Future asymptotics of vacuum Bianchi type VI_0 solutions

Joint work with J. Mark Heinzle

Mathematical Relativity in Lisbon, June 18, 2009

Particle horizons

> ...if the 3°K background radiation were last scattered at a redshift $z = 7$, then the radiation coming to us from two directions in the sky separated by more than about 30° was last scattered by regions of plasma whose prior histories had no causal relationship. [...] Robertson-Walker models therefore give no insight into why the observed microwave radiation from widely different angles in the sky has very precisely ($\lesssim 0.2\%$) the same temperature. *Charles Misner*

Bianchi IX solutions a solution? Compare the flat Kasner metric:

$$-dt^2 + t^2 dx^2 + dy^2 + dz^2 \tag{1}$$

Localization

Let (M, g) be a Lorentz manifold, where

$$M = (t_-, t_+) \times \Sigma.$$

Initial singularity: $t = t_-$.

Let γ be an inextendible causal curve and

$$\mathcal{A}_t = J^+(\gamma) \cap (\{t\} \times \Sigma).$$

Localization:

$$\lim_{t \to t_-} \operatorname{diam}(\mathcal{A}_t) = 0.$$

Importance

Singularity theorems $\Rightarrow$ generic singularities.

Curvature blow up, strong cosmic censorship? PDE perspective necessary!?

Causal structure important when taking the PDE perspective.

BKL and related conjectures: localization assumed.

Bianchi class A vacuum spacetimes

Metric:

$$g = -dt^2 + \sum_{i=1}^{3} a_i^2(t)\xi^i \otimes \xi^i$$

on $I \times G$, where

- G is a 3–dimensional unimodular Lie group,
- I is an interval,
- the ξ^i are the duals of a basis e_i of the Lie algebra $\mathfrak{g}$,
- the a_i are smooth positive functions on I.

Variables of Ellis and MacCallum

Define the structure constants γ_{ij}^k by

$$[e_i, e_j] = \gamma_{ij}^k e_k.$$

Then G is unimodular if and only if $\gamma_{ik}^k = 0$. In that case,

$$\gamma_{ij}^k = \epsilon_{ijl} \nu^{lk},$$

where the matrix ν can be assumed to be diagonal:

$$\nu = \text{diag}(\nu_1, \nu_2, \nu_3).$$

Variables of Ellis and MacCallum, continued

The Bianchi class A types can be characterized by the ν_i's:

Table 1: Bianchi class A.

Type	ν_1	ν_2	ν_3
I	0	0	0
II	+	0	0
VI_0	0	+	−
VII_0	0	+	+
VIII	−	+	+
IX	+	+	+

Variables of Ellis and MacCallum, continued

Changing basis to an orthonormal one,

$$\bar{e}_i = \frac{1}{a_i} e_i \quad \text{(no summation)},$$

then

$$\nu_i \longrightarrow n_i(t).$$

Variables: n_i and θ_i (the diagonal components of the second fundamental form).

Variables of Wainwright and Hsu

Let

$$\theta = \sum_{i=1}^{3} \theta_i, \quad \sigma_i = \theta_i - \frac{1}{3}\theta.$$

The Wainwright–Hsu variables are given by

$$N_i = \frac{n_i}{\theta}, \quad \Sigma_i = \frac{\sigma_i}{\theta}.$$

Furthermore, the time coordinate is changed according to

$$\frac{d}{d\tau} = \frac{3}{\theta}\frac{d}{dt}.$$

Equations of Wainwright and Hsu

$$
\begin{aligned}
\Sigma_i' &= -2(1-\Sigma^2)\Sigma_i - {}^3S_i, \\
N_i' &= 2(\Sigma^2 + \Sigma_i)\, N_i
\end{aligned}
$$

(no summation), where

$$
\begin{aligned}
\Sigma^2 &= \tfrac{1}{6}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2), \\
{}^3S_i &= \tfrac{1}{3}\left[N_i(2N_i - N_j - N_k) - (N_j - N_k)^2\right],
\end{aligned}
$$

where (ijk) is an even permutation of (123).

Constraints: $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$ and

$$
\Sigma^2 + \tfrac{1}{12}\Big[N_1^2 + N_2^2 + N_3^2 - 2N_1N_2 - 2N_2N_3 - 2N_3N_1\Big] = 1\,.
$$

Localization, revisited

Types of interest: VIII and IX.

If the manifold is given by

$$M = (t_-, t_+) \times G$$

and t_- corresponds to the singularity, there is localization if and only if

$$\sum_{i=1}^{3} \int_{t_-}^{t_0} \frac{1}{a_i(t)} dt < \infty$$

In the case of Bianchi type VIII and IX, this condition can be reformulated to

$$\int_{t_-}^{t_0} (\sqrt{|n_1 n_2|} + \sqrt{|n_2 n_3|} + \sqrt{|n_3 n_1|}) dt < \infty,$$

Localization, revisited

In terms of Wainwright–Hsu variables and time, the condition reads

$$\boxed{\int_{-\infty}^{\tau_0}(\sqrt{|N_1N_2|}+\sqrt{|N_2N_3|}+\sqrt{|N_3N_1|})d\tau<\infty.}$$

Local rotational symmetry leads to divergence (there are Cauchy horizons).

Generic Bianchi IX solutions

In the case of generic Bianchi IX solutions, the integrand, say I, converges to zero (but not monotonically).

Behaviour: None of the N_i converge to zero, but the products do.

Problem: For each product, there is a region of the phase space where the behaviour of that particular product is essentially oscillatory. These regions are "close" to the *flat Kasner solutions*, cf. Misner's argument.

Generic Bianchi VIII solutions

In this case it's not even known whether the integrand converges to zero or not.

Bianchi IX – almost monotone convergence.

Bianchi VIII – almost monotone convergence cannot hold. Proof based on a study of the future asymptotics of Bianchi type VI_0.

Expected behaviour

The constraint can be written

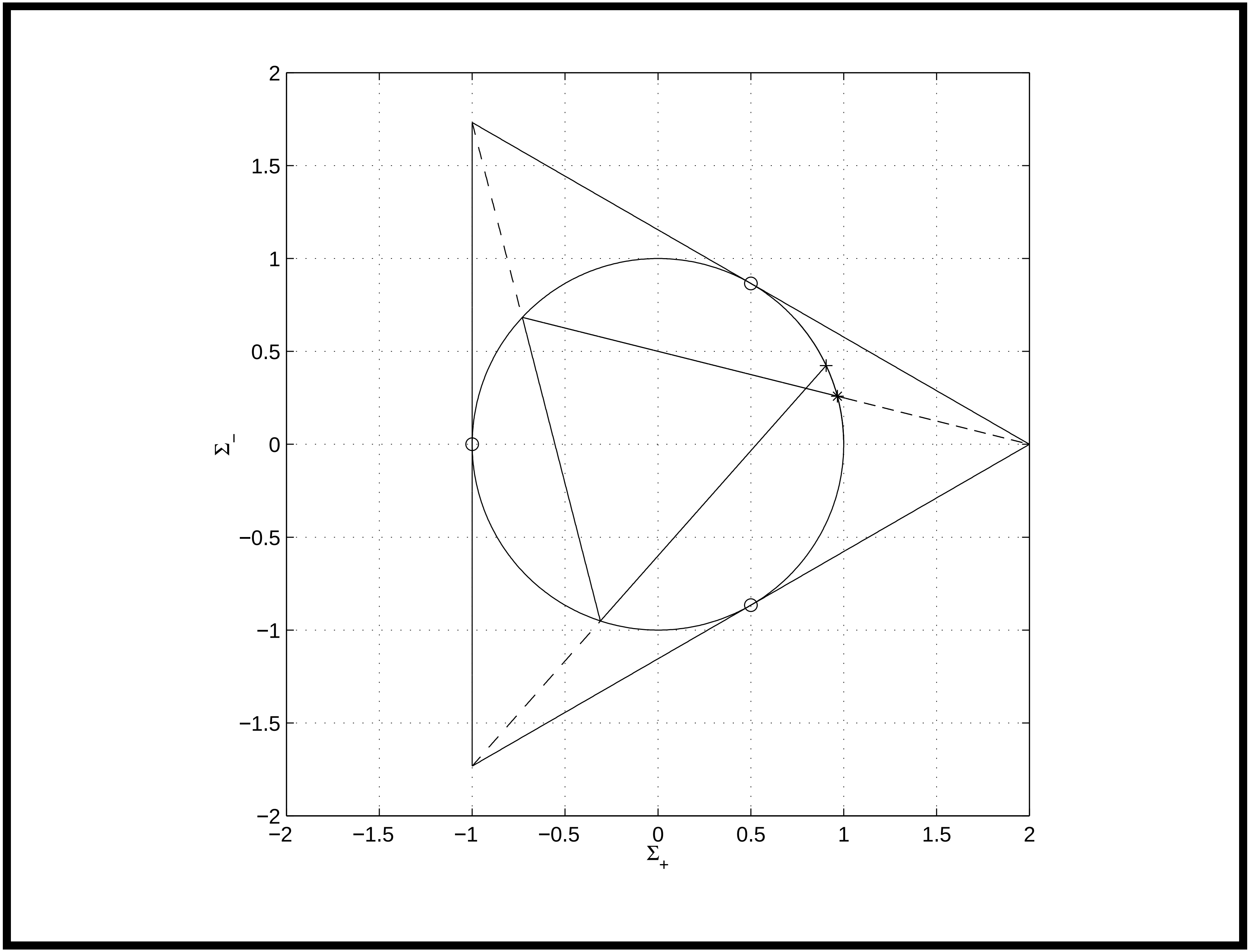
$$\overline{\Sigma}_+^2 + \overline{\Sigma}_-^2 + \frac{3}{4}\Big[N_1^2 + N_2^2 + N_3^2 - 2N_1N_2 - 2N_2N_3 - 2N_3N_1\Big] = 1. \quad (2)$$

Bianchi type I $\leftrightarrow$ Kasner circle.

Bianchi type II $\leftrightarrow$ six half ellipsoids.

Attractor:

$\mathcal{A} = \{(\overline{\Sigma}_+, \overline{\Sigma}_-, N_1, N_2, N_3) : (2) \text{ holds}, \ |N_1N_2| + |N_2N_3| + |N_3N_1| = 0\}.$



2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
Σ₊
Σ₋

Relevance of the future asymptotics of Bianchi type VI_0

Bianchi type VI_0 solutions converge to the attractor to the future.

Bianchi type VI_0 solutions are on the boundary of Bianchi VIII.

Thus, perturbing the corresponding initial data initial data leads to Bianchi type VIII solutions that start out close to the attractor and go away from it to the past.

Thus there is no almost monotone convergence to the attractor in the case of Bianchi type VIII.

Conclusion, introduction

If one wants to prove that

$$\int_{-\infty}^{\tau_0} (\sqrt{|N_1 N_2|} + \sqrt{|N_2 N_3|} + \sqrt{|N_3 N_1|}) d\tau < \infty.$$

holds for Bianchi class A spacetimes, the most problematic type is Bianchi type VIII, and the most problematic region is where the solution is well approximated by Bianchi type VI_0 solutions.

Bianchi type VI_0

Bianchi type VI_0 converge to a special point on the Kasner circle to the future.

If

$$\overline{\Sigma}_- = \frac{\sqrt{3}}{2}(\Sigma_2 - \Sigma_3)$$

and $N_2 > 0$, $N_3 < 0$, then

$$Z_{-1} = \frac{\frac{4}{3}\overline{\Sigma}_-^2 + (N_2 + N_3)^2}{-N_2 N_3}.$$

Monotonically decreasing, measures distance to Bianchi type II behaviour. **Main result:**

$$\lim_{\tau \to \infty} Z_{-1}(\tau) = 0.$$

Technical details

Introducing the variables

$$\Sigma_+ = \frac{\Sigma_1 + \Sigma_2}{2} = -\frac{\Sigma_3}{2}, \quad \Sigma_- = -\frac{\Sigma_1 - \Sigma_2}{2\sqrt{3}},$$

and (assuming $N_1 > 0$ and $N_2 < 0$)

$$N_+ = \frac{N_1 + N_2}{2\sqrt{3}}, \quad N_- = \frac{N_1 - N_2}{2\sqrt{3}},$$

the constraint reads

$$\Sigma_+^2 + \Sigma_-^2 + N_-^2 = 1.$$

Since $N_- > 0$, we obtain

$$N_- = \sqrt{1 - \Sigma_+^2 - \Sigma_-^2}.$$

Equations

Given the above observations, we can rewrite the equations as an unconstrained system

$$\Sigma_+' = -2\big(1 + \Sigma_+\big)\big(1 - \Sigma_+^2 - \Sigma_-^2\big) \tag{3a}$$

$$\Sigma_-' = -2\left[\Sigma_-(1 - \Sigma_+^2 - \Sigma_-^2) - \sqrt{3}\, N_+\sqrt{1 - \Sigma_+^2 - \Sigma_-^2}\right] \tag{3b}$$

$$N_+' = 2\left[N_+\Sigma_+ + N_+(\Sigma_+^2 + \Sigma_-^2) - \sqrt{3}\,\Sigma_-\sqrt{1 - \Sigma_+^2 - \Sigma_-^2}\right] \tag{3c}$$

on

$$\boldsymbol{B}_{\mathrm{VI}_0} = \Big\{\big(\Sigma_+, \Sigma_-, N_+\big)\,\big|\,\Sigma_+^2 + \Sigma_-^2 + N_+^2 < 1\Big\}\,. \tag{4}$$

The function ζ

The function

$$\zeta = -1 - \frac{(\Sigma_1 - \Sigma_2)^2 + (N_1 - N_2)^2}{4N_1 N_2} = \frac{N_+^2 + \Sigma_-^2}{1 - \Sigma_+^2 - \Sigma_-^2 - N_+^2}$$

is of central importance. It is either always zero or always non–zero.

$\zeta = 0$ corresponds to an orbit called Γ.

ζ is a constant multiple of the function Z_{-1} introduced earlier.

Main result

Consider a solution in $\boldsymbol{B}_{\mathrm{VI}_0} \setminus \Gamma$. Then there is an r_0 and a ζ_0 such that

$$\Sigma_+ = -1 + \tfrac{1}{4}\tau^{-1}\left[1 + O(\tau^{-1/2})\right], \tag{5a}$$

$$\Sigma_- = r_0 \tau^{-3/4}\cos\vartheta(\tau)\left[1 + O(\tau^{-1/2})\right], \tag{5b}$$

$$N_+ = -r_0 \tau^{-3/4}\sin\vartheta(\tau)\left[1 + O(\tau^{-1/2})\right], \tag{5c}$$

$$N_- = \frac{1}{\sqrt{2}}\tau^{-1/2}\left[1 + O(\tau^{-1/2})\right], \tag{5d}$$

$$\vartheta(\tau) = 2\sqrt{6}\,\tau^{1/2} + O(\log\tau), \tag{5e}$$

$$\zeta(\tau) = \zeta_0 \tau^{-1/2}\left[1 + O(\tau^{-1/2})\right]. \tag{5f}$$

Idea of the proof

It's convenient to change variables to $(\tilde{\Sigma}_+, \zeta, \vartheta)$, where

$$\tilde{\Sigma}_+ = 1 + \Sigma_+ \,, \tag{6a}$$

$$\Sigma_- = \frac{\sqrt{1-\Sigma_+^2}}{\sqrt{1+\zeta^{-1}}} \cos\vartheta \,, \tag{6b}$$

$$N_+ = -\frac{\sqrt{1-\Sigma_+^2}}{\sqrt{1+\zeta^{-1}}} \sin\vartheta \,, \tag{6c}$$

and time coordinate according to

$$\frac{d}{d\sigma} = \frac{1}{\tilde{\Sigma}_+} \frac{d}{d\tau}.$$

Note: $(\Sigma_+, \Sigma_-, N_+, N_-) \to (-1, 0, 0, 0)$.

Time coordinates

Lemma 1 *$\tau \to \infty$ corresponds to $\sigma \to \infty$.*

Proof. Since

$$\tilde{\Sigma}_+' = -2(1 - \Sigma_+^2 - \Sigma_-^2)\tilde{\Sigma}_+,$$

and $\tilde{\Sigma}_+ \to 0$ as $\tau \to \infty$, we have $1 - \Sigma_+^2 - \Sigma_-^2 \notin L^1([\tau_0, \infty))$. Since

$$0 < 1 - \Sigma_+^2 - \Sigma_-^2 \leq 1 - \Sigma_+^2 = (2 - \tilde{\Sigma}_+)\tilde{\Sigma}_+ \leq 2\tilde{\Sigma}_+,$$

we infer that $\tilde{\Sigma}_+ \notin L^1([\tau_0, \infty))$. Since $\tilde{\Sigma}_+$ is positive and

$$\sigma(\tau) = \sigma(\tau_0) + \int_{\tau_0}^{\tau} \tilde{\Sigma}_+(s)\, ds,$$

the lemma follows. □

Equations

Using $(\tilde{\Sigma}_+, \zeta, \vartheta)$ as variables and σ as 'time', the dynamical system becomes

$$\frac{d\tilde{\Sigma}_+}{d\sigma} = -2\tilde{\Sigma}_+(2-\tilde{\Sigma}_+)\left[1+\zeta\sin^2\vartheta\right](1+\zeta)^{-1}\,, \tag{7a}$$

$$\frac{d\vartheta}{d\sigma} = 2\sqrt{3}\,\tilde{\Sigma}_+^{-1/2}\,(2-\tilde{\Sigma}_+)^{1/2}(1+\zeta\sin^2\vartheta)^{1/2}(1+\zeta)^{-1/2}+\sin 2\vartheta\,, \tag{7b}$$

$$\frac{d\zeta}{d\sigma} = -4\zeta\cos^2\vartheta\,. \tag{7c}$$

Note that

$$\zeta(\sigma) = \zeta(\sigma_0)\exp\left[-\int_{\sigma_0}^{\sigma} 4\cos^2\big(\vartheta(s)\big)ds\right].$$

Oscillatory behaviour

It's of central importance to estimate

$$g(\sigma) := \int_{\sigma_0}^{\sigma} \cos^2\big(\vartheta(s)\big)\, ds\,.$$

Lemma 2 *Consider a solution* $\big(\tilde{\Sigma}_+(\sigma), \vartheta(\sigma), \zeta(\sigma)\big)$ *of the dynamical system* (7) *in* $\boldsymbol{B}_{\mathrm{VI}_0}\backslash\Gamma$*. Then there exists a* $\beta > 0$ *such that*

$$\frac{\sigma}{2} - \beta \;\leq\; g(\sigma) \;\leq\; \frac{\sigma}{2} + \beta \tag{8}$$

for all $\sigma \geq \sigma_0$.