{\mu\nu}\Xi_{\mu\nu} &= \Big(\frac{V'}{2} + \frac{n-1}{2r}V\Big)(Y\cdot\phi)^2 + \frac{n-1}{r}(K\cdot\phi)(Y\cdot\phi) - \frac{n-3}{2r^3}|\mathring{\nabla}\phi|^2 - \frac{n-1}{2r}m^2\phi^2 \\ &= \frac{V'}{2}\Big(Y\cdot\phi + \frac{n-1}{rV'}(K\cdot\phi)\Big)^2 - \frac{(n-1)^2}{2r^2V'}(K\cdot\phi)^2 - \frac{n-3}{2r^3}|\mathring{\nabla}\phi|^2 - \frac{n-1}{2r}m^2\phi^2. \end{aligned}$$

Now as $V'(r) > 0$ for $r \geqslant r_c$ (global redshift), it follows that the first summand in the last expression is nonnegative, and so we obtain the inequality

$$T^{\mu\nu}\Xi_{\mu\nu} \geqslant -\frac{(n-1)^2}{2r^2V'}(K\cdot\phi)^2 - \frac{n-3}{2r^3}|\mathring{\nabla}\phi|^2 - \frac{n-1}{2r}m^2\phi^2. \quad (12)$$

(e) In this final step, we obtain the desired red-shift estimates, and complete the proof of Theorem 5.3.

Suppose that r_0 is fixed. As $r^2V'(r) > 0$ for all $r \in [r_c, r_0]$, we have $\min_{r\in[r_c,r_0]} r^2V'(r) > 0$.

Thus $-\frac{(n-1)^2}{2r^2V'} \geqslant -\frac{(n-1)^2}{\min_{r\in[r_c,r_0]} r^2V'(r)} =: -C_1(r_0)$.

Similarly, for $r \in [r_c, r_0]$, $\frac{1}{r^3} \leqslant \frac{1}{r_c^3}$, and so $-\frac{n-3}{2r^3} \geqslant -\frac{n-3}{2r_c^3} =: -C_2(r_c)$.

Also, for $r \in [r_c, r_0]$, $-\frac{(n-1)}{2r}m^2 \geqslant -\frac{(n-1)}{2r_c}m^2 =: -C_3(r_c)$.

So (12) gives

$$\begin{aligned} T^{\mu\nu}\Xi_{\mu\nu} &\geqslant -C_1(r_0)(K\cdot\phi)^2 - C_2(r_c)|\mathring{\nabla}\phi|^2 - C_3(r_c)\phi^2 \\ &\geqslant -\underbrace{\max\{C_1(r_0), C_2(r_c), C_3(r_c)\}}_{=:C(r_c,r_0)>0}\cdot\Big((K\cdot\phi)^2 + |\mathring{\nabla}\phi|^2 + \phi^2\Big). \end{aligned}$$

Given an $r_1 \in (r_c, r_0)$, and a $T > 0$, we define the set $\mathcal{D} = \{r = r_1\} \cap \{-T \leqslant t \leqslant T\}$. Applying the divergence theorem to the region $\mathcal{T} = D^+(\mathcal{D}) \cap \{r \leqslant r_0\}$, and noticing that the flux across the future null boundaries is less than or equal to 0, we obtain, after passing the limit $T \to \infty$, that

$$\widetilde{E}(r_1) - \widetilde{E}(r_0) \geqslant -\int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} C(r_0, r_c)\Big((K\cdot\phi)^2 + |\mathring{\nabla}\phi|^2 + \phi^2\Big) r^{n-1} dt d\Omega dr. \tag{13}$$

But

$$\begin{aligned} \widetilde{E}(r) &= \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \Big(V^2\phi'^2 + \dot{\phi}^2 + \frac{V}{r^2}|\mathring{\nabla}\phi|^2 + m^2V\phi^2\Big) dt d\Omega \\ &\quad + \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \Big(V(Y\cdot\phi)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + m^2\phi^2\Big) r^{n-1} dt d\Omega. \end{aligned}$$

In particular,

$$\begin{aligned} \int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} (K\cdot\phi)^2 r^{n-1} dt d\Omega dr &\leqslant \int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} \dot{\phi}^2 r_0^{n-1} dt d\Omega dr \\ &\leqslant \int_{r_1}^{r_0} 2\widetilde{E}(r) r_0^{n-1} dr = 2r_0^{n-1}\int_{r_1}^{r_0}\widetilde{E}(r)dr. \end{aligned}$$

Also, $$\begin{aligned} \int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} |\mathring{\nabla}\phi|^2 r^{n-1} dt d\Omega dr &\leqslant \int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} \frac{|\mathring{\nabla}\phi|^2}{r^2} r^{n-1} r^2 dt d\Omega dr \\ &\leqslant \int_{r_1}^{r_0} 2\widetilde{E}(r) r_0^2 dr = 2r_0^2\int_{r_1}^{r_0}\widetilde{E}(r)dr. \end{aligned}$$

Finally, $$\int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} \phi^2 r^{n-1} dt d\Omega dr \leqslant \int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} m^2\phi^2 r^{n-1}\frac{1}{m^2} dt d\Omega dr$$
$$\leqslant \int_{r_1}^{r_0} 2\widetilde{E}(r)\frac{1}{m^2}dr = \frac{2}{m^2}\int_{r_1}^{r_0}\widetilde{E}(r)dr.$$

Using the above three estimates, it follows from (13) that

$$\widetilde{E}(r_1)-\widetilde{E}(r_0) \geqslant -\int_{r_1}^{r_0} C(r_0,r_c)\left(2r_0^{n-1}+2r_0^2+\frac{2}{m^2}\right)\widetilde{E}(r)dr = -\int_{r_1}^{r_0} k(r_0,r_c)\widetilde{E}(r)dr, \quad (14)$$

where
$$k(r_0,r_c) := C(r_0,r_c)\left(2r_0^{n-1}+2r_0^2+\frac{2}{m^2}\right).$$

Now suppose that r_2 is such that $r_c < r_1 < r_2 < r_0$. If we redo all of the above steps in order to obtain (14), but with r_2 replacing r_0, we obtain

$$\widetilde{E}(r_1)-\widetilde{E}(r_2) \geqslant -\int_{r_1}^{r_2} k(r_2,r_c)\widetilde{E}(r)dr, \quad (15)$$

where
$$k(r_2,r_c) = C(r_2,r_c)\left(2r_2^{n-1}+2r_2^2+\frac{2}{m^2}\right).$$

But

$$\begin{aligned} k(r_2,r_c) &= C(r_2,r_c)\left(2r_2^{n-1}+2r_2^2+\frac{2}{m^2}\right) \leqslant C(r_2,r_c)\left(2r_0^{n-1}+2r_0^2+\frac{2}{m^2}\right) \\ &= \max\{C_1(r_2),\, C_2(r_c),\, C_3(r_c)\}\cdot\left(2r_0^{n-1}+2r_0^2+\frac{2}{m^2}\right). \end{aligned}$$

We have $C_1(r_2) = \dfrac{(n-1)^2}{\min\limits_{r\in[r_c,r_2]} r^2V'(r)} \leqslant \dfrac{(n-1)^2}{\min\limits_{r\in[r_c,r_0]} r^2V'(r)} = C_1(r_0)$, since $[r_c,r_2]\subset[r_c,r_0]$. Hence

$$\begin{aligned} k(r_2,r_c) &\leqslant \max\{C_1(r_2),\, C_2(r_c),\, C_3(r_c)\}\cdot\left(2r_0^{n-1}+2r_0^2+\frac{2}{m^2}\right) \\ &\leqslant \max\{C_1(r_0),\, C_2(r_c),\, C_3(r_c)\}\cdot\left(2r_0^{n-1}+2r_0^2+\frac{2}{m^2}\right) = k(r_0,r_c). \end{aligned}$$

So from (15), we get $\widetilde{E}(r_1)-\widetilde{E}(r_2) \geqslant -\int_{r_1}^{r_2} k(r_2,r_c)\widetilde{E}(r)dr \geqslant -\int_{r_1}^{r_2} k(r_0,r_c)\widetilde{E}(r)dr.$

Consequently, for all $r_2\in[r_1,r_0)$, $\widetilde{E}(r_2) \leqslant \widetilde{E}(r_1) + \int_{r_1}^{r_2} k(r_0,r_c)\widetilde{E}(r)dr.$

By the integral form of Grönwall's inequality (see e.g. [35, Thm. 1.10]), we obtain for all $r_2\in[r_1,r_0)$ that $\widetilde{E}(r_2) \leqslant \widetilde{E}(r_1)e^{\int_{r_1}^{r_2} k(r_0,r_c)dr} = \widetilde{E}(r_1)e^{k(r_0,r_c)\cdot(r_2-r_1)}$. Passing the limit as

$r_2 \nearrow r_0$ yields $\widetilde{E}(r_0) \leqslant \widetilde{E}(r_1) e^{k(r_0,r_c)\cdot(r_0-r_1)}$, and this holds for all $r_1 \in (r_c, r_0)$. Now passing the limit as $r_1 \searrow r_c$, we obtain $\widetilde{E}(r_0) \leqslant \widetilde{E}(r_c) e^{k(r_0,r_c)\cdot(r_0-r_c)}$. Consequently,

$$E(r_0) \leqslant \widetilde{E}(r_0) \leqslant \widetilde{E}(r_c) e^{k(r_0,r_c)\cdot(r_0-r_c)} \lesssim \widetilde{E}(r_c) \lesssim \|\phi\|^2_{H^1(\mathcal{CH}_1^+)} + \|\phi\|^2_{H^1(\mathcal{CH}_2^+)} < +\infty.$$

Commuting with the Killing vector fields $\frac{\partial}{\partial t}$ and L_{ij}, we see that the hypothesis from Theorem 5.3, namely, $\|\phi\|_{H^k(\mathcal{CH}_1^+)} < +\infty$ and $\|\phi\|_{H^k(\mathcal{CH}_2^+)} < +\infty$, for some $k > \frac{n}{2} + 2$, yields also that $\|\phi\|_{H^k(\{r=r_0\})} \lesssim \|\phi\|_{H^k(\mathcal{CH}_1^+)} + \|\phi\|_{H^k(\mathcal{CH}_2^+)} < +\infty$. We now show that this justifies the assumption used in the previous two subsections. For simplicity, we only consider one of the energies

$$\mathcal{E}(r) = \frac{1}{2} \int_{\mathbb{R}\times S^{n-1}} \left(\psi'^2 + \frac{1}{V^2}\dot{\psi}^2 + \frac{1}{r^2 V} |\mathring{\nabla}\psi|^2 + \theta\psi^2 \right) dt d\Omega.$$

(The proof of the finiteness of $\widetilde{\mathcal{E}}(r_0)$ is entirely analogous.) As $\psi = r^\kappa \phi$, we obtain finiteness of the last summand, namely

$$\begin{aligned} \int_{\mathbb{R}\times S^{n-1}} \theta(r_0) \left(\psi(r_0,\cdot)\right)^2 dt d\Omega &= \theta(r_0) r_0^{2\kappa} \int_{\mathbb{R}\times S^{n-1}} \left(\phi(r_0,\cdot)\right)^2 dt d\Omega \\ &\leqslant \theta(r_0) r_0^{2\kappa} \|\phi(r_0,\cdot)\|^2_{H^1(\{r=r_0\})} < +\infty. \end{aligned}$$

We have

$$\int_{\mathbb{R}\times S^{n-1}} \left(\phi'(r_0,\cdot)\right)^2 dt d\Omega = \frac{1}{(V(r_0))^2} \int_{\mathbb{R}\times S^{n-1}} \left(V(r_0,\cdot)\phi'(r_0,\cdot)\right)^2 dt d\Omega \leqslant \frac{2E(r_0)}{(V(r_0))^2} < +\infty.$$

Using $\psi'(r_0,\cdot) = \kappa r_0^{\kappa-1}\phi(r_0,\cdot) + r_0^\kappa \phi'(r_0,\cdot)$ and $\phi(r_0,\cdot) \in H^1(\{r = r_0\})$, we conclude that $\psi(r_0,\cdot) \in L^2(\mathbb{R}\times S^{n-1})$, that is,

$$\int_{\mathbb{R}\times S^{n-1}} \left(\psi'(r_0,\cdot)\right)^2 dt d\Omega < +\infty.$$

We also have

$$\int_{\mathbb{R}\times S^{n-1}} \frac{(\dot{\psi}(r_0,\cdot))^2}{\left(V(r_0)\right)^2} dt d\Omega = \frac{r_0^{2\kappa}}{(V(r_0))^2} \int_{\mathbb{R}\times S^{n-1}} (\dot{\phi}(r_0,\cdot))^2 dt d\Omega \leqslant \frac{r_0^{2\kappa}}{(V(r_0))^2} \|\phi(r_0,\cdot)\|^2_{H^1(\{r=r_0\})} < \infty.$$

Finally,

$$\int_{\mathbb{R}\times S^{n-1}} \frac{|\mathring{\nabla}\psi(r_0,\cdot)|^2}{r_0^2 V(r_0)} dt d\Omega = \frac{r_0^{2\kappa}}{r_0^2 V(r_0)} \int_{\mathbb{R}\times S^{n-1}} |\mathring{\nabla}\phi(r_0,\cdot)|^2 dt d\Omega \lesssim \|\phi(r_0,\cdot)\|^2_{H^1(\{r=r_0\})} < \infty.$$

Thus each summand in the expression for $\mathcal{E}(r_0)$ is finite.

This completes the proof of Theorem 5.3.

6 Decay in RNdS when $m = 0$, the wave equation

In this section, we will prove Theorem 6.3. This will be done by using a similar method to the one used to prove Rendall's Conjecture (Theorem 3.2) in Section 3. Our Theorem 6.3 is an improvement to [9, Theorem 2], which we recall below.

Theorem 6.1.

Suppose that

- $\delta > 0$,
- $M > 0$,
- $e \geqslant 0$,
- $n > 2$,
- (M, g) *is the* $(n+1)$*-dimensional subextremal Reissner-Nordström-de Sitter solution given by the metric* $g = -\frac{1}{V}dr^2 + Vdt^2 + r^2 d\Omega^2$*, where* $V = r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1$*, and* $d\Omega^2$ *is the metric of the unit* $(n-1)$*-dimensional sphere* S^{n-1}*,*
- $k > \frac{n}{2} + 2$*, and*
- ϕ *is a smooth solution to* $\Box_g \phi = 0$ *such that*
$$\|\phi\|_{H^k(\mathcal{CH}_1^+)} < +\infty \quad \text{and} \quad \|\phi\|_{H^k(\mathcal{CH}_2^+)} < +\infty,$$
where $\mathcal{CH}_1^+ \simeq \mathcal{CH}_2^+ \simeq \mathbb{R} \times S^{n-1}$ *are the two components of the future cosmological horizon, parameterised by the flow parameter of the global Killing vector field* $\frac{\partial}{\partial t}$*.*

Then there exists a r_0 *large enough so that for all* $r \geqslant r_0$*,* $\|\partial_r \phi(r, \cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim r^{-3+\delta}$*.*

Using a method similar to the one we used to show Rendall's conjecture in Theorem 3.2, we can improve the almost-exact bound of $r^{-3+\delta}$ to r^{-3}.

Remark 6.2. As observed in [9, Remark 1.4], this decay rate bound of r^{-3} for $\partial_r \phi$ is in fact the decay rate one would expect in light of Rendall's conjecture. We had noted in Remark 5.2, that one may compare the RNdS metric in the cosmological region with large r in the $(r, t, \ldots)$-coordinates with the metric of de Sitter spacetime in flat FLRW form in $(\tau, x, \ldots)$-coordinates, with $r \sim e^\tau$. Having settled Rendall's Conjecture in de Sitter spacetime in flat FLRW form, namely by Theorem 3.2, we know that $\partial_\tau \phi \sim e^{-2\tau}$. Also,
$$\frac{dr}{d\tau} = e^\tau.$$
Thus we expect that
$$\partial_r \phi \sim \frac{\partial_\tau \phi}{\partial_\tau r} \sim \frac{e^{-2\tau}}{e^\tau} = e^{-3\tau} = (e^{-\tau})^3 \sim \frac{1}{r^3}.$$

Thus our improved version of Theorem 6.1 is the following result.

Theorem 6.3.

Suppose that

- $M > 0$,
- $e \geqslant 0$,
- $n > 2$,
- (M, g) *is the* $(n+1)$*-dimensional subextremal Reissner-Nordström-de Sitter solution given by the metric* $g = -\frac{1}{V}dr^2 + Vdt^2 + r^2 d\Omega^2$, *where* $V = r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1$, *and* $d\Omega^2$ *is the metric of the unit* $(n-1)$*-dimensional sphere* S^{n-1},
- $k > \frac{n}{2} + 2$, *and*
- ϕ *is a smooth solution to* $\Box_g \phi = 0$ *such that*

$$\|\phi\|_{H^k(\mathcal{CH}_1^+)} < +\infty \quad \text{and} \quad \|\phi\|_{H^k(\mathcal{CH}_2^+)} < +\infty,$$

where $\mathcal{CH}_1^+ \simeq \mathcal{CH}_2^+ \simeq \mathbb{R} \times S^{n-1}$ *are the two components of the future cosmological horizon, parameterised by the flow parameter of the global Killing vector field* $\frac{\partial}{\partial t}$.

Then there exists a r_0 *large enough so that for all* $r \geqslant r_0$, $\|\partial_r \phi(r, \cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim r^{-3}$.

Proof. We have the following estimates. There exists an r_0 large enough so that for all $r \geqslant r_0$,

$$\|\ddot{\phi}(r, \cdot)\|_{L^\infty(\mathbb{R}, S^{n-1})} \lesssim 1, \quad \text{and} \quad \|\mathring{\Delta}\phi(r, \cdot)\|_{L^\infty(\mathbb{R}, S^{n-1})} \lesssim 1.$$

These can be showed by following [9, §3.2]. For the details, we refer the reader to Step 1 of the proof of [28, Theorem 5.3].

Next, we will write the wave equation in new coordinates which 'equalises' the magnitude of the coefficient weights for the r and t coordinates in the matrix of the metric. To this end, we define $\rho = \int_{r_0}^{r} \frac{1}{V(r)} dr$. Then $\frac{d\rho}{dr} = \frac{1}{V(r)}$ and $V(r)\frac{d}{dr} = \frac{d}{d\rho}$.

With a slight abuse of notation, we write $V(\rho) := V(r(\rho))$. We have

$$g = -\frac{1}{V}dr^2 + Vdt^2 + r^2 d\Omega^2 = -Vd\rho^2 + Vdt^2 + (r(\rho))^2 d\Omega^2.$$

The wave equation $\Box_g \phi = 0$ can be rewritten as $\partial_\mu(\sqrt{-g}\, \partial^\mu \phi) = 0$, which becomes $\partial_\mu(Vr^{n-1}\partial^\mu \phi) = 0$. Separating the differential operators with respect to the $(\rho, t, \dots)$ coordinates, we obtain $\partial_\rho(r^{n-1}\partial_\rho \phi) = r^{n-1}\ddot{\phi} + Vr^{n-3}\mathring{\Delta}\phi$. Integrating from $\rho_0 := \rho(r_0) = 0$

to $\rho = \rho(r)$, we obtain

$$r^{n-1}\partial_\rho \phi - r_0^{n-1} \left(\partial_\rho \phi\right)\big|_{\rho=\rho_0} = \int_0^\rho \left(r^{n-1}\ddot{\phi} + V r^{n-3} \mathring{\Delta}\phi\right) d\rho,$$

and so $r^{n-1} V \partial_r \phi = r_0^{n-1} V(r_0) \left(\partial_r \phi\right)\big|_{r=r_0} + \int_0^\rho \left(r^{n-1}\ddot{\phi} + V r^{n-3} \mathring{\Delta}\phi\right) d\rho$, that is,

$$\partial_r \phi = \left(\frac{r_0}{r}\right)^{n-1} \frac{V(r_0)}{V(r)} \left(\partial_r \phi\right)\big|_{r=r_0} + \frac{1}{r^{n-1} V} \int_0^\rho \left(r^{n-1}\ddot{\phi} + V r^{n-3} \mathring{\Delta}\phi\right) d\rho.$$

Hence

$$\begin{aligned} \|\partial_r \phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \leqslant{} & \left(\frac{r_0}{r}\right)^{n-1} \frac{V(r_0)}{V(r)} \|(\partial_r \phi)(r_0,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \\ & + \frac{1}{r^{n-1} V} \int_{\rho_0}^\rho \left(r^{n-1} \|\ddot{\phi}(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} + V r^{n-3} \|\mathring{\Delta}\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})}\right) d\rho. \end{aligned}$$

Using the fact that $V \sim r^2$ for $r \geqslant r_0$, with r_0 large enough, and the estimates from Step 1 above, we obtain

$$\begin{aligned} \|\partial_r \phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim \frac{A}{r^{n+1}} + \frac{B}{r^{n+1}} \int_{\rho_0}^\rho \left(r(\rho)\right)^{n-1} d\rho & \lesssim \frac{A}{r^{n+1}} + \frac{B}{r^{n+1}} \int_{r_0}^r r^{n-1} \frac{1}{V(r)} dr \\ & \lesssim \frac{A}{r^{n+1}} + \frac{B'}{r^{n+1}} \int_{r_0}^r r^{n-3} dr. \end{aligned}$$

Recalling that $n > 2$, we have $\|\partial_r \phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim \frac{A}{r^{n+1}} + \frac{B'}{r^{n+1}} \frac{1}{(n-2)} (r^{n-2} - r_0^{n-2}) \lesssim \frac{1}{r^3}$.

This completes the proof of Theorem 6.3. $\square$

7 Conclusions and outlook

In this thesis, we obtained exact decay rates for solutions to the Klein-Gordon equation in two expanding cosmological spacetimes, namely the de Sitter universe in flat FLRW form, and the cosmological region of the RNdS model. This was achieved by using energy methods, assuming that initial data for the Cauchy problem has finite higher order energies. We also improved a previously established decay rate of the time derivative of the solution to the wave equation, in an expanding de Sitter universe in flat FLRW form, proving Rendall's conjecture. A similar improvement was also given for the wave equation in the cosmological region of the RNdS spacetime.

A natural question for future investigation that arises is whether we can obtain similar results assuming weaker regularity, in terms of the order of the H^k-norm, of the initial data.

A Fourier modes (de Sitter in flat FLRW form)

In this appendix, we give the details of the Fourier modal analysis that motivates the specific estimates given in Theorem 4.1, starting with spatially periodic solutions to the Klein-Gordon equation.

Let $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. Suppose that the 'spatially periodic' $\phi : \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}$ satisfies the Klein-Gordon equation (4).

Writing $\phi = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}}(t) e^{i\langle \mathbf{k},\mathbf{x}\rangle}$, (4) yields $-\ddot{c}_{\mathbf{k}} - \frac{n\dot{a}}{a}\dot{c}_{\mathbf{k}} + \frac{1}{a^2}\delta^{pq} ik_p ik_q c_{\mathbf{k}} - m^2 c_{\mathbf{k}} = 0$, that is,

$$\ddot{c}_{\mathbf{k}} + \frac{n\dot{a}}{a}\dot{c}_{\mathbf{k}} + \frac{k^2}{a^2}c_{\mathbf{k}} + m^2 c_{\mathbf{k}} = 0,$$

where $k^2 := \langle \mathbf{k}, \mathbf{k}\rangle$. So

$$\frac{d}{dt}(a^n\dot{c}_{\mathbf{k}}) = na^{n-1}\dot{a}\dot{c}_{\mathbf{k}} + a^n\ddot{c}_{\mathbf{k}} = na^{n-1}\dot{a}\dot{c}_{\mathbf{k}} + a^n\left(-\frac{n\dot{a}}{a}\dot{c}_{\mathbf{k}} - \left(\frac{k^2}{a^2} + m^2\right)c_{\mathbf{k}}\right) = -a^n\left(\frac{k^2}{a^2} + m^2\right)c_{\mathbf{k}},$$

that is,

$$\frac{d}{dt}(a^n\dot{c}_{\mathbf{k}}) + a^{n-2}(k^2 + m^2a^2)c_{\mathbf{k}} = 0. \tag{16}$$

Let $\tau = \int \frac{1}{a(t)}dt$. Then $\frac{d}{dt} = \frac{1}{a}\frac{d}{d\tau}$, and so (16) becomes (with $\frac{d}{d\tau} =: \,'\,$)

$$\frac{1}{a}\left(a^n\frac{1}{a}c'_{\mathbf{k}}\right)' + a^{n-1}(k^2 + m^2a^2)c_{\mathbf{k}} = 0,$$

that is,

$$(a^{n-1}c'_{\mathbf{k}})' + a^{n-1}(k^2 + m^2a^2)c_{\mathbf{k}} = 0. \tag{17}$$

Defining $d_{\mathbf{k}}$ by $c_{\mathbf{k}} =: a^{-\frac{n-1}{2}}d_{\mathbf{k}}$, we have $c'_{\mathbf{k}} = a^{-\frac{n-1}{2}}d'_{\mathbf{k}} - \frac{n-1}{2}a^{-\frac{n-1}{2}-1}a'd_{\mathbf{k}}$. (17) yields

$$\left(a^{\frac{n-1}{2}}d'_{\mathbf{k}} - \frac{n-1}{2}a^{\frac{n-1}{2}-1}a'd_{\mathbf{k}}\right)' + a^{\frac{n-1}{2}}(k^2 + m^2a^2)d_{\mathbf{k}} = 0,$$

that is,

$$d''_{\mathbf{k}} + \left(k^2 + m^2a^2 - \frac{(n-1)}{2}\frac{a''}{a} - \frac{(n-1)(n-3)}{4}\left(\frac{a'}{a}\right)^2\right)d_{\mathbf{k}} = 0. \tag{18}$$

Now if $a(t) = e^t$, then we may take $\tau = -e^{-t}$, so that $-t = \log(-\tau)$, that is, $-\tau = e^{-t}$. We remark that relative to our earlier use of conformal coordinates in (3) on page 15, we are taking $t_0 = +\infty$ for simplicity.

We then have $a = e^{-\log(-\tau)} = -\dfrac{1}{\tau}$, $a' = \dfrac{1}{\tau^2}$, $a'' = -\dfrac{2}{\tau^3}$. Hence (18) becomes

$$d''_{\mathbf{k}}+\left(k^2+\frac{m^2}{\tau^2}-\frac{n-1}{2}\Big(-\frac{2}{\tau^3}\Big)\Big(-\frac{\tau}{1}\Big)-\frac{(n-1)(n-3)}{4}\Big(\frac{1}{\tau^2}\frac{(-\tau)^2}{1}\Big)\right)d_{\mathbf{k}}=0,$$

that is,

$$d''_{\mathbf{k}} + \left(k^2 - \frac{\mu}{\tau^2}\right) d_{\mathbf{k}} = 0, \tag{19}$$

where $\mu := n - 1 + \dfrac{(n-1)(n-3)}{4} - m^2$. The general solution to this equation is[14] given by

$$C_1\sqrt{\tau}\, J_\nu(|k|\tau) + C_2\sqrt{\tau}\, Y_\nu(|k|\tau), \tag{20}$$

where ν satisfies $\nu^2 = \dfrac{1}{4} + \mu = \dfrac{n^2}{4} - m^2$. Here J_ν is the Bessel function of the first kind,

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\Big(\frac{z}{2}\Big)^{2m+\nu},$$

and Y_ν is the Bessel function of the second kind,

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)},$$

where the right hand side is replaced by its limiting value if ν is an integer. Without loss of generality, in the solution (20), we may only consider ν such that $\mathrm{Re}(\nu) \geqslant 0$.

We note that as $t \to \infty$, $-\tau = e^{-t} \searrow 0$, and so $\tau \nearrow 0$. We now use the asymptotic expansions of $J_\nu(z)$ and $Y_\nu(z)$ as $z \nearrow 0$ (see e.g. [1, 9.1.7-9]):

<u>1</u>° If $\nu \neq 0$ (that is, $m \neq \pm\frac{n}{2}$), then as $\tau \nearrow 0$, we have

$$\begin{aligned} J_\nu(|k|\tau) &= C(-\tau)^\nu + O(|\tau|), \\ Y_\nu(|k|\tau) &= A(-\tau)^\nu + B(-\tau)^{-\nu} + C(-\tau)^{2-\nu} + O(|\tau|). \end{aligned}$$

So as $\tau \nearrow 0$ or $t \to \infty$, we have

$$\begin{aligned} d_{\mathbf{k}} &= A(-\tau)^{\frac{1}{2}+\nu} + B(-\tau)^{\frac{1}{2}-\nu} + C(-\tau)^{\frac{5}{2}-\nu} + O(|\tau|) \\ &= Ae^{(-\frac{1}{2}-\nu)t} + Be^{(-\frac{1}{2}+\nu)t} + Ce^{(-\frac{5}{2}+\nu)t} + O(e^{-t}). \end{aligned}$$

But $c_{\mathbf{k}} = a^{-\frac{n}{2}+\frac{1}{2}} d_{\mathbf{k}} = e^{(-\frac{n}{2}+\frac{1}{2})t} d_{\mathbf{k}}$, and so

$$c_{\mathbf{k}} = Ae^{(-\frac{n}{2}-\nu)t} + Be^{(-\frac{n}{2}+\nu)t} + Ce^{(-\frac{n}{2}-2+\nu)t} + O(e^{-\frac{n+1}{2}t})$$

as $t \to +\infty$. We recall that $\mathrm{Re}(\nu) \geqslant 0$, and so keeping only the dominating term, we have $|c_{\mathbf{k}}| = C'e^{-\left(\frac{n}{2}-\mathrm{Re}(\nu)\right)t} + O(e^{-\frac{n+1}{2}t})$ as $t \to +\infty$. Thus we expect ϕ to satisfy

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim \begin{cases} a^{-\frac{n}{2}} & \text{if } |m| > \frac{n}{2}, \\ a^{-\frac{n}{2}+\sqrt{\frac{n^2}{4}-m^2}} & \text{if } |m| < \frac{n}{2}. \end{cases}$$

[14]See for example [38, p.95]. For the relevant notation, see also [38, pages 82,100,101].

$\underline{2}^\circ$ If $\nu = 0$ (that is, $m = \pm\frac{n}{2}$), then as $\tau \nearrow 0$, we have

$$\begin{aligned} J_\nu(|k|\tau) &= C + O(|\tau|), \\ Y_\nu(|k|\tau) &= C\log(-\tau) + O(|\tau|). \end{aligned}$$

This implies that $d_{\mathbf{k}} = (A + Bt)e^{-\frac{1}{2}t} + O(e^{-t})$ as $t \to +\infty$.

Hence $|c_{\mathbf{k}}| = (A + Bt)e^{-\frac{n}{2}t} + O(e^{-\frac{n+1}{2}t})$. Thus we expect ϕ to satisfy

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim a^{-\frac{n}{2}}\log a \quad \text{if} \quad m = \pm\frac{n}{2}.$$

Summarising, ϕ is expected to have the decay

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim \begin{cases} a^{-\frac{n}{2}} & \text{if } |m| > \frac{n}{2}, \\ a^{-\frac{n}{2}}\log a & \text{if } |m| = \frac{n}{2}, \\ a^{-\frac{n}{2}+\sqrt{\frac{n^2}{4}-m^2}} & \text{if } |m| < \frac{n}{2}. \end{cases}$$

This motivates the decay estimates in Theorem 4.1.

B A technical lemma

In this appendix, we prove the technical result we had used in the proof of Theorem 4.1, in Section 4.

Lemma B.1. *If $f, g \in H^1(\mathbb{R}^n)$, then* $\lim_{r\to+\infty} \int_{S_r} fg\,d\sigma_r = 0$.

Proof. By the Cauchy-Schwarz inequality, $\left|\int_{S_r} fg\,d\sigma_r\right|^2 \leqslant \int_{S_r} |f|^2 d\sigma_r \cdot \int_{S_r} |g|^2 d\sigma_r$.

So it is enough to show that $\lim_{r\to\infty} \int_{S_r} |f|^2 d\sigma_r = 0$. Suppose this does not hold.

Then there exists an increasing sequence $(r_k)_k$ such that $r_k \overset{k\to\infty}{\longrightarrow} \infty$, and there exists an $\epsilon > 0$ such that for each k,

$$\int_{S_{r_k}} |f|^2 d\sigma_{r_k} > \epsilon.$$

(The plan is to use the trace theorem to fatten these S_{r_k}-slices to 'annuli' A_k and obtain $\|f\|^2_{H^1(A_k)} > \tilde{\epsilon} > 0$ for all k, giving rise to the contradiction that

$$\infty > \|f\|^2_{H^1(\mathbb{R}^n)} > \sum_k \|f\|^2_{H^1(A_k)} > \sum_k \tilde{\epsilon} = +\infty.$$

So we will construct a subsequence $(r_{k_m})_m$ of $(r_k)_k$ and a sequence $(\delta_m)_m$ of positive numbers such that $r_{k_1} < r_{k_1} + \delta_1 < r_{k_2} < r_{k_2} + \delta_2 < r_{k_3} < \cdots$, and such that for the 'annuli' $A_m := \{\mathbf{x} : r_{k_m} < |\mathbf{x}| < r_{k_m} + \delta_m\}$, we have $\|f\|^2_{H^1(A_m)} > \widetilde{\epsilon}$. We will need to keep track of the constants in the trace theorems on our annuli A_m, and we will use the following [22, p.41].)

Theorem B.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Then for $f \in H^1(\Omega)$ and for all $\epsilon \in (0,1)$,*

$$\|f\|^2_{L^2(\partial\Omega)} \leqslant \frac{\|\boldsymbol{\mu}\|_{C^1(\overline{\Omega})}}{\delta}\left(\epsilon^{1/2}\|\nabla f\|^2_{L^2(\Omega)} + (1+\epsilon^{-1/2})\|f\|^2_{L^2(\Omega)}\right),$$

where $\boldsymbol{\mu} \in C^1(\overline{\Omega}, \mathbb{R}^n)$ is such that $\boldsymbol{\mu}\cdot\mathbf{n} \geqslant \delta$ on $\partial\Omega$, $\mathbf{n}$ being the outer normal vector.

If Ω is an annulus $A = \{\mathbf{x} : r < \|\mathbf{x}\| < R\}$ (which is clearly bounded, open, and also it has the Lipschitz boundaries which are the two spheres S_r and S_R), then with $\boldsymbol{\mu}(\mathbf{x}) = \mathbf{x}$, we have

$$\boldsymbol{\mu}\cdot\mathbf{n} = \|\mathbf{x}\| = \left\{\begin{array}{c} R \text{ on } S_R, \\ r \text{ on } S_r \end{array}\right\} \geqslant r =: \delta.$$

Also, if we take $\epsilon = 1/4$, then $\|f\|^2_{L^2(S_r)} \leqslant \|f\|^2_{L^2(\partial A)} \leqslant 3\dfrac{\|\boldsymbol{\mu}\|_{C^1(\overline{A})}}{r}\|f\|^2_{H^1(A)}$. As

$$\|\boldsymbol{\mu}\|_{C^1(\overline{A})} = \max_{\overline{A}}\|\boldsymbol{\mu}\| + \max_{\overline{A}}|\nabla\cdot\boldsymbol{\mu}| = R + n,$$

we obtain $\|f\|^2_{L^2(S_r)} \leqslant 3\dfrac{R+n}{r}\|f\|^2_{H^1(A)}$. Now we will construct $(r_{k_m})_m$ and $(\delta_m)_m$.

We choose k_1 such that $r_{k_1} > n$. Let δ_1 be such that $0 < \delta_1 < r_{k_1} - n$. Then for the annulus $A_1 := \{\mathbf{x} : r_{k_1} < \|\mathbf{x}\| < r_{k_1} + \delta_1\}$, we have

$$\|f\|^2_{H^1(A_1)} \geqslant \frac{r_{k_1}/3}{(r_{k_1}+\delta_1)+n}\|f\|^2_{L^2(S_{r_{k_1}})} \geqslant \frac{1/3}{1+\frac{\delta_1+n}{r_{k_1}}}\epsilon > \frac{1/3}{1+1}\epsilon = \frac{\epsilon}{6} =: \widetilde{\epsilon}.$$

Suppose $r_{k_1}, \ldots, r_{k_m}, \delta_1, \ldots, \delta_m$ possessing the desired properties have been constructed. Choose k_{m+1} such that $r_{k_{m+1}} > r_{k_m} + \delta_m$. Let δ_{m+1} be such that $0 < \delta_{m+1} < r_{k_{m+1}} - n$. Then for the annulus $A_{m+1} := \{\mathbf{x} : r_{k_{m+1}} < \|\mathbf{x}\| < r_{k_{m+1}} + \delta_{m+1}\}$, we have

$$\|f\|^2_{H^1(A_{m+1})} > \frac{r_{k_{m+1}}/3}{(r_{k_{m+1}}+\delta_{m+1})+n}\|f\|_{L^2(S_{r_{k_{m+1}}})} \geqslant \frac{1/3}{1+\frac{\delta_{m+1}+n}{r_{k_{m+1}}}}\epsilon > \frac{\epsilon}{6} = \widetilde{\epsilon}.$$

This completes the induction step.

So we have arrived at the contradiction that $+\infty > \|f\|^2_{H^1(\mathbb{R}^n)} \geqslant \sum_m \|f\|^2_{H^1(A_m)} \geqslant \sum_m \widetilde{\epsilon} = +\infty$.

This shows that our original assumption was incorrect, and so $\lim_{r\to\infty}\int_{S_r}|f|^2 d\sigma_r = 0$. □

An analogous result also holds for the cylinder $\mathbb{R} \times S^{n-1}$. This was used in the proof of our Theorem 5.3.

Lemma B.3. *If $f, g \in H^1(\mathbb{R} \times S^{n-1})$, then* $\lim_{t\to+\infty} \int_{S^{n-1}} fgd\Omega = 0 = \lim_{t\to-\infty} \int_{S^{n-1}} fgd\Omega.$

Proof. (Sketch.) The proof is based on the same idea as the above, but is somewhat simpler, since the radius of S^{n-1} doesn't change, and the constants one has in the trace theorem for a 'cylindrical band' of the form $(a, b) \times S^{n-1}$ already work, as opposed to having to keep careful track, via Theorem B.2, of the constants in the earlier case when the radii of the S_r^{n-1} were changing. Proceeding in the same way as in the previous lemma, we assume that

$$\neg \left(\lim_{t\to+\infty} \int_{S^{n-1}} |f|^2 d\Omega = 0 \right),$$

and so there exists an $\epsilon > 0$ and a sequence $(t_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} t_k = +\infty$, and

$$\lim_{k\to+\infty} \int_{S^{n-1}} |f(t_k, \cdot)|^2 d\Omega > \epsilon.$$

In order to fatten the 'circle' $\{t_k\} \times S^{n-1}$ to a cylindrical band $I = (t_k, t_k + \delta) \times S^{n-1}$, while keeping the L^2-norm of f on the band uniformly (in k) bigger than a fixed positive quantity, one can use the inequality $\|f(t_k, \cdot)\|_{L^2(S^{n-1})} \leqslant C\|f\|_{H^1(I\times S^{n-1})}$. This follows from [36, Prop. 4.5, p.287], by taking $\Omega = [t_k, t_k + \delta] \times S^{n-1}$. The rest of the proof is the along the same lines. ☐

C Sharpness of bound when $|m| = \frac{n}{2}$ in Theorem 4.1

In this appendix, we will show the sharpness of the bound for the key estimate of the proof of the $|m| = \frac{n}{2}$ case of Theorem 4.1 (which was about the decay of the solution to the Klein-Gordon equation in the de Sitter universe in flat FLRW form).

Let us recall this bound: for all $t \geqslant t_0$, $\|\phi(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim a^{-\frac{n}{2}} \log a$.

If $|m| = \frac{n}{2}$, then with $\psi := a^{\frac{n}{2}}\phi$, we had seen that $\ddot{\psi} - \frac{1}{a^2}\Delta\psi = 0$.

We will give a solution ψ satisfying $\|\psi(t, \cdot)\|_{L^2(\mathbb{R}^n)} \sim A + Bt$ as $t \to \infty$, showing that $\|\phi(t, \cdot)\|_{L^2(\mathbb{R}^n)} \sim (A + Bt)a^{-\frac{n}{2}}$ as $t \to \infty$, and so the bound $\|\phi(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (A + Bt)a^{-\frac{n}{2}}$ for all large t cannot be improved.

We want

$$\ddot{\psi} - \frac{1}{e^{2t}}\Delta\psi = 0. \tag{21}$$

Taking the Fourier transform with respect to only the (spatial) $\mathbf{x}$-variable, and denoting

$$\widehat{\psi}(t, \boldsymbol{\xi}) := \int_{\mathbb{R}^n} \psi(t, \mathbf{x}) e^{i\langle\boldsymbol{\xi},\mathbf{x}\rangle} d^n\mathbf{x},$$

(21) becomes

$$\frac{\partial^2}{\partial t^2}\widehat{\psi}(t,\boldsymbol{\xi}) + \frac{\|\boldsymbol{\xi}\|^2}{e^{2t}}\widehat{\psi}(t,\boldsymbol{\xi}) = 0, \tag{22}$$

which is a family of ordinary differential equations in t, parameterised by $\boldsymbol{\xi} \in \mathbb{R}^n$. For a fixed $\boldsymbol{\xi} \in \mathbb{R}^n$, the general solution to the ODE (22) is given by

$$\widehat{\psi}(t,\boldsymbol{\xi}) = C_1(\boldsymbol{\xi}) \cdot J_0(\|\boldsymbol{\xi}\|e^{-t}) + C_2(\boldsymbol{\xi}) \cdot Y_0(\|\boldsymbol{\xi}\|e^{-t}),$$

where J_0 is the Bessel function of first kind and of order 0, and Y_0 is the Bessel function of second kind and of order 0. In order to construct our ψ, we will make special choices of C_1 and C_2.

We recall [1, (9.1.7-8)] that $J_0(z) \sim 1$, and $Y_0(z) \sim \dfrac{2}{\pi}\log z$ as $z \searrow 0$ ($z \in \mathbb{R}$).

Now as $t \to \infty$, $e^{-t} \searrow 0$, and so from the above limiting behaviour of J_0 and Y_0, we obtain that as $t \to \infty$,

$$\widehat{\psi}(t,\boldsymbol{\xi}) \;\sim\; C_1(\boldsymbol{\xi}) \cdot 1 + C_2(\boldsymbol{\xi}) \cdot \left(\frac{2}{\pi}\log(\|\boldsymbol{\xi}\|e^{-t})\right) = C_1(\boldsymbol{\xi}) + \frac{2}{\pi}C_2(\boldsymbol{\xi})\log\|\boldsymbol{\xi}\| - \frac{2}{\pi}t \cdot C_2(\boldsymbol{\xi}).$$

By Plancherel's identity [36, Prop. 3.2], $\|\widehat{\psi}(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)}$. Since we want the linear behaviour in t of $\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)}$, we keep C_2 nonzero, but may take $C_1 \equiv 0$. Then as $t \to \infty$, $\widehat{\psi}(t,\boldsymbol{\xi}) = C_2(\boldsymbol{\xi}) \cdot Y_0(\|\boldsymbol{\xi}\|e^{-t})$. In order to have $\widehat{\psi}(t,\cdot)$ (and so also $\psi(t,\cdot)$) in $L^2(\mathbb{R}^n)$ for all t, we choose C_2 to have a sufficiently fast decay.

We recall [1, §9.2.2] that $Y_0(z) = \sqrt{\dfrac{2}{\pi z}}\left(\sin\left(z - \dfrac{\pi}{4}\right) + O\Big(\dfrac{1}{|z|}\Big)\right)$ as $z \to \infty$ ($z \in \mathbb{R}$). So

$$Y_0(\|\boldsymbol{\xi}\|e^{-t}) = \sqrt{\frac{2}{\pi\|\boldsymbol{\xi}\|e^{-t}}}\left(\sin\left(\|\boldsymbol{\xi}\|e^{-t} - \frac{\pi}{4}\right) + O\Big(\frac{1}{\|\boldsymbol{\xi}\|e^{-t}}\Big)\right)$$

as $\|\boldsymbol{\xi}\| \to +\infty$ (and t is kept fixed). So to arrange $\widehat{\psi}(t,\cdot) \in L^2(\mathbb{R}^n)$ for all t, we may take

$$C_2(\boldsymbol{\xi}) := \frac{\|\boldsymbol{\xi}\|}{(\|\boldsymbol{\xi}\|^2+1)^{1+\frac{n}{4}}}.$$

(Also this choice makes $\boldsymbol{\xi} \mapsto C_2(\boldsymbol{\xi})\log\|\boldsymbol{\xi}\| \in L^2(\mathbb{R}^n)$, which will be needed below.)

Then $\widehat{\psi}(t,\cdot) \in L^2(\mathbb{R}^n)$ for all t.

Also, as $t \to \infty$, $\widehat{\psi}(t,\boldsymbol{\xi}) \sim \dfrac{2}{\pi}\Bigg(\underbrace{\dfrac{\|\boldsymbol{\xi}\|}{(\|\boldsymbol{\xi}\|^2+1)^{1+\frac{n}{4}}}\log\|\boldsymbol{\xi}\|}_{=:f\in L^2(\mathbb{R}^n)} - t\underbrace{\dfrac{\|\boldsymbol{\xi}\|}{(\|\boldsymbol{\xi}\|^2+1)^{1+\frac{n}{4}}}}_{=:g\in L^2(\mathbb{R}^n)}\Bigg)$, and

$$\|\widehat{\psi}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \geqslant \frac{2}{\pi}\Big(t\underbrace{\|g\|_{L^2(\mathbb{R}^n)}}_{\neq 0} - \|f\|_{L^2(\mathbb{R}^n)}\Big) \geqslant 0$$

for large t.

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