Dynamics of linearized cosmological perturbations

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Introduction

Stars are clumped to form galaxies and larger structures

Can these structures be explained by a theoretical model?

The dominant force on these scales is gravity

The appropriate description is given by the Einstein equations

Matter is very often modelled by the Euler equations

Well-known homogeneous solutions of the Einstein-Euler system

FLRW (Friedmann-Lemaitre-Robertson-Walker) models

Structures to be modelled are by definition inhomogeneous

Thus it is necessary to go beyond the FLRW models

Usual procedure is to linearize about the homogeneous models

Common practise in applied mathematics

Problem: the system is diffeomorphism invariant

This leads to gauge invariance on the linearized level

Which perturbations are physically meaningful?

The basic equation

Practical approach here:

Take an equation from the astrophysics literature and analyse its properties. Do not worry about the derivation.

Real-valued function Φ defined on $(0, \infty) \times T^3$.

The equation is $\qquad (\Lambda = 0!)$

$$\Phi'' + \frac{6(1+w)}{1+3w}\frac{1}{\eta}\Phi' = w\Delta\Phi \tag{1}$$

w is a parameter belonging to $[0, 1]$. Equation of state $p = w\epsilon$.

The unknown Φ is an analogue of the Newtonian potential

Cf. Euler-Poisson-Darboux equation

We were not able to profit from that fact

A related equation is the polarized Gowdy equation

$$P_{tt} + t^{-1} P_t = P_{xx}$$

Its solutions define a class of solutions of the Einstein equations

They have been studied in detail analytically

Asymptotics for $\eta \to 0$

Any smooth solution has an asymptotic expansion

$\Phi(\eta, x) \sim \sum_k (\Phi_{k,0}(x) + \Phi_{k,1}(x) \log \eta)\eta^k$

k belongs to an increasing sequence of real numbers tending to infinity, $k \geq -2\nu$ where $\nu = (5 + 3w)/(1 + 3w)$

All coefficients are determined by $\Phi_{-2\nu,0}$ and $\Phi_{0,0}$

For prescribed $\Phi_{-2\nu,0}$ and $\Phi_{0,0}$ there is a unique solution

Solutions parametrized by asymptotic data

Methods of proof:

1. Energy estimates (Isenberg/Moncrief, 1990, for Gowdy)

$E_1(\eta) = \frac{1}{2}\int_{T^3} |\Phi'|^2 + w|\nabla\Phi|^2$

$\frac{d}{d\eta}(\eta^{2(2\nu+1)}E_1) \geq 0$ whence $\eta^{2(2\nu+1)}E_1$ bounded in past

Commute with spatial derivatives and apply Sobolev

The gives bounds for all derivatives

Putting this into the equation gives desired expansion

2. Fuchsian methods

Reduce equation to first order system

Derive Fuchsian system by subtracting finitely many terms

Consider first analytic case

Apply theorem of Kichenassamy/ADR on Fuchsian systems

Do more subtraction to get system which is symmetric hyperbolic

Approximate smooth data by analytic data

Show convergence of solutions using energy estimates

Asymptotics for $\eta \to \infty$

Solution can be split as

$$\Phi = \bar{\Phi} + \tilde{\Phi}$$

where $\bar{\Phi}$ depends only on t and $\tilde{\Phi}$ has mean zero

Any smooth solution has asymptotic expansions

$$\bar{\Phi}(\eta, x) = \Phi_{-2\nu,0}(x)\eta^{-2\nu} + \Phi_{0,0}(x)$$

$$\tilde{\Phi}(\eta, x) = \eta^{-\nu-\frac{1}{2}}(W(\eta, x) + o(1)), \qquad W'' = w\Delta W$$

The part $\tilde{\Phi}$ of the solution determined uniquely by W

The solution W of the wave equation can be prescribed

Solutions parametrized by asymptotic data

Asymptotics and uniqueness for polarized Gowdy proved by Jurke (2003)

Existence result for polarized Gowdy proved by Ringström (2005)

Methods of proof:

1. Energy estimates and compactness (Arzelà-Ascoli)

Define $\psi = \eta^{\nu+\frac{1}{2}}\Phi$.

Then $\psi'' = w\Delta\psi + \left(\nu^2 - \frac{1}{4}\right)\eta^{-2}\psi$. Let

$E_2(\eta) = \frac{1}{2}\int_{T^3} |\psi'|^2 + w|\nabla\psi|^2$. Then

$\tilde{E}_2' = 2\left(\nu^2 - \frac{1}{4}\right)\eta^{-2}\int_{T^3}\psi\psi'$

Using Gronwall's inequality E_2 is bounded in the future

Spatial derivatives can be handled similarly

Pointwise bounds can be obtained

Any sequence of time translates has a convergent subsequence

This gives a candidate for W

Prove existence theorem for a given W (energy estimates again)

Show that when applied to the candidate it produces the correct solution

2. Fourier decomposition in space

The last proof is done by doing a Fourier transform in space

This gives an ODE (mode equation)

$$\widehat{\psi}'' = -w|k|^2\widehat{\psi} + \left(\nu^2 - \tfrac{1}{4}\right)\eta^{-2}\widehat{\psi}$$

Asymptotics can be analysed directly

Alternatively can be transformed to Bessel's equation

Do not use mode equation for existence statements

More general equations of state

The basic equation is

$$\Phi'' + 3(1 + f'(\epsilon))\mathcal{H}\Phi' + (2\mathcal{H}' + (1 + 3f'(\epsilon))\mathcal{H}^2)\Phi - f'(\epsilon)\Delta\Phi = 0.$$

Equation of state is $p = f(\epsilon)$

The quantity $\mathcal{H}$ satisfies

$$\mathcal{H}' = -\frac{1}{2}\left(1 + \frac{3f(\epsilon)}{\epsilon}\right)\mathcal{H}^2$$

First determine asymptotics of $\mathcal{H}$ and then those of Φ

Define $E_3 = \frac{1}{2}\int (\Phi')^2 + \frac{df}{d\epsilon}|\nabla\Phi|^2 + \Lambda\mathcal{H}^2\Phi^2$

where Λ is a well-chosen constant.

Under some general assumptions on f a suitably rescaled version of E_3 is monotone.

Gives basic boundedness statements in limit $\eta \to 0$

More if equation of state is asymptotically linear for $\epsilon \to \infty$

$f(\epsilon) \sim w\epsilon + \sum_{j=1}^{\infty} f_j \epsilon^{a_j}$, $\epsilon \to \infty$

Ex. polytropic case $\epsilon = m + Knm^{\frac{n+1}{n}}$, $p = Km^{\frac{n+1}{n}}$

Expanding direction with more general equation of state

$f(\epsilon) \sim w\epsilon + \sum_{j=1}^{\infty} f_j \epsilon^{a_j}$, $\epsilon \to 0$, $\sigma = a_1 - 1$

Change of variable to eliminate term with Φ', $\frac{d\tau}{d\eta} = \sqrt{f'(\epsilon)}$.

For $w > 0$ the change of time variable is asymptotically linear

For $w = 0$ the relation is $\tau = C_1 \eta^{1-3\sigma} + C_2 + \ldots$

Basic boundedness statements for $\eta \to \infty$ using another energy

Asympotics for $\sigma < \frac{1}{3}$ similar to linear case, with 'warped' W

For $\sigma > \frac{1}{3}$ time coordinate τ tends to a finite limit for $\eta \to \infty$.

While waves propagate for ever in the case $\sigma > \frac{1}{3}$ they 'freeze' at late times for $\sigma > \frac{1}{3}$.

This is reminiscent of the behaviour of perturbations of de Sitter space.

Conclusions and outlook

1. For a linear equation of state it has been shown that solutions can be parametrized by asymptotic data in either time direction

2. Higher order expansions at late times?

3. For more general equations of state partial results have been obtained. Bifurcation at $\sigma = \frac{1}{3}$

4. In cosmology two-fluid models are common (system of equations)

5. Central questions in the applications apparently concern intermediate asymptotics. Mathematical formulation?