Future stability of models of the universe

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Introduction

The standard models of the universe

- satisfy the **cosmological principle** (i.e., they are spatially homogeneous and isotropic),
- are spatially flat,
- have matter content consisting of ordinary matter, dark matter and dark energy.
Current model of the universe: NASA/WMAP Science Team.

Hans Ringström  On the topology and future stability of the universe
Questions

Since the universe is not exactly spatially homogeneous and isotropic, it is natural to ask: **are the standard models future stable?**

Since the assumption of spatial homogeneity and isotropy yields strong restrictions on the allowed topologies, it is also of interest to ask: **what are the restrictions on the global topology of the universe imposed by the constraint that what we observe seems to be close to a standard model?**
Matter models

**Perfect fluids:** Matter described by energy density $\rho$ and pressure $p$. *Dust:* $p = 0$. *Radiation:* $p = \frac{\rho}{3}$.

**Vlasov matter:** collection of particles, where

- the particles all have unit mass,
- collisions are neglected,
- the particles follow geodesics,
- collection described statistically by a distribution function.
Einstein’s equations

Einstein’s equations:

\[ G + \Lambda g = T, \]

where

- \((M, g)\) is a Lorentz manifold,
- \(G = \text{Ric} - \frac{1}{2} Sg\) is the Einstein tensor,
- \(\Lambda\) is the cosmological constant,
- \(T\) is the stress energy tensor.
Vlasov matter, mathematical structures

In the Vlasov setting, the relevant mathematical structures are

- the **mass shell** $P$; the future directed unit timelike vectors in $(M, g)$,
- the **distribution function** $f : P \rightarrow [0, \infty)$,
- the **stress energy tensor**
  \[
  T_{\alpha\beta} |_{\xi} = \int_{P_{\xi}} f p_\alpha p_\beta \mu p_\xi,
  \]
- the **Vlasov equation**
  \[
  \mathcal{L} f = 0.
  \]
The Einstein-Vlasov system

The Einstein-Vlasov system consists of the equations

\[ G + \Lambda g = T, \]
\[ \mathcal{L}f = 0 \]

for \( g \) and \( f \). Note that the second equation corresponds to the requirement that \( f \) be constant along timelike geodesics.
Standard models

Spatial homogeneity, isotropy and flatness imply that the metric takes the form

\[ g = -dt^2 + a^2(t)\bar{g}. \]

The stress energy tensor of the Vlasov matter then takes perfect fluid form, and the energy density and pressure are given by

\[
\rho_{\text{Vl}}(t) = \int_{\mathbb{R}^3} \bar{f}(a(t)\bar{q})(1 + |\bar{q}|^2)^{1/2} d\bar{q},
\]

\[
p_{\text{Vl}}(t) = \frac{1}{3} \int_{\mathbb{R}^3} \bar{f}(a(t)\bar{q}) \frac{|\bar{q}|^2}{(1 + |\bar{q}|^2)^{1/2}} d\bar{q},
\]

where \( \bar{f} \), a function on \( \mathbb{R}^3 \), is the initial datum for the distribution function.
Approximating fluids

Figure: An illustration of an initial datum for the distribution function which is appropriate when approximating a standard model.
Induced initial data

Let \((M, g, f)\) be a solution and \(\Sigma\) be a spacelike hypersurface in \((M, g)\). Then the \textbf{initial data induced on} \(\Sigma\) consist of

- the Riemannian metric induced on \(\Sigma\) by \(g\), say \(\bar{g}\),
- the second fundamental form induced on \(\Sigma\) by \(g\), say \(\bar{k}\),
- the induced distribution function \(\bar{f} : T\Sigma \rightarrow [0, \infty)\).

Here

\[
\bar{f} = f \circ \text{proj}_{\Sigma}^{-1},
\]

where \(\text{proj}_{\Sigma} : P_{\Sigma} \rightarrow T\Sigma\) represents projection orthogonal to the normal.
Constraint equations

Due to the fact that \((M, g, f)\) solve the Einstein-Vlasov system, \((\bar{g}, \bar{k}, \bar{f})\) have to solve the constraint equations:

\[\bar{r} - \bar{k}_{ij}\bar{k}^{ij} + (\text{tr}\bar{k})^2 = 2\Lambda + 2\rho^\text{Vl},\]  
\[\nabla^j\bar{k}_{ji} - \nabla_i(\text{tr}\bar{k}) = -\bar{J}_i^\text{Vl}.\]

Here

\[\rho^\text{Vl}(\xi) = \int_{T\xi\Sigma} \bar{f}(\bar{p})[1 + \bar{g}(\bar{p}, \bar{p})]^{1/2}\bar{\mu}_{\bar{g},\xi},\]  
\[\bar{J}^\text{Vl}(\bar{X}) = \int_{T\xi\Sigma} \bar{f}(\bar{p})\bar{g}(\bar{X}, \bar{p})\bar{\mu}_{\bar{g},\xi}.\]
Abstract initial data

**Abstract initial data** consist of

- an $n$-dimensional manifold $\Sigma$,
- a Riemannian metric $\bar{g}$ on $\Sigma$,
- a symmetric covariant 2-tensor field $\bar{k}$ on $\Sigma$,
- a function $\bar{f} : T\Sigma \to [0, \infty)$ (belonging to a suitable function space),

all assumed to be smooth and such that the constraint equations (1) and (2) are satisfied.
Developments

Given abstract initial data, a development is

- a solution $(M, g, f)$ to the Einstein-Vlasov system, and
- an embedding $i : \Sigma \to M$

such that the initial data induced on $i(\Sigma)$ by $(M, g, f)$ correspond to the abstract initial data.
Maximal globally hyperbolic development

Definition
Given initial data, a \textit{maximal globally hyperbolic development} of the data is a globally hyperbolic development \((M, g, f)\), with embedding \(i : \Sigma \rightarrow M\), such that if \((M', g', f')\) is any other globally hyperbolic development of the same data, with embedding \(i' : \Sigma \rightarrow M'\), then there is a map \(\psi : M' \rightarrow M\) which is a diffeomorphism onto its image such that \(\psi^* g = g'\), \(\psi^* f = f'\) and \(\psi \circ i' = i\).
Function spaces

If $\Sigma$ is a compact manifold, $\mathcal{D}_\mu^\infty(T\Sigma)$ denotes the space of smooth functions $f : T\Sigma \to \mathbb{R}$ such that

$$
\| \tilde{f} \|_{H^{l}_{\nabla_1,\mu}} = \left( \sum_{i=1}^{j} \sum_{|\alpha|+|\beta| \leq l} \int_{\tilde{x}_i(U_i) \times \mathbb{R}^n} \langle \tilde{\bar{\rho}} \rangle^{2\mu+2|\beta|} \tilde{\chi}_i(\tilde{\xi})(\partial_{\tilde{\xi}}^\alpha \partial_{\tilde{\bar{\rho}}}^\beta \tilde{f}_{\tilde{x}_i})^2(\tilde{\xi}, \tilde{\bar{\rho}}) d\tilde{\xi} d\tilde{\bar{\rho}} \right)^{1/2}
$$

is finite for every $l \geq 0$, where

$$
\langle \tilde{\bar{\rho}} \rangle = (1 + |\tilde{\bar{\rho}}|^2)^{1/2}.
$$
Previous results

- Stability of de Sitter space in $3 + 1$-dimensions, etc., Helmut Friedrich, ’86, ’91.
- Stability of even dimensional de Sitter spaces, Michael Anderson ’05.
- Stability in the non-linear scalar field case, H.R. ’08.
- Einstein-Euler with a positive cosmological constant, Igor Rodnianski and Jared Speck, *preprint* ’09.
Bianchi initial data

Let

- $G$ be a 3–dimensional Lie group,
- $5/2 < \mu \in \mathbb{R}$,
- $\bar{g}$ and $\bar{k}$ be a left invariant Riemannian metric and symmetric covariant 2–tensor field on $G$ respectively,
- $\bar{f} \in \mathcal{D}_\mu^\infty(TG)$ be left invariant and non-negative.

Then $(G, \bar{g}, \bar{k}, \bar{f})$ are referred to as **Bianchi initial data** for the Einstein–Vlasov system, assuming they satisfy the constraints.
Future stability of spatially locally homogeneous solutions

Let \((G, \bar{g}_{bg}, \bar{k}_{bg}, \bar{f}_{bg})\) be Bianchi initial data for the Einstein–Vlasov system, where

- the universal covering group of \(G\) is not isomorphic to \(SU(2)\),
- \(\text{tr} \bar{k}_{bg} = \bar{g}_{bg}^{ij} \bar{k}_{bg,ij} > 0\).

Assume that there is a cocompact subgroup \(\Gamma\) of the isometry group of the initial data. Let \(\Sigma\) be the compact quotient. Then there is an \(\epsilon > 0\) such that if \((\Sigma, \bar{g}, \bar{k}, \bar{f})\) are initial data satisfying

\[
\|\bar{g} - \bar{g}_{bg}\|_{H^5} + \|\bar{k} - \bar{k}_{bg}\|_{H^4} + \|\bar{f} - \bar{f}_{bg}\|_{H^4_{V1,\mu}} \leq \epsilon,
\]

then the maximal Cauchy development of \((\Sigma, \bar{g}, \bar{k}, \bar{f})\) is future causally geodesically complete.
What is the shape of the universe?
Minkowski space; non-silent causality

Let $\gamma(t) = (t, 0, 0, 0)$. Then $\gamma$ is an observer in Minkowski space.
How much of the $t = 0$ hypersurface does $\gamma$ see?

Figure: The causal past of $\gamma(t)$ intersected with the causal future of the $t = 0$ hypersurface for $t = 1/2$, $t = 1$ and $t = 2$. 
de Sitter space; silent causality

Consider the metric

\[ g = -dt^2 + e^{2t} \tilde{g}. \]

\[ \begin{array}{ccc}
x^2 & 0 & 0.5 \\
-0.5 & -0.5 & 0 \\
0 & 0.5 & 0.5 \\
\end{array} \]

\[ \begin{array}{ccc}
\gamma^2 & 0 & 4 \\
-1 & -1 & 1 \\
0 & 0.5 & 0.5 \\
\end{array} \]

**Figure:** The causal past of \( \gamma(t) \) intersected with the causal future of the \( t = 0 \) hypersurface for \( t = 1/2 \) and for all \( t \).
Assume that

- the observational data indicate that, to our past, the universe is well approximated by one of the standard models,
- interpreting the data in this model, we only have information concerning the universe for $t \geq t_0$,
- there is a big bang,
- analogous statements apply to all observers in the universe (with the same $t_0$).

The question is then: what conclusions are we allowed to draw concerning the global spatial topology of the universe?
Ingredients

Assume we are given

- a standard model, characterised by an existence interval $I$, a scale factor $a$ etc.,
- a $t_0 \in I$, which represents the time to the future of which we wish the approximation to be valid,
- an $l \in \mathbb{N}$, specifying the norm with respect to which we measure proximity to the standard model,
- an $\epsilon > 0$, characterising the size of the distance,
- a closed 3-manifold $\Sigma$. 
Construction

There is a solution \((M, g, f)\) with the following properties:

- \((M, g, f)\) is a maximal Cauchy development,
- \((M, g)\) is future causally geodesically complete,
- there is a Cauchy hypersurface, say \(\bar{S}\), in \((M, g)\), diffeomorphic to \(\Sigma\),
- given an observer \(\gamma\) in \((M, g)\), there is a neighbourhood, say \(U\), of
  \[ J^- (\gamma) \cap J^+ (\bar{S}) \]
  such that the solution in \(U\) is \(\epsilon\)-close to the standard model in a solid cylinder of the form \([t_0, \infty) \times \bar{B}_R(0)\),
- all timelike geodesics in \((M, g)\) are past incomplete,
- the solution is stable with these properties.
Further reading...