Boundaries of spacetimes: causal and conformal approaches

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In honour **Aureliano de Mira Fernandes**

Aims:

- Different type of boundaries for spacetimes –and their problems
- History and recent (final!) definition of the causal boundary
- Relation with the conformal boundary

J.L. Flores, J.Herrera, MS., *On the final definition of the causal boundary and its relation with the conformal boundary*, in preparation.

MS, Nonlinear Anal. 09, arXiv:0812.0243v1

1.- Boundaries for spacetimes

1.1- Boundaries in Differential Geometry

Depending on the type of properties one would like to study, different "boundaries" may be attached to a space:

- Boundary associated to a Cauchy completion.
- Busemann boundary, containing the asymptotic directions in a Hadamard manifold.
- "Point at infinity" which compactifies the complex plane...

1.2- Geroch's g-boundary, Schmidt's b-boundary

Involve classes of curves and are non-conformally invariant:

- Mathematically elegant and systematic constructions such that:

  (i) every incomplete geodesic in the original spacetime terminates at a point

  (ii) geodesically continuous

- Drawback: a generic example (M,g) by Geroch, Liang and Wald (JMP'82) shows that minimal conditions (i), (ii) yield a rather undesirable topology

  Example:

  $M = (\mathbb{R}^2 \backslash \{r\}) \times \mathbb{R}^2$, $g = \Omega\langle\cdot,\cdot\rangle_{\mathbb{L}^2} + \langle\cdot,\cdot\rangle_{\mathbb{R}^2}$, for appropriate $\Omega > 0$. Undesirable property:

  boundary point r not T_1 related to some points in M.

Alternatives: conformal and causal boundaries –not so elegant mathematically

2.- Penrose conformal boundary

2.1- Generalities

- Importance because of some particular cases:

  - Lorentz-Minkowski $\mathbb{L}^n$ in Einstein static $\mathbb{R} \times \mathbb{S}^{n-1}$ –applications for asymptotic flatness, black holes, etc.
  - AdS-CFT correspondence –even though the limitations of the boundary led to the causal one for plane waves (Berenstein-Nastase '02, Marolf-Ross '02,'03, Flores-MS '08)

- General Defn. (?): find (if possible) an open conformal embedding $i : M \hookrightarrow M_0$ of your spacetime M in a ("aphysical") spacetime M_0 with (say, compact) closure $\overline{i(M)}$ and take as boundary $\partial(i(M)) \subset M_0$.

- Drawbacks: non-general and (except if plus restrictive) non-intrinsic.
- Some improvements but no general solution with further notions:

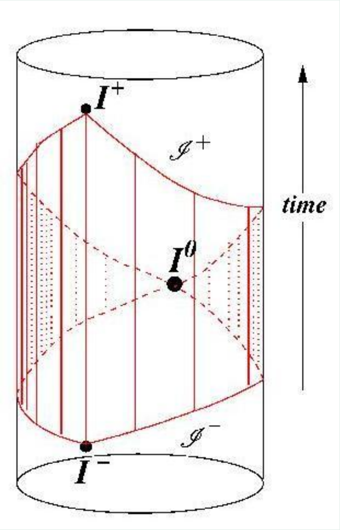
  Ashtekar & Hansen intrinsic conditions for asymp. flatness (JMP'78)

  Scott & Szekeres *abstract boundary* (JGP'94),

  García-Parrado & Senovilla *isocausal extensions* (CQG'03).

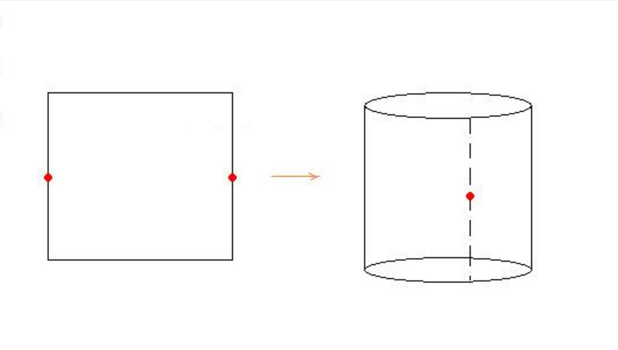
2.2- The conformal boundary fauna

Model:



Notice that the boundary is *not* smooth at $i^{\pm}$ and, what is more, recall i^0:

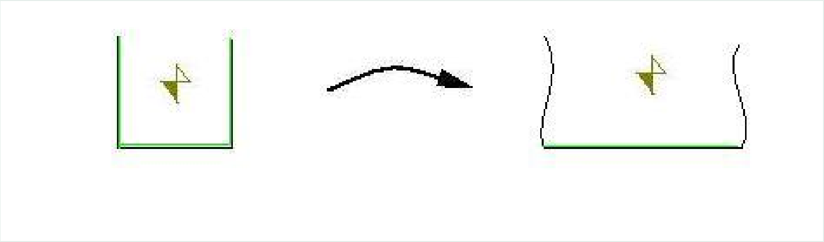
if we admit i^0, must we admit this?



Recall: implicitly one is using the induced topology.

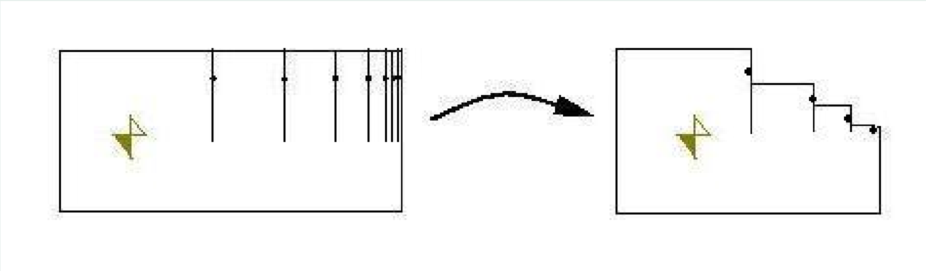
If we drop i^0, the completion is not compact.

Are we sure that the conformal boundary is “complete”?



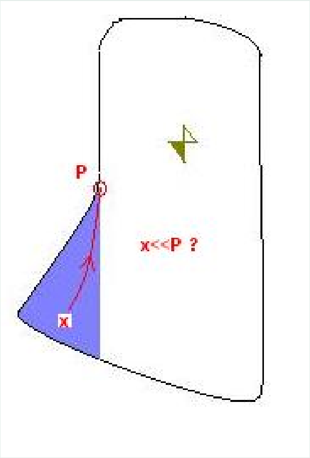
Even if the completion is compact:

is the conformal boundary truly conformally invariant?



Even in the smooth (and compact) case,

must we admit non-open chronological futures in the completion?



Recall: implicitly one is assuming that the induced chronology is a consistent chronology, is this true?

Necessity of criteria to admit the conformal boundary as admissible:

- as a point set,
- topologically and
- chronologically

3- Intuition for the c-boundary

Causal boundary ∂M of a spacetime M:

- Purpose: attach a boundary endpoint $P \in \partial M$ to any inextensible future or past directed timelike curve γ
- Basic idea: the boundary point would be represented by $P = I^{-}(\gamma)$ or $P = I^{+}(\gamma)$.
- Essential structure: conformal structure (Causality).
- No direct information on true singularities but on
  - limit causal dependence,
  - points at infinity or
  - conformally invariant singularities

.

- Intuitive picture: ∂M would be the disjoint union of

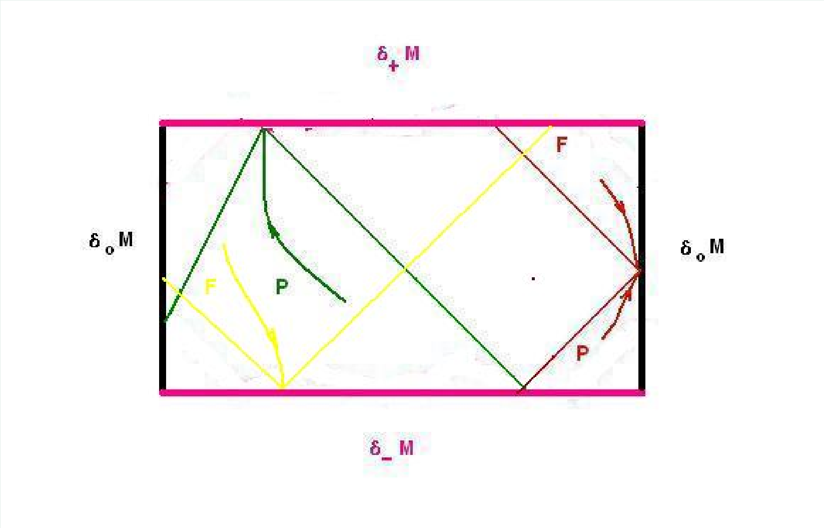
  1. *Future infinity* $\partial_+ M$, reached by future-directed timelike curves but no past-directed ones,
  2. *Past infinity* $\partial_- M$, dual to the former, and
  3. *Timelike boundary* $\partial_0 M$, whose points are reached by both, future and past directed curves.

     $\partial_0 M$ may represent

     – naked singularities,
     – the boundary of a removed region in a bigger spacetime
     – in general, losses of global hyperbolicity ($J^+(p) \cap J^-(q)$ is not compact)

Ideal scenario for differential equations:

– Initial conditions as some sort of limit on $\partial_- M$
– Boundary conditions on $\partial_0 M$


δ+ M
δo M
F
P
F
P
δo M
δ_ M

BUT recall, again one has to provide satisfactory definitions for

1. ∂M as a point set
2. $\overline{M} = M \cup \partial M$ as a topological set
3. $\overline{M}$ as a "chronological set" endowed (at least) with a binary relation $\overline{\ll}$ which extends the chronological one $\ll$ on M.

4.- The old problems of the c-boundary

4.1- Starting causal boundaries: GKP construction (Proc. London '72)

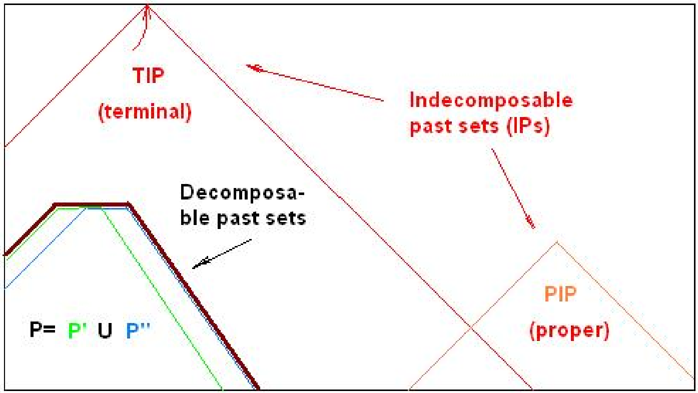
General construction starting (in principle) at a strongly causal spacetime M: intrinsic, systematic and unique

(a) *Construction: nice part.*

- $\hat{M}$: set of all the IP's (indecomposable past sets). Such a P may be:
  - *proper*: PIP, $P = I^-(p)$ for some $p \in M$ (M identificable to the set of PIP's, as the spacetime is past distinguishing) or
  - *terminal*: TIP, $P = I^-[\gamma]$ for some inextendible future-directed timelike curve

    The set of all TIPs is the *future boundary* $\hat{\partial} M$.
- Analogously $\check{M}$, IF, PIF, TIF $\check{\partial} M$.
- Precompletion $M^\sharp$: M with preboundary points $\hat{\partial} M \cup \check{\partial} M$

  $M^\sharp = (\hat{M} \cup \check{M})/\sim$ where $I^+(p) \sim I^-(p), \forall p \in M$.


TIP
(terminal)
Indecomposable
past sets (IPs)
Decomposa-
ble past sets
PIP
(proper)
P= P' U P''

(b) *Construction: not so nice part*

- Topologize $M^\sharp$ with a generalization of Alexandrov topology: sub-base F^{int}, F^{ext}, P^{int}, P^{ext}.

  $F^{int} = \{P \in \hat{M} : P \cap F \neq \emptyset\}$, for each $F \in \check{M}$.

  $F^{ext} = \{P \in \hat{M} : P = I^-[\omega] \Rightarrow I^+[\omega] \not\subset F\}$, for each $F \in \check{M}$.

- Causal completion:

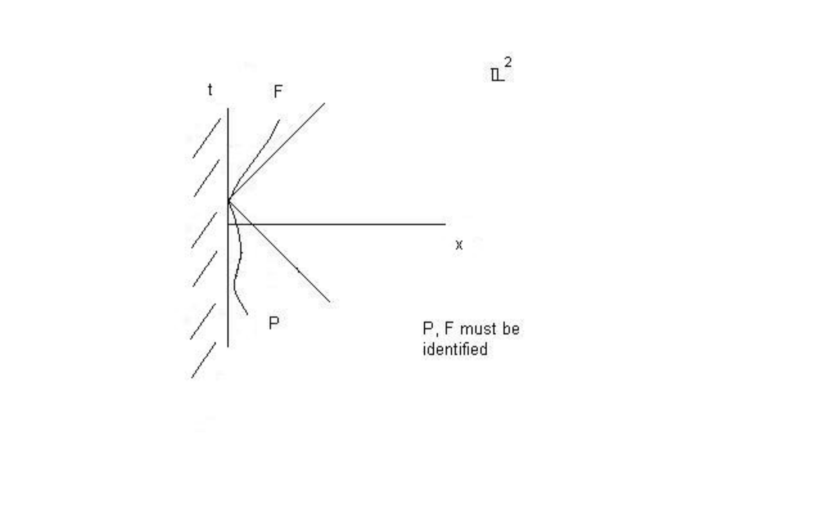
  $\bar{M} = M^\sharp / R_H$

  where R_H is the minimum identification of *preboundary points* to obtain a Hausdorff space

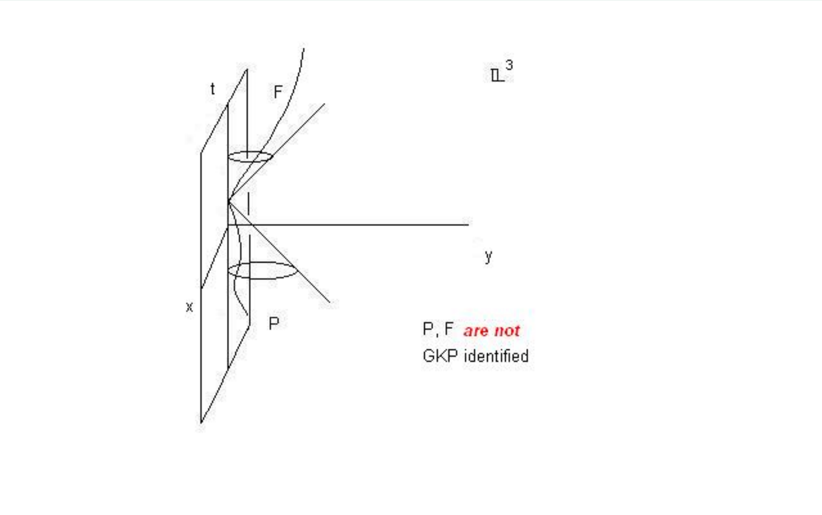
  $\longrightarrow$ one such identification (and, then, the intersection of all them) exist if M is stably causal but no in a general strongly causal spacetime (Szabados, CQG'88).

(c) *Problems*:

The relation of equivalence R_H was introduced because obvious identifications must be carried out to recover the natural boundary in examples such as $\mathbb{R} \times \mathbb{R}^+ \subset \mathbb{L}^2$ $(x > 0)$ $(x = 0)$



But GKP construction does not recover well the boundary of $\mathbb{R}^2 \times \mathbb{R}^+ \subset \mathbb{L}^3$ ($y > 0$)!!



$P = I^-[\rho], F = I^+[\gamma]$ "finish" at the same point with $y = 0$ but are not identified!! (Kuang, Liang, JMP'87).

More problems:

- Taub spacetime is static, but its “GKP singularity” is only a point, not a line (Kuang, Li, Liang, PRD’86).
- For $\mathbb{L}^n$, the topology of the (causal part of the) Penrose conformal boundary does not agree GKP topology (Harris CQG’00).
- Restriction to stably causal spacetimes...

4.3- First modifications: Budic & Sachs (JMP'74), Rácz (PRD'87)

How to ammend these problems?:

1. Rácz: GKP topology is unappropriate

   maintain the identifications but redefine the topology: F^{int}, F^{ext} are defined only when F is a PIF, and make it generate also IFs (and analogously for P)

2. Budic & Sachs: GKP identifications are unappropriate (as they are *indirect* and impose a priori Hausdorfness)

   maintain GKP topology but identify $P \in \hat{M}, F \in \check{M}$ in $M^\sharp$ if $P \sim_{BS} F$ i.e.:

$$P = \downarrow F (:= \mathrm{Int} \cap_{x \in F} I^-(x)) \text{ and } F = \uparrow P$$

   Of course $I^-(p) \sim_{BS} I^+(p)$ for all $p \in M$.

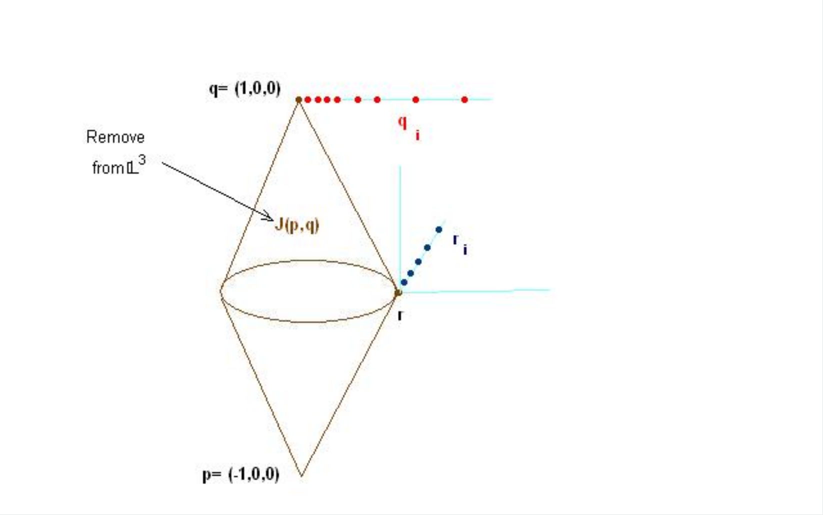
   + Other ingredients introduced, with expected good properties for causally continuous spacetimes

"Counterexample" by Kuang & Liang (JMP'88, PRD'92):

Causally continuous spacetime $M = \mathbb{L}^3 \backslash J(p,q) \;\; p = (-1, 0.0), q = (1,0,0):$

$\{q_i\} \not\to q$ in $\underline{\overline{M}}$ for Budic & Sachs construction

$\{r_i\} \not\to r$ in $\overline{M}$ for Rácz topology



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4.4- Szabados reformulation (CQG '88,'89)

Refine Budic & Sachs identification in $M^\sharp$:

$$P \sim_S F \iff \begin{cases} F \quad \text{is included and is maximal in} \quad \uparrow P \\ P \quad \text{is included and is maximal in} \quad \downarrow F \end{cases}$$

(plus a second identification).

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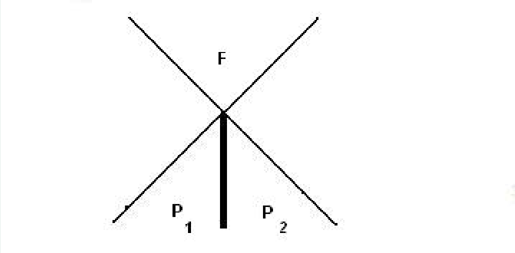
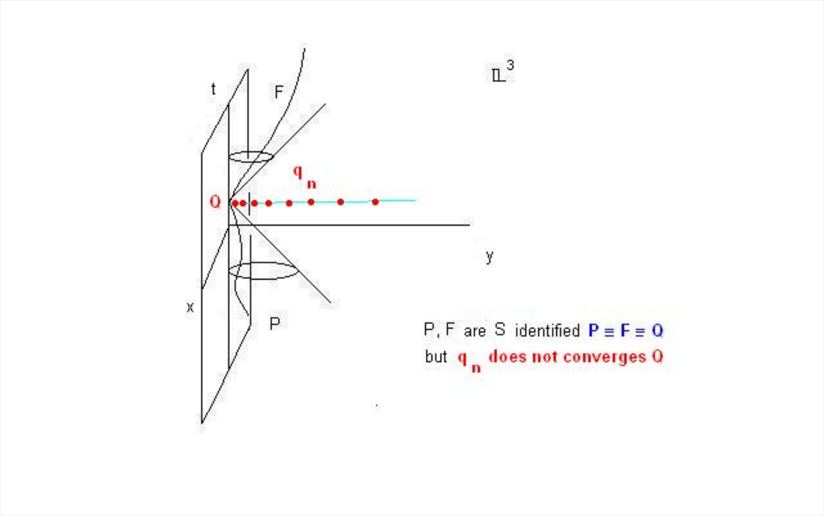


$\downarrow F = P_1 \cup P_2$; thus $P_i \not\sim_{BS} F$ for $i = 1, 2$ but $P_1 \sim_S F \sim_S P_2$ (Szabados imposes that the three sets represent a single boundary point)

Nevertheless, the topology is again induced "à la GKP"
...and this wills not be good:

Kuang & Liang (PRD '92): Szabados construction does not recover well the topology for the completion of $\mathbb{R}^2 \times \mathbb{R}^+ \subset \mathbb{L}^3$ $(y > 0)$!!



Now, $P = I^-[\rho] \sim_S F = I^+[\gamma]$ yield the same boundary point Q... but $\{q_n\} \not\to Q$!!

4.5.- Kuang and Liang dramatic claim

We are inclined to believe that the whole project of constructing a singular boundary has to be given up (PRD'92).

5.-Reconstructing the boundary I: Harris' universal construction (CQG '00)

Consider only the (non-problematic) future part $M, \hat{M}, \hat{\partial} M$:

$(M, \ll)$ and $\hat{M}$ (with an extension of $\ll^c$ of the original $\ll$) are particular cases of *chronological spaces*

(a) *Category of chronological spaces:* **(NOTE THIS!)**

- $(X, \ll)$: X set, $\ll$ transitive and anti-reflexive such that:
  - No isolated points: each x satisfies $x \ll y$ or $y \ll x$ for some y
  - Existence of a *chronologically dense* numerable subset $\mathscr{D}$: $\forall x \ll y \, \exists d \in \mathscr{D} : x \ll d \ll y$.
- Natural extension of spacetime definitions. F. ex: IPs, future-timelike curves replaced by future *chronological chains*: $x_1 \ll \cdots \ll x_n \ll \ldots$
- $(X, \ll)$ (past-)regular: $I^-(x)$ is IP for all $x \in X$ (as in spacetimes).
- Natural functor in this category: maps preserving the chronology

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(b) *Chronological completion of* $(X, \ll)$:

Claim. Up to some technicalities regarding the definition of $\ll^c$ ("past determination") *GKP construction is the minimal (universal, categorically unique) way of "future-completing" a past-regular, past-distinguishing chronological set.*

In particular, this happens for strongly causal spacetimes, yielding a support for the "nice part" of GKP construction (preboundary points and extended causal relations to $M^\sharp$).

(c) *New type of topology:* **(NOTE THIS!)**

No GKP (Alexandrov) topology but a *limit operator* $\hat{L}$:

- Definition: for any sequence $\sigma = \{x_n\}_n \subset X$:

$$x \in \hat{L}(\sigma) \Leftrightarrow \begin{cases} y \ll x \Rightarrow y \ll x_n \\ I^-(x) \subsetneq P(\in \hat{M}) \Rightarrow \exists z \in P : z \not\ll x_n \end{cases}$$

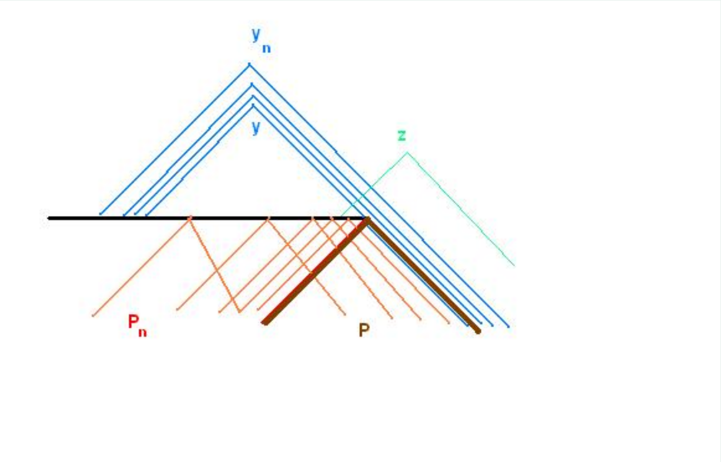
(for large n)

- Flores reformulation:

$$x \in \hat{L}(\sigma) \Leftrightarrow \begin{cases} I^-(x) \subset LI\{I^-(x_n)\} \\ I^-(x) \text{ is a maximal IP in } LS\{I^-(x_n)\} \end{cases}$$

- *$\hat{L}$ allows to define closed subsets and, thus, a topology on $\hat{M}$* –with a number of nice properties.

  But only categorical properties for particular subcategories (remarkably, when the boundary is spacelike)



In $\mathbb{L}^2 \setminus \{x \leq 0\}$ each point of the removed semi-axis yields a past boundary point.
Now: (a) $I^-(y)$ and P lie in $\hat{L}(\{y_n\}_n)$ (i.e. $y \in M$ and $P \in \hat{\partial} M$ are not T_2 related),
(b) if $I^-(z)$ is open in $\overline{M}$, the sequence $\{P\}_n \subset \hat{M}$ would not converge to P.

(d) *Extra bonus:*

Tools for computing the boundary (TIPs, TIFs) for product $I(\subseteq \mathbb{R}) \times S$ spacetimes (and then static standard, Robertson-Walker...) –refined in Harris & Flores CQG'07.

Reconstructing the boundary II: Marolf-Ross' "recipe" (CQG'03)

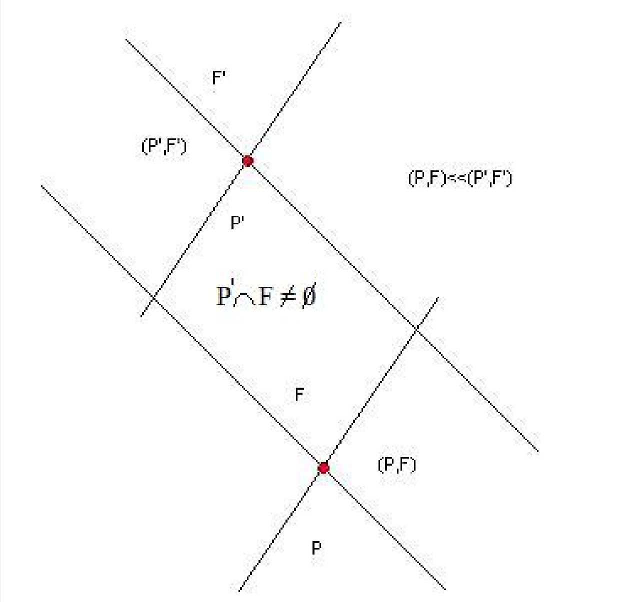
New viewpoint on:

- Completion $\bar{M}$: set of pairs (P,F), **(NOTE THIS!)** $P \in \hat{M}, F \in \check{F}$ s.t.: $P \sim_S F$

  (Each $p \in M$ identified to $(I^-(p), I^+(p))$ so $M \subset \bar{M}$).

- Chronological relation $\bar{\ll}$:

  $(P,F)\bar{\ll}(P',F') \Longleftrightarrow F \cap P' \neq \emptyset$ (introduced by Szabados, but now free of inconsistencies, **(NOTE THIS!)**)



Remarkably, **no spurious relations introduced**: $\bar{\ll} = \ll$ on M.

(No Harris problems on past-determination; consistency for Szabados)

- Topology: *two* reasonable (and technical) options –the less coarse preferred.

  Closed subsets: $L^{+}(S) := Cl_{FB}(S \cup L^{+}_{IF}(S))$ with $\in L^{+}_{IF}(S) \Leftrightarrow P \subset I^{+}[S]$

  (plus analogous -).

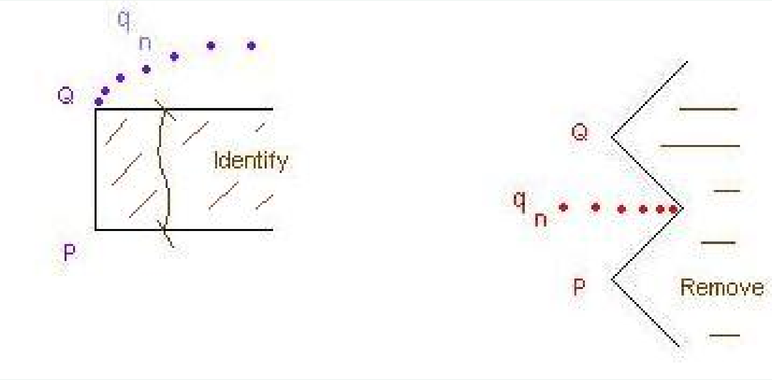
  Essentially, $L^{+}(S)$ is a limit operator which means

$$\lim P_n = Q \leftrightarrow \begin{cases} Q \subset LI(P_n) \\ M \backslash \bar{Q} \subset LI(M \backslash \bar{P}_n) \end{cases}$$

  then, given $\{x_n\} \subset M$:

$$(I^{-}(x_n), I^{+}(x_n)) \longrightarrow (P, F) \Longleftrightarrow \begin{cases} P = \lim_n I^{-}(x_n) \\ F = \lim_n I^{+}(x_n) \end{cases}$$

Some nice properties –even though no T_1, this situation is “controlled”
BUT: no definitive argument for the choice of the topology



Less coarse topology: $\{q_n\} \not\to Q, P$
Coarsest topology: $\{q_n\} \to Q, P$

Reconstructing the boundary III: Flores revision (CMP'07)

- **Ingredients**

1. Harris

   - Framework of chronological sets $(X, \ll)$
   - Decomposition operator: $\mathrm{dec}(P) = \cup P_\alpha, P_\alpha$ indecomposable [$\mathrm{dec}(F) = \cup F_\alpha, F_\alpha$ indecomposable].
   - Limit operator for sequences $\hat{L}$ (non-necessarily unique limit) [and analogous $\check{L}$]

2. Szabados (Budic & Sachs) relation (but **no a priori**):

$$P \sim_S F \Longleftrightarrow \begin{cases} F & \text{is included and is maximal in} \quad \uparrow P \\ P & \text{is included and is maximal in} \quad \downarrow F \end{cases}$$

3. Marolf & Ross:

- Pairs past/future: $(P,F) \subset X_p \times X_f$ (X_p: past sets; X_f :future sets)
- Chronological relation $(P,F)\overline{\ll}(P',F') \Longleftrightarrow F \cap P' \neq \emptyset$

- **Approach**

  1. New viewpoint:

     Consider any chronological set:

     - For any chain $\eta = \{x_n\}$ [future $x_n \ll x_{n+1}$], notion of endpoint (P, F) (previous to any consideration of limit):
       $P = I^-(\eta)$, dec$F = \check{L}(\eta)$
     - For completions $\bar{X}$ of X:

       Defn: $X \subset \bar{X} \subset X_p \times X_f$ which is *complete*: each chain has a endpoint

       (Th: completion is a *chronological set* and *complete*).

     One has many completions, each one $(\bar{X}, \bar{\ll})$ a (complete) chronological space

2. Topology for any chronological space (**NOTE THIS!**):

define a natural *chronological topology* (extension of Harris' idea *using also the future part*):

- L "limit" operator on sequences $\sigma = \{x_n\} \subset Y$:

$$x \in L(\sigma) \Longleftrightarrow \begin{cases} \mathrm{dec}I^-(x) \subset \hat{L}(\sigma) \\ \mathrm{dec}I^+(x) \subset \check{L}(\sigma) \end{cases}$$

- this defines the closed sets.

Essentially, the topology is the natural choice so that

- The endpoint of a chain η is a limit point of the sequence
- (For strongly causal spacetimes:) the topology of X agrees the restriction of $\overline{X}$.

- **Results**

  1. For any completion $(\bar{X}, \overline{\ll})$: first natural properties

     (a) The completion is a (weakly) distinguishing chronological set

     (b) $i : X \hookrightarrow \bar{X}$ is a chronological map, with image chronologically dense

     (c) no new (spurious) relation introduced by $\overline{\ll}$ in $i(X)$

     (d) The completion is a complete chronological set!

2. Notion of *chronological* completions **(NOTE THIS!)** (the *minimal* ones –essentially, removing a point of $\overline{X}$ destroys completeness):

   They exist! and, for strongly causal spacetimes:

   (a) the boundary is closed

   (b) the limits and endpoints of chains agree

   (c) $\overline{X}$ is T_1 [not fulfilled by Marolf-Ross topologies]

   (d) Non-Hausdorff related points lie in $\partial\overline{X}$ [not fulfilled by Harris]

3. For *strongly causal* spacetimes, **S-relation emerges**, i.e.,it is not imposed (**NOTE THIS!)**

   Theorem Let M be a strongly causal spacetime. A completion $\overline{M} \subset M_p \times M_f$ is minimal if and only if its boundary ∂M satisfies the following properties:

   (i) Every TIP and TIF in M is the component of some pair in ∂M.

   (ii) If $(P,F) \in \partial M$ and $P \neq \emptyset \neq F$, then P is a TIP, F is a TIF, and $P \sim_S F$.

   (iii) If $(P,F) \in \partial M$ and $F = \emptyset$ (resp. $P = \emptyset$) then P (resp. F) is a (non-empty) terminal indecomposable set and is not S-related to any other set.

   (iv) If $(P,F_1),(P,F_2) \in \partial M$ and $F_1 \neq F_2$ (resp., $(P_1,F),(P_2,F) \in \partial M$ and $P_1 \neq P_2$) then F_i (resp. P_i), $i = 1,2$, does not appear in another pair of ∂M.

4. About Marolf & Ross completion (MR): It is not minimal, even if there exist a unique minimal completion

6.- Our final choice (Flores, Herrera, MS, in progress)

Admissible notions for the completions:

- Admissible as point sets: completions considered as pairs (IP, IF) (as in Marolf-Ross), under under Szabados relation (as justified by Flores).

  $\rightsquigarrow$ Marolf-Ross completion, which includes all these pairs is singled out as the maximum one (in fact, the unique univocally determined as the minimal ones are not unique).

- Extended chronological relations: those $\tilde{\ll}$ which includes:

  the ones in the spacetime: $x \ll y \Rightarrow x \tilde{\ll} y$ and

  the ones which define the boundary: $x \in P \Rightarrow x \tilde{\ll} (P,F)$, $y \in F \Rightarrow (P,F) \tilde{\ll} y$

  $\rightsquigarrow$ Marolf-Ross/Szabados chronology $(P,F) \overline{\ll} (P',F') \Leftrightarrow F \cap P' \neq \emptyset$ is singled out as (i) the minimum extension, and also as (ii) the unique which is **admissible** (in the sense does not introduce spurious relations in M) for any topology with M dense and open chronological futures and pasts $I^{\pm}(P,F)$.

- Admissible topologies: the coarsest such that $I^{\pm}(P,F)$ are open (this is also a way to understand Alexandrov topology!).

  $\rightsquigarrow$ All of them coincide around points with $P \neq \emptyset \neq F$ and Flores/Harris topology based on $\hat{L}, \check{L}$ is singled out as the unique T_1 such that the compatibility properties under limit sets also hold when P or F are empty.

Moreover: additional new properties of good behavior of the topology (ANII, determination of local compactness etc.)

7.- Coming back to the conformal boundary

- General notion. *Conformal envelopment* $i : M \hookrightarrow M_0$ in a (“*aphysical*”) strongly causal spacetime M_0. *Conformal completion* of M (w.r.t. i) is the closure $\overline{M}_i := \overline{i(M)} \subset M_0$, and the *conformal boundary* the complement $\partial_i M := \overline{i(M)} \setminus i(M)$.

- Relevant part. Focus in the *accessible boundary* $\partial_i^* M$ (and accessible completion M_i^*): points reachable as endpoints of timelike curves in M.

  1. If $\overline{M}_i^*$ is an embedded manifold with C^1 boundary: all the points accessible $\partial_i^* M = \partial_i M$.
  2. Do not care on the existence of points such as spacelike infinity i^0.

- Chronological completeness: Any inextensible (future or past) timelike curve in M have an endpoint in the conformal boundary.

  1. Fulfilled if $\overline{M}_i^*$ is compact.
  2. Fulfilled by the (non-compact) standard conformal completion $\mathbb{L}^n \hookrightarrow \mathbb{R} \times \mathbb{S}^{n-1}$ *with* i^0 *removed*.

- **Proposition.** If $\partial_i M^*$ is chr. complete, well defined natural projections

$$\hat{\pi} : \hat{\partial} M \rightarrow \partial^* M, \quad P = I^-(\gamma) \mapsto \lim \gamma$$

$$\check{\pi} : \check{\partial} M \rightarrow \partial^* M, \quad F = I^+(\gamma) \mapsto \lim \gamma$$

- Causal and (accessible) conformal completions identifiable $\overline{M} \equiv \overline{M}_i^*$ if:

$$\pi : \partial M \rightarrow \partial_i^* M, \qquad \pi((P,F)) = \begin{cases} \hat{\pi}(P) & \text{if} \quad P \neq \emptyset \\ \check{\pi}(F) & \text{if} \quad F \neq \emptyset \end{cases}$$

satisfies:

1. It is well-defined, i.e.: $(P,F) \in \partial M$ and $P \neq \emptyset \neq F \quad \Rightarrow \hat{\pi}(P) = \check{\pi}(F)$
2. π is bijective
3. Its extension to the completions $\overline{\pi} : \overline{M} \rightarrow \overline{M}_i^*$ is both, a homemorphism and a chronological isomorphism.

- Sufficient conditions to ensure $\overline{M} \equiv \overline{M}_i$

  1. Completion $\overline{M}_i$ is a closed (perhaps non-compact) embedded manifold with C^1 boundary
  2. Either global hyperbolicity or some technical condition of accessibility –type: $TIPs$ and $TIFs$ are generated by timelike curves in M *extensible as timelike* at the boundary point $z \in \partial_i M$ (under refinement).

     [Difficulty: a smooth timelike curve may be continuously but non-smoothly extendible to the boundary and then must be understood in the aphysical spacetime as a continuous causal curve. It is not enough for these curves to be almost everywhere smooth and future or past timelike but also to be H^1. The proximity of the boundary may interfere with its past/future.]

in particular, the typical null conformal boundary of asymptotically flat spacetimes agrees the causal one (as well as the boundaries in typical exact solutions -Schwarzschild, de Sitter, Robertson-Walker...)