Marginally outer trapped surfaces: evolution and properties under spacetime symmetries

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Work done in collaboration with several authors:
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Outline

Introduction (I)

- Understanding the formation and evolution of black holes $\rightarrow$ Fundamental problem in General Relativity.
- The definition of black hole requires global information (existence of $\mathscr{I}^+$, strong asymptotic predictability, ...).
- Should be deduced (or contradicted) from solving the long time behaviour. Do black holes form in a gravitational collapse? (Weak Cosmic Censorship Conjecture).
- From an evolutionary point of view: Need of a local-in-time alternative (defined with finite or, better, instantaneous temporal information).
- Important for many reasons:
  - Numerical evolutions: Need of defining and evolving "black holes" at each time step (the global future spacetime is usually never constructed; even if so, fully unpractical to find the event horizon a posteriori...).
  - Tool for addressing cosmic censorship: A good quasi-local replacement should imply that black holes do form or else cosmic censorship fails...

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- Marginally outer trapped surfaces (MOTS): Null geodesics in the outer orthogonal direction are neither converging nor diverging $\rightarrow$ signal the presence of a strong gravitational field.

MOTS are widely believed to be good replacements of black holes.

Confirmation of this requires addressing many issues, like:

- How do MOTS evolve in time?
  - How and when do marginally outer trapped surfaces form?
  - Can they vanish once they have formed?
  - Do they evolve smoothly in a spacetime? Can they become singular before the spacetime does?
  - Is there an outermost such surface in any spacetime slice? Does it jump or evolve smoothly?
- What is the late time behaviour of the MOTS? Does an event horizon form?
- If a black hole really forms: do the MOTS always approach the event horizon? At which rate?

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- If MOTS are quasi-local replacements of black holes, then it should be true that any stationary spacetime containing a MOTS should be a stationary black hole, i.e.

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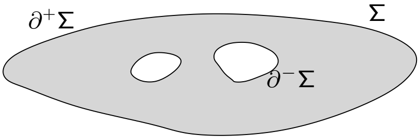

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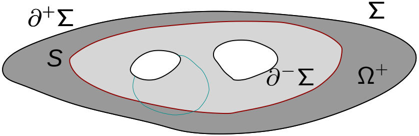

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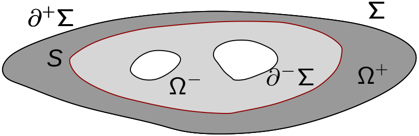

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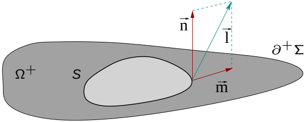
- The complementary $\Omega^- = \Sigma \setminus \overline{\Omega^+}$: interior domain.
- Choice: the outer normal $\vec{m}$ points towards Ω^+.
- Outer null vector: $\vec{l} = \vec{n} + \vec{m}$.



MOTS embedded in a spacelike hypersurface

- p : mean curvature of $S \subset \Sigma$, q: trace of K_{ij} along S.

Definition (Weakly outer trapped surface and MOTS)

A **weakly outer trapped surface** *$S \subset \Sigma$ is a bounding surface in (Σ, g_{ij}, K_{ij}) satisfying $\theta_l \equiv p + q \leq 0$.*
A **marginally outer trapped surface (MOTS)** *$S \subset \Sigma$ is a bounding surface satisfying $p + q = 0$.*

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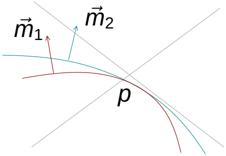
Lemma (maximum principle for MOTS)

Two MOTS which touch at one point p and such that the outer normals agree on p must coincide everywhere.



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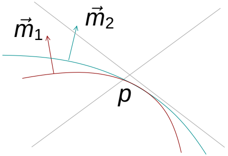
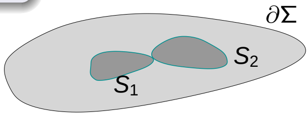
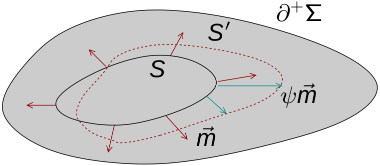


- The maximum principle allows MOTS to touch each other (if the normals do not match).



Stability operator

- For any function ψ on S, we can calculate the first order variation of θ_l along $\psi\vec{m}$.
- Defines an important operator.



Definition

The stability operator L_m along the direction $\vec{m}$ is defined as $\mathbf{L_m}\psi \equiv \delta_{\psi\vec{m}}\theta_l$.

- Explicitly:

$$L_m\psi = -\Delta_S\psi + 2s^A D_A\psi + \left(\frac{1}{2}R_S - Y - s^A s_A + D_A s^A\right)\psi,$$

  - $s_A \equiv K_{ij} m^i e^j_A$ normal-tangential component of the second fundamental form.
  - $Y \equiv G_{\mu\nu} l^\mu n^\nu + (1/2)\kappa_l^2$, where κ_l^2 is the distortion along $\vec{l}$.

Properties of the stability operator:

- Elliptic operator, not self-adjoint in general.
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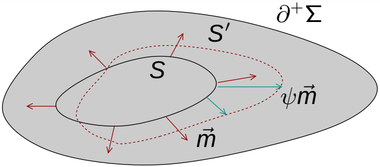

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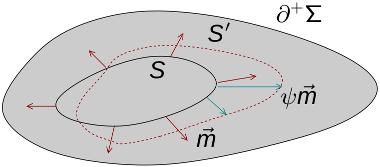

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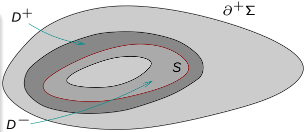

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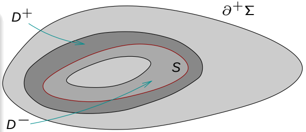

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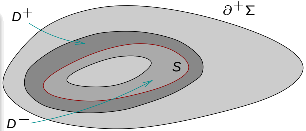

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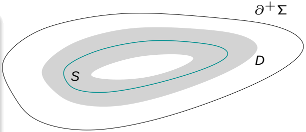
S strictly stable MOTS, $\exists$ a two-sided neighbourhood D of S such that no weakly outer trapped surface in D can enter the exterior part D^+.



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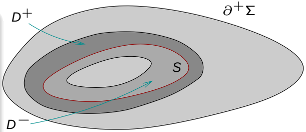

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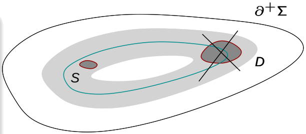

Existence of MOTS

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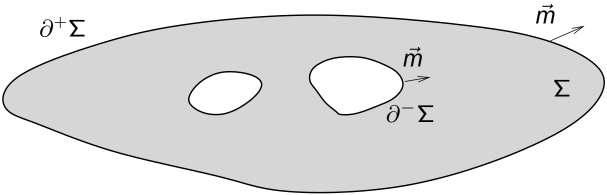

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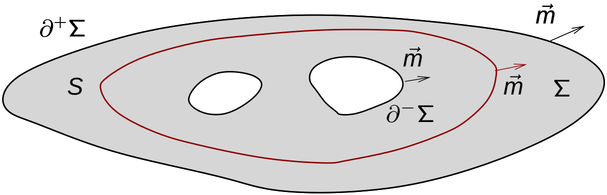

Theorem (Schoen 2004; Andersson & Metzger, 2007)

Let $(\Sigma, \gamma_{ij}, K_{ij})$ *be three-dimensional, compact and satisfying* $\theta_+(\partial^-\Sigma) < 0$, $\theta_+(\partial^+\Sigma) > 0$. *Then there exists a stable MOTS S in* Σ.

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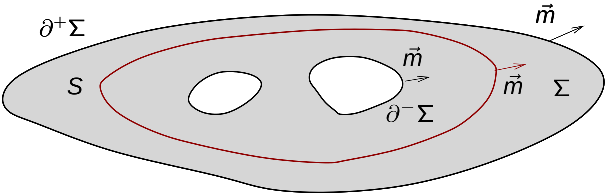

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- Proof uses the Jang equation (quasi-linear PDE): Suitable solutions blow up on the boundary and on the MOTS in-between.
- Assume from now on that $\partial^+\Sigma$ is outer untrapped and $\partial^-\Sigma$ is outer trapped (or empty).

The trapped region

- Recall: a bounding surface S is weakly outer trapped if $\theta_l \leq 0$.

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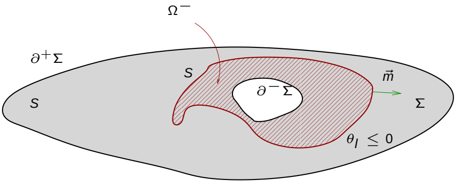

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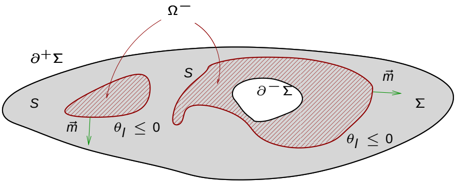

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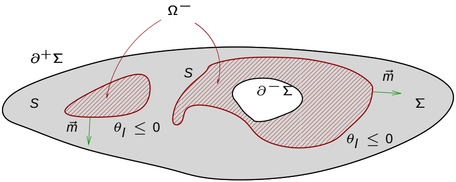

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Theorem (Andersson & Metzger 2007)

If the trapped region is not empty, the $\partial\mathcal{T}$ is a smooth stable MOTS.

- Consequence: An initial data set containing a bounding weakly outer trapped surface has a unique outermost MOTS.
- Outermost: any other bounding MOTS S' is enclosed by $\partial\mathcal{T}$.

The untrapped region (I)

- A bounding surface S is weakly outer untrapped if $\theta_l \geq 0$.
- Signal regions of “weak” gravitational field.
- Natural to ask whether there is an innermost one (i.e. “entering” most in the strong field regime)

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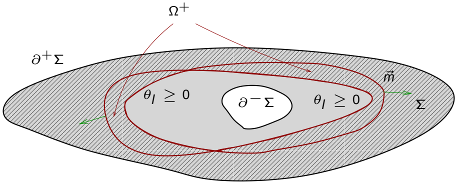

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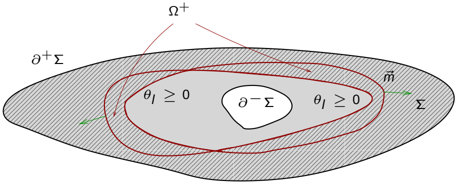

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Theorem (M, 2009)

If the untrapped region is not empty, the $\partial\mathcal{U}$ is a smooth stable MOTS.

- Consequence of Andersson & Metzger theorem, by playing with the time and space orientations.

The untrapped region (II)

Assume $\partial^- \Sigma \neq \emptyset$ and outer trapped. Then $\partial \mathcal{T} \neq \emptyset$ and $\partial \mathcal{U} \neq \emptyset$.

Lemma

$\partial \mathcal{U}$ is the innermost MOTS in Σ (i.e. any other bounding MOTS encloses $\partial \mathcal{U}$).

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- Consequence of the maximum principle:
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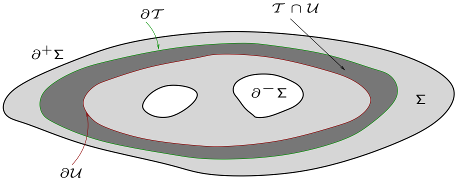

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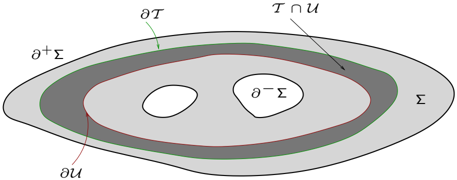

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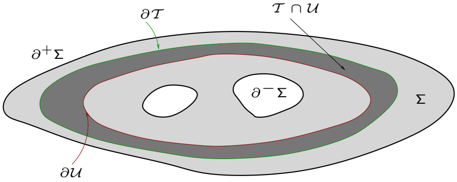

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New ideas required

Spacetime persistence of outermost MOTS

- Assume a spacetime (M, g^4) and a 3+1 decomposition $\{\Sigma_t\}$.
- A natural question: Do MOTS persist into the future?
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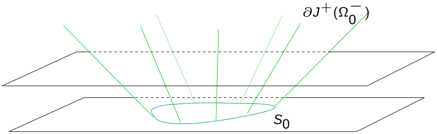

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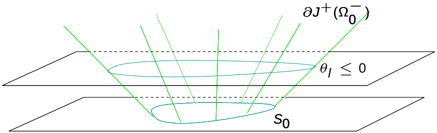

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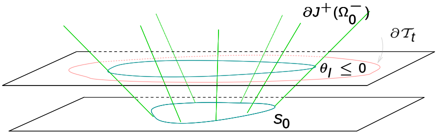
- $S_t \equiv \partial \mathcal{T}_t$ is a smooth MOTS.



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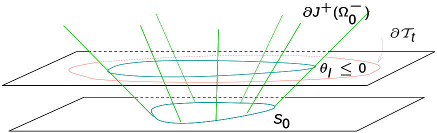


- $S_t \equiv \partial\mathcal{T}_t$ is a smooth MOTS.
- The proof says nothing about the smoothness of the collection $\{S_t\}$.

Is the evolution smooth? Are there jumps? How many?

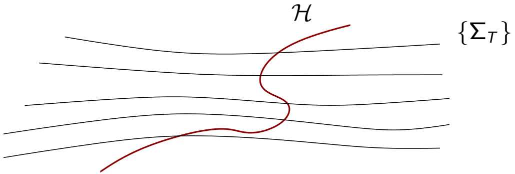
Local time evolution of MOTS

Definition (Marginally outer trapped tube (MOTT))

A marginally outer trapped tube is a smooth hypersurface $\mathcal{H}$ foliated by MOTS.

Definition

Given a 3+1 foliation $\{\Sigma_t\}$, a MOTT adapted to $\{\Sigma_t\}$ is a MOTT such that $S_t \equiv \mathcal{H} \cup \Sigma_t$ is a smooth MOTS (or empty) for all t.



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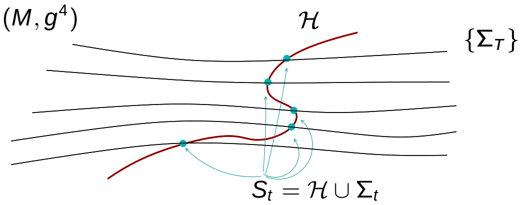

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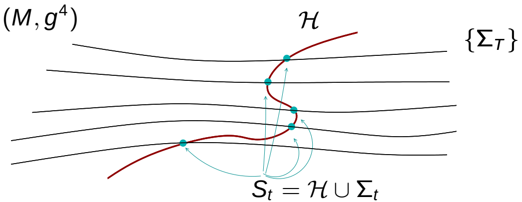


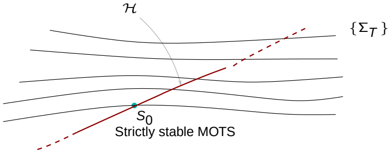
- In these definitions, the outer null normals must define a smooth field on $\mathcal{H}$ ("outer" must be consistently defined).
- Important issue: Given a MOTS in an initial slice, under which conditions does it propagate smoothly to the next slices to form an adapted MOTT?

Local time evolution of MOTS (II)

Theorem (Andersson, M, Simon, 2005)

Let $(\Sigma_0, g_{ij}, K_{ij})$ be a partial Cauchy surface and S_0 a strictly stable MOTS in Σ_0 (not necessarily bounding). Then, for any 3+1 foliation $\{\Sigma_t\}$ containing Σ_0 there exists an adapted MOTT containing S_0.

- Proof is by using the implicit function theorem.
- The existence is local in time (it exists for some time interval).
- If the foliation changes, the MOTT changes $\rightarrow$ MOTT passing though a given MOTS are highly non unique.



- For a fixed foliation, $\mathcal{H}$ is unique at least as long as $\{S_t\}$ is strictly stable.
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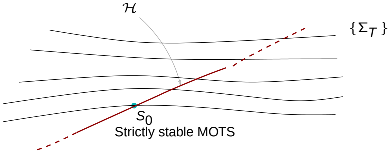

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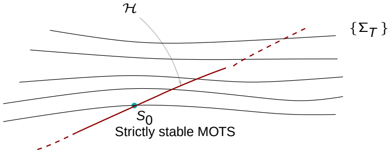

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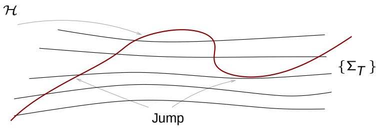

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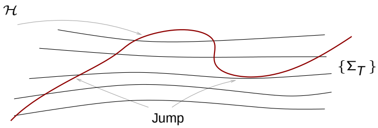

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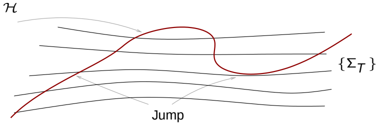

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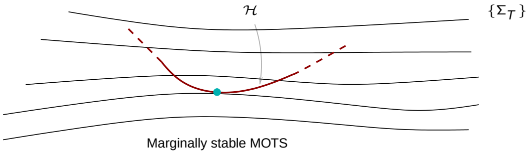

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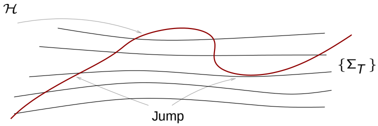

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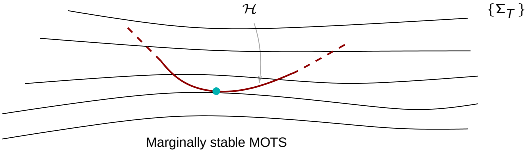

- It would be of interest to drop the genericity condition.

Outer distance bound

- Important property of $\partial \mathcal{T}$:

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Let (Σ, g_{ij}, K_{ij}) *be compact, with* $\partial\Sigma = \partial^+\Sigma \cup \partial^-\Sigma$ *satisfying* $\theta_l(\partial^+\Sigma) > 0$ *and* $\theta_l(\partial^-\Sigma) < 0$*. Then,* $\exists\, \delta > 0$ *s.t. the outermost MOTS* $S \subset \Sigma$ *satisfies* $d^+ \geq \delta$*.*

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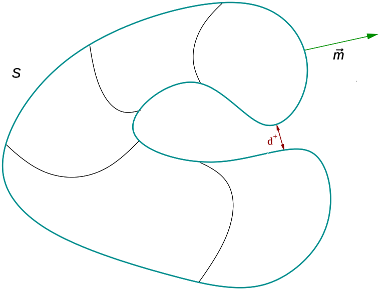

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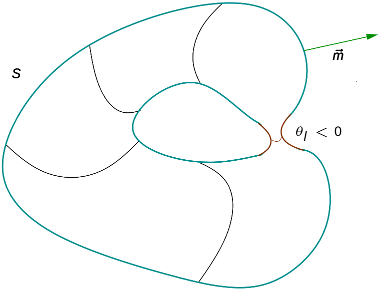

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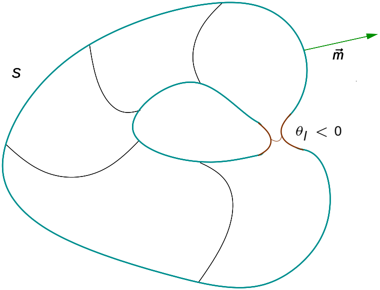
- We have a weakly outer trapped surface outside S, contradiction.
- The proof needs to smooth out the surface after the surgery while maintaining $\theta_l \leq 0$ everywhere.



Application: coalescence of "black holes"

- If two MOTS approach each other so that their distance approaches zero, a new MOTS arises enclosing them before they make contact.

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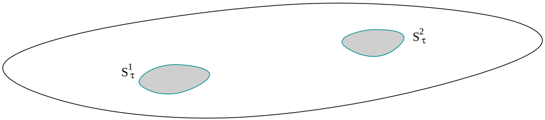

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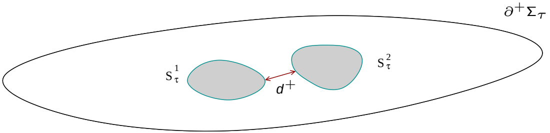

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- If two MOTS approach each other so that their distance approaches zero, a new MOTS arises enclosing them before they make contact.

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Let (Σ_t, g_t, K_t) for $t \in [0, T]$ be 3+1 foliation of a spacetime, with Σ_t compact and $\partial\Sigma_t = \partial^-\Sigma_t \cup \partial^+\Sigma_t$ satisfying $\theta_l(\partial^+\Sigma) > 0$ and $\theta_l(\partial^-\Sigma) < 0$. Assume $\exists$ a MOTS S_t with at least two connected components S_t^1 and S_t^2 satisfying $\mathrm{dist}(S_t^1, S_t^2) \to 0 \quad \text{as} \quad t \to T$.
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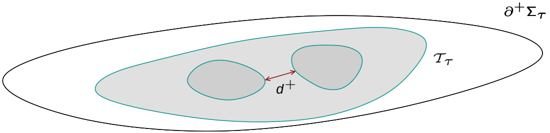

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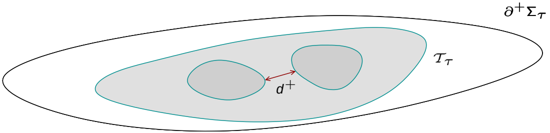

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What is the regularity of the outermost MOTT?

Regularity of outermost MOTT (I)

- Notation: $\partial\mathcal{T}_t$ trapped region on Σ_t. How do they compare to each other?
- All hypersurfaces $\Sigma_t = \Sigma \times t$ of the foliation can be identified with Σ (e.g. by normal projection) $\longrightarrow$ We can view all $\partial\mathcal{T}_t$ (t in a sufficiently small interval) as surfaces on a single Σ.

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Assume NEC, then

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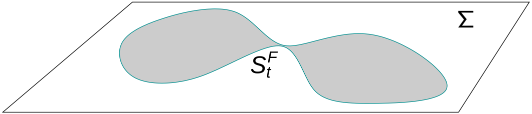

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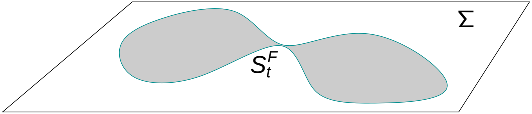

- Proof: Uses compactness results of MOTS, Andersson & Metzger, 2005.

Regularity of outermost MOTT (II)

Definition (Jump times)

Assume $\tau \in (0, T)$. If $S^P_\tau \neq \partial \mathcal{T}_\tau$ then τ is called past jump time.
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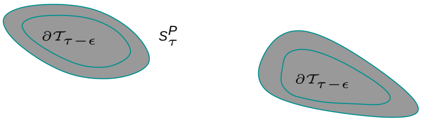

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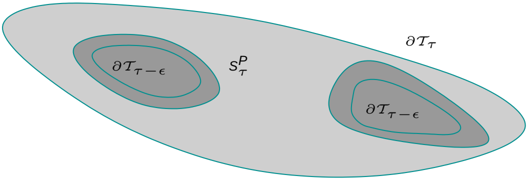

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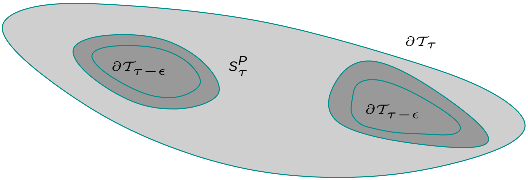

- The maximum principle implies that on a past jump, all points do jump.
- The volume covered by the jump is finite $\to$ there can be at most countably many past jump times.

Regularity of outermost MOTT (III)

Properties of future jumps:

- Under NEC, S_τ^F encloses $\mathcal{T}_\tau \longrightarrow$ future jumps are also outwards.
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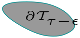

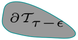

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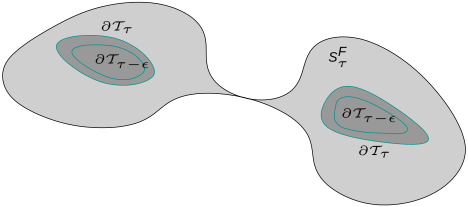

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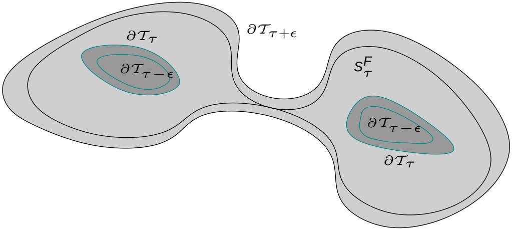

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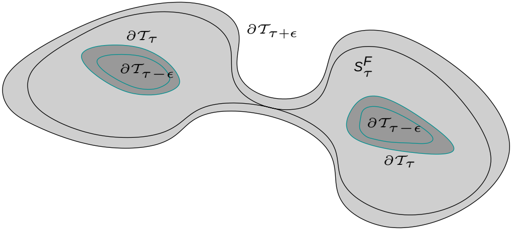

- Jump covers a finite volume $\longrightarrow$ at most countably many future jump times.
- Behaviour allowed by the known compactness theorems. However, do future jumps really exist?
- A priori, a jump time can be both future and past.

Regularity of outermost MOTT (IV)

Notation: $\mathcal{B} \equiv \bigcup_{t\in[0,T]} \partial\mathcal{T}_t \subset \Sigma \times [0,T]$, $\quad \forall I \subset [0,T]$, $\mathcal{B}_I = \bigcup_{t\in I} \partial\mathcal{T}_t \subset \mathcal{B}$

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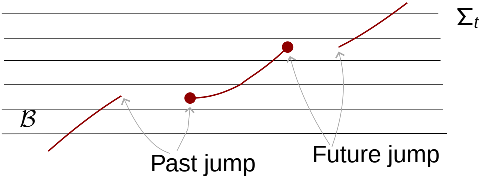

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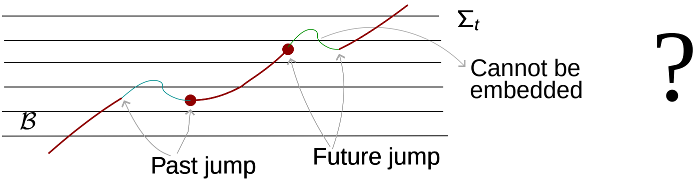

If this outermost MOTT part of a smooth tube connecting the jumps?

Stationary MOTS and black holes

- In the equilibrium situation (stationary spacetimes), the behaviour of MOTS and MOTT should simplify enormously.
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Do black holes uniqueness theorems hold for stationary spacetimes containing MOTS?

- Start with the static case
  - Uniqueness of rotating black holes is a considerably harder problem.
- An interesting result along these lines: **Miao Theorem**.

Theorem (Miao 2005)

Let (Σ, g, V) be a static, vacuum, 3-dimensional asymptotically Euclidean manifold with a smooth, compact boundary $\partial\Sigma$ with vanishing mean curvature. Then (Σ, g, V) is isometric to one-half of the spatial Schwarzschild manifold.

MOTS and symmetries

Aim: Generalize this result in two directions:

- Allow for non-time symmetric initial data, i.e. MOTS instead of minimal surfaces.
- Allow for other matter models.

Difficulty:

- The static black hole uniqueness proofs use the "quotient" metric (metric on the hypersurface orthogonal to $\vec{\xi}$) and the existence of a compact surface without boundary where $\lambda \equiv -(\vec{\xi}, \vec{\xi}) = 0$ (the Killing horizon).

Need to understand MOTS in spacetimes with isometries, and more generally in spacetimes with symmetries.

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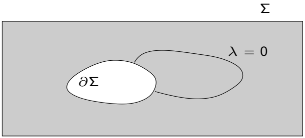

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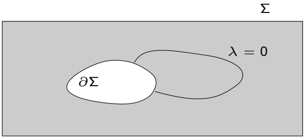

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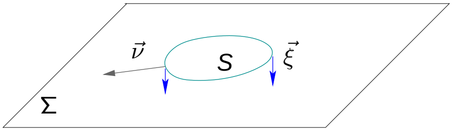

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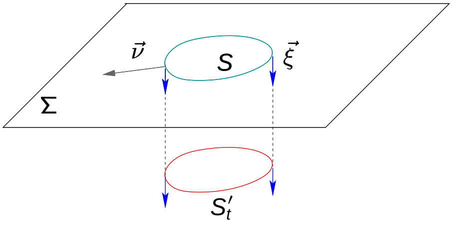

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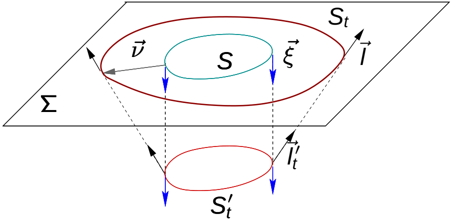

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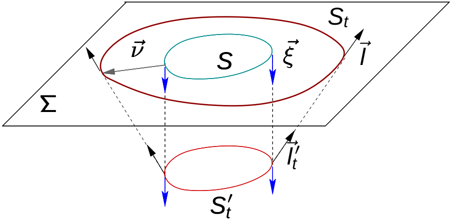

- The Raychaudhuri equation $\frac{d\theta_l}{d\beta} = -W \leq 0$ implies that S_t is a **weakly outer trapped surface** $\longrightarrow$ contradiction with locally outermost.

Maximum principle for the stability operator

- Main ingredients of the argument:
  (i) $\delta_{\vec{\nu}}\theta_l \leq 0$ (the procedure gives a weakly outer trapped surface).
  (ii) The vector $\vec{\nu}$ points outside S everywhere.
- Condition (ii) can be relaxed substantially and still get a contradiction.
- The idea is to use the properties of the elliptic operator L_m (recall that **locally outermost** implies **stable**).

Lemma (Maximum principle, Andersson, M, Simon, 2008)

Let S be a stable MOTS on Σ and Q a function on S.

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Let S be a stable MOTS on Σ and Q a function on S.

- *If $L_m Q \leq 0$ and not identically zero, then $Q < 0$.*
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Maximum principle for the stability operator

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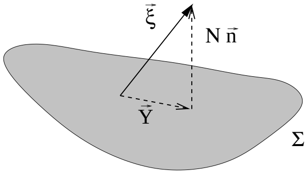
What about generalizing point (i)?

Deformation tensor

- Decompose $\vec{\xi} = N\vec{n} + \vec{Y}$
- From the geometric construction: $\vec{\nu} = -N\vec{l} + \vec{\xi}$



- The variation $\delta_{\vec{\nu}}\theta_l$ has two parts:
  - $\delta_{-N\vec{l}}\theta_l \rightarrow$ Raychaudhuri equation.
  - $\delta_{\vec{\xi}}\theta_l = 0$ because $\vec{\xi}$ is a Killing vector $\rightarrow$ **not at all** obvious from the explicit form of the stability operator ( ).
- To generalize to other symmetries, the expression should be optimal for Killing vectors (i.e. should give explicitly zero in that case).

Definition

The **deformation tensor** *of any vector field $\vec{\xi}$ is $a_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$*

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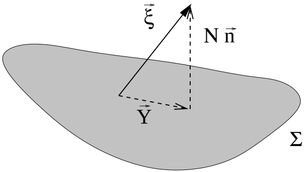

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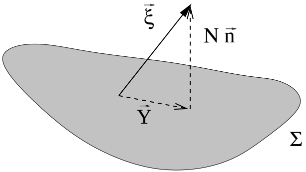

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Let S be a stable MOTS in a hypersurface Σ of a spacetime (M, g) which admits a conformal Killing vector $\vec{\xi}$, $\mathcal{L}_{\xi} g_{\mu\nu} = 2\phi g_{\mu\nu}$ (including homotheties $\phi = C$, and isometries $\phi = 0$).

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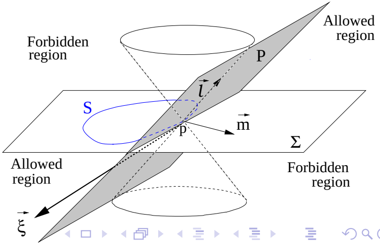

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- So, except for an exceptional case (marginally stable and both shear and matter "density" along $\vec{l}$ identically zero), $\vec{\xi} \propto \vec{l}$ (hence null everywhere).
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- Assuming staticity, the results before can be substantially strengthened:

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Let (Σ, g_{ij}, K_{ij}) be a compact initial data set with outer untrapped boundary $\partial^+\Sigma$ (i.e. $\theta_l(\partial^+\Sigma) > 0$) satisfying NEC and admitting a static Killing vector $\vec{\xi} = N\vec{n} + \vec{Y}$ which is timelike near $\partial^+\Sigma$. Let $\mathcal{V}$ be the connected component of $\{p; \vec{\xi}(p) \text{timelike}\}$ containing $\partial^+\Sigma$. If

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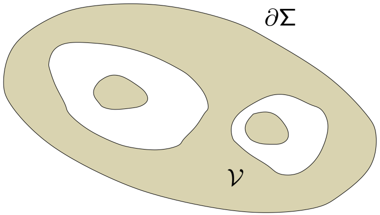

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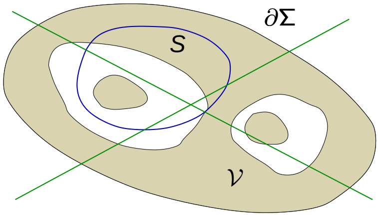

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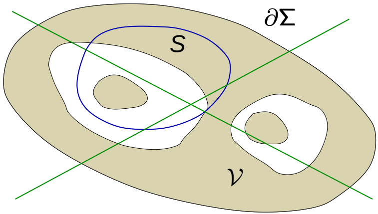
- Condition (i) is a technical restriction. It excludes the coexistence of portions of a Killing horizon of "black hole" and "white hole" type on Σ.



Applications to black holes uniqueness

- This “non-penetration” theorem is an important ingredient for a “black hole uniqueness” result for MOTS in the static case.

Theorem (Carrasco, M, 2009)

Let (Σ, g_{ij}, K_{ij}) be an asymptotically flat initial data set satisfying NEC, with a non-empty compact boundary $\partial\Sigma$ and admitting a static Killing vector $\vec{\xi}$ which is timelike in the asymptotic end. Assume

(i) *The boundary is a MOTS, $\theta_l(\partial\Sigma) = 0$.*

(ii) *The matter model is such that the Bunting & Masood-ul-Alam doubling method gives uniqueness for black holes.*

(iii) *$\vec{\xi}$ is null if and only if $\vec{\xi} = 0$.*

Then, (Σ, g_{ij}, K_{ij}) is a slice of such a unique spacetime.

- Conditions (i) and (ii) are natural. However, condition (iii) is technical and should be dropped.
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- Continue with the local time evolution of MOTS.
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  - Crucial step: Is there an unstable MOTS between two stable ones?
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An application for FLRW spacetimes

- FLRW spacetimes

$$
\begin{aligned}
g_{FLRW} &= -dt^2 + a^2(t)\left[dr^2 + \chi^2(r;k)d\Omega^2\right], \\
\chi(r;k) &= \{\sin r, r, \sinh r\} \text{ for } k = \{1, 0, -1\}
\end{aligned}
$$

  admit a timelike conformal Killing vector $\vec{\xi} = \partial_t$.
- Use this vector to get restrictions on stable MOTS.

Theorem

There exists no stable MOTS in any spacelike hypersurface of a FLRW spacetime (M, g_{FLRW}) satisfying

$$
\frac{\dot{a}^2(t)+k}{a(t)} > 0, \qquad -\frac{\dot{a}^2(t)+k}{a(t)} \leq \ddot{a}(t) \leq \frac{\dot{a}^2(t)+k}{a(t)}.
$$

- In terms of the total pressure and energy density, the conditions on $a(t)$ read $\rho > 0$, $\rho \geq 3p$ and $\rho + p \geq 0$
- Particular case: NEC + accelerated expansion ($\ddot{a} > 0$). Back