Physical properties of the double-Kerr solution

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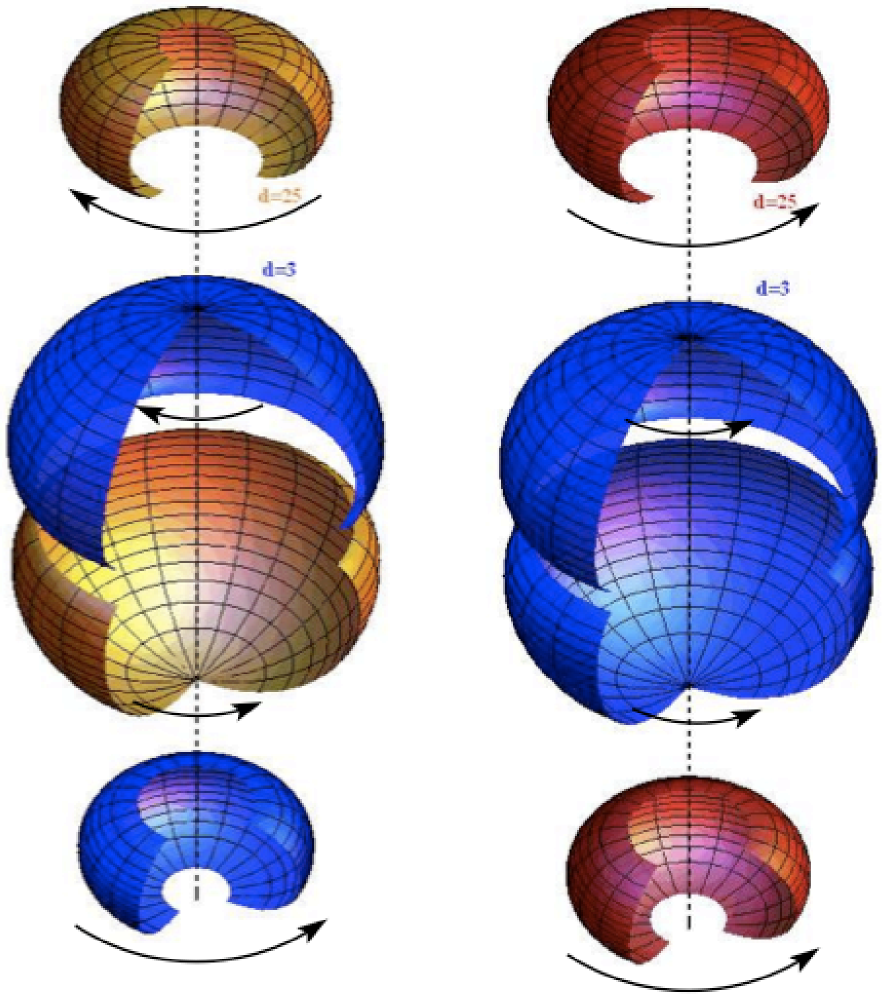
0805.1206 (hep-th), JHEP 0807:009 (2008),
0808.3941 (gr-qc), JHEP 0810:017 (2008),
0903.0264 (gr-qc), PRD 79:123508 (2009)

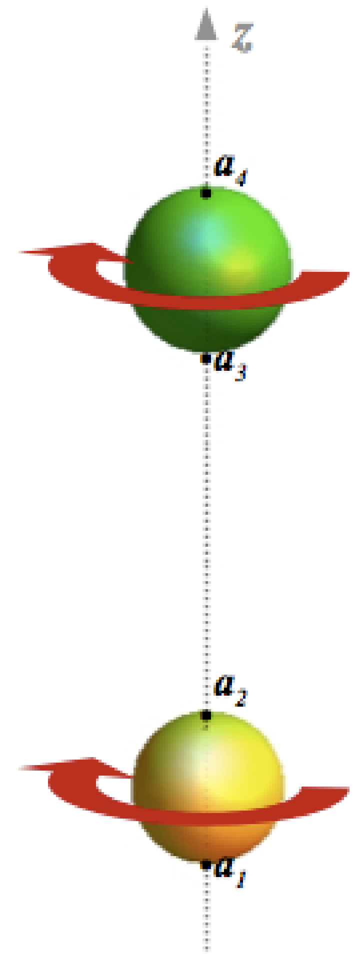
with C. Rebelo, M. Costa and M. Zilhão

"Mathematical Relativity in Lisbon",
June 19th 2009



Motivation:

1) Probing the existence of multi-black holes equilibrium configurations, without supersymmetry.



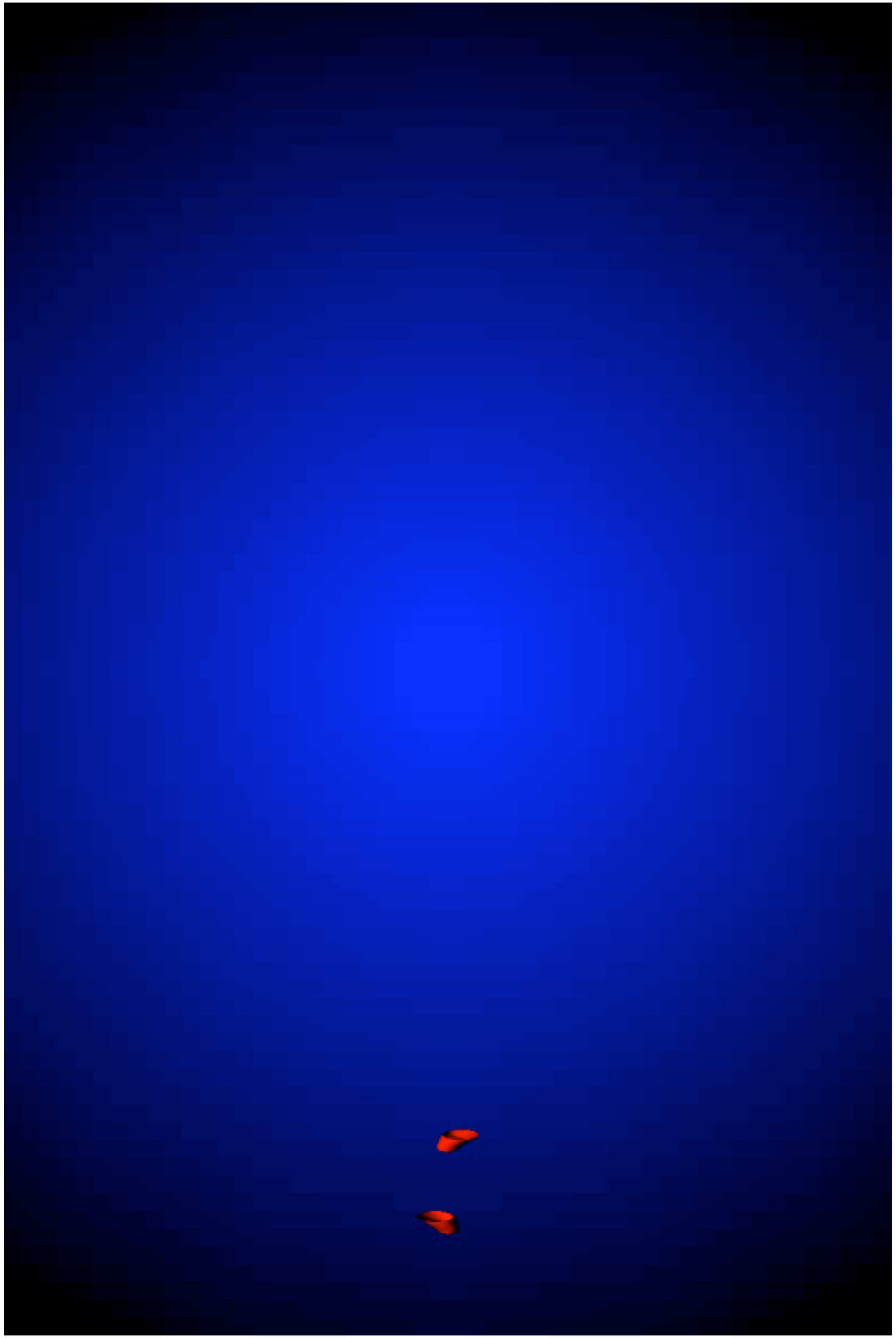
"Can we have a regular (on and outside an event horizon), asymptotically flat, vacuum solution of multiple black holes with (topologically) spherical horizons in D=5?"

Is there a regular double-Myers-Perry configuration?

Motivation:

2) Study interactions of astro-physical black holes in a quasi-static approximation

"Can we learn something to complement approximation methods and numerical simulations?"



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Thanks to Marcus Thierfelder and Helvi Witek

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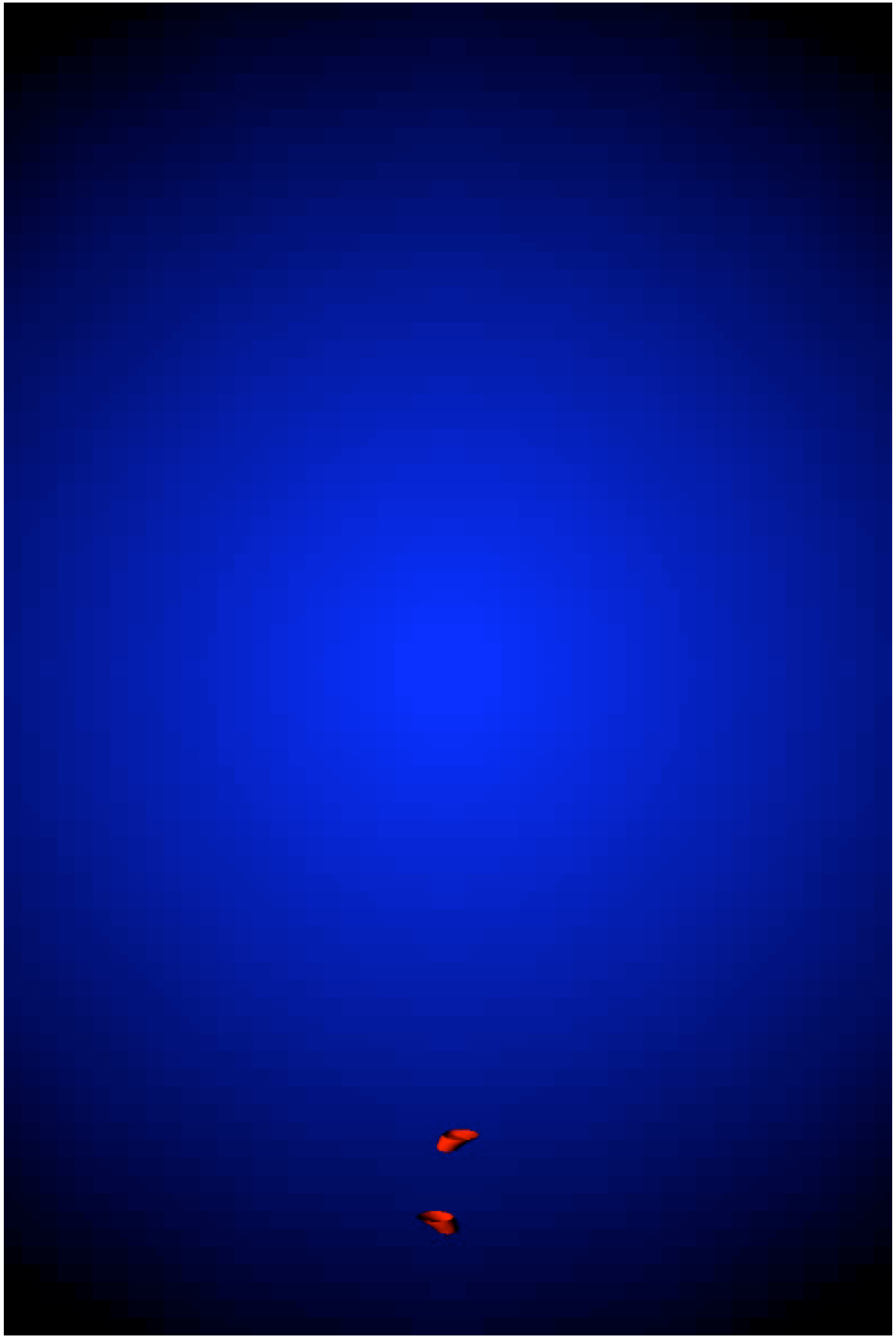

Outline:

1) Generating the solution

- Weyl solutions
- Inverse scattering technique
- Counter-rotating and co-rotating double Kerr

2) Physical properties

- Mutual rotational dragging: angular velocities and extremality condition
- Horizon geometry
- Ergoregions

3) Final remarks

1) Generating the solution

In D dimensions, assuming the existence of D-2 commuting Killing vector fields:

$$ds^2 = g_{ab}(\rho, z)dx^a dx^b + e^{2\nu(\rho,z)}(d\rho^2 + dz^2) \quad a, b = 1 \dots D-2$$

In vacuum one can take: $det g = -\rho^2$

Defining the (D-2) dimensional square matrices: $\mathbf{U} = \rho \mathbf{g}_{,\rho}\mathbf{g}^{-1}, \quad \mathbf{V} = \rho \mathbf{g}_{,z}\mathbf{g}^{-1}$

The vacuum Einstein equations become: $\mathbf{U}_{,\rho} + \mathbf{V}_{,z} = 0$

(equations for a completely integrable system: the principal chiral field model, a non-linear sigma model with group GL(D-2,R)), and

$$\partial_\rho \nu = -\frac{1}{2\rho} + \frac{\mathrm{Tr}(\mathbf{U}^2 - \mathbf{V}^2)}{8\rho}$$

$$\partial_z \nu = \frac{\mathrm{Tr}(\mathbf{UV})}{4\rho}$$

which determines $\nu(\rho, z)$ by a line integral. Belinskii and Zakharov '78, 79

Static case: Weyl solutions

Take the Killing vectors to be mutually orthogonal. Then we are considering **Weyl solutions** Weyl '17; Emparan and Reall '01

$$ds^2 = -e^{2U_0}dt^2 + \sum_{a=1}^{D-3} e^{2U_a}(dx^a)^2 + e^{2\nu}(d\rho^2 + dz^2)$$

The coefficients of each Killing direction, are harmonic in an auxiliary 3 dimensional space: an effective Newtonian problem for each Killing direction

$$\Delta_{E^3} U_a = 0$$

the remaining conditions encode the non-linearity; they read

$$\partial_\rho \nu = -\frac{1}{2\rho} + \frac{\rho}{2}\sum_{a=0}^{D-3}\left[(\partial_\rho U_a)^2 - (\partial_z U_a)^2\right] , \quad \partial_z \nu = \rho \sum_{i=0}^{D-3} \partial_\rho U_a \partial_z U_a$$

and

$$det g = -\rho^2 \qquad \text{becomes} \qquad \sum_{a=0}^{D-3} U_a = \log \rho$$

Note: As $U_a \to -\infty$ the Killing vector has a set of fixed points.

Static case: Weyl solutions and Rod Structure

For each Killing direction we have an effective Newtonian problem for which we can specify sources. The interesting ones have rod sources and there is a nice pictorial way to build interesting solutions. Recall:

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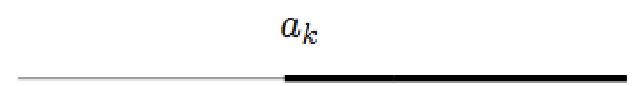
The potential for a semi-infinite rod along $[a_k, +\infty)$ with linear density ϱ is



$$U = \varrho \log \mu_k \ , \qquad \mu_k = \sqrt{\rho^2 + (z - a_k)^2} - (z - a_k)$$

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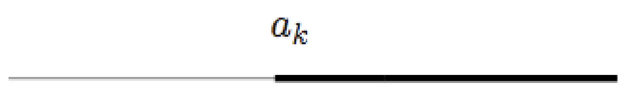

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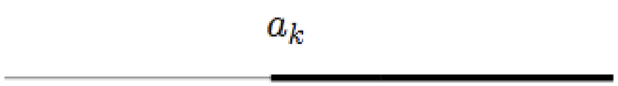

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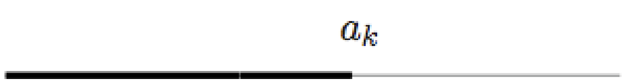

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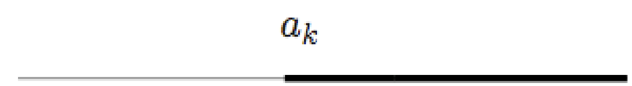

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Note: $\mu_k, \bar{\mu}_k \to 0$ at the source; thus $U_a \to -\infty$ at the source.

Static case: 4D examples of Rod Structure

Method:

1) For each Killing direction one chooses rod sources
2) At the rod source, the Killing direction will have a set of fixed points
3) Regularity at the axis requires that $\varrho = 1/2$
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4D Minkowski space

z axis for each case

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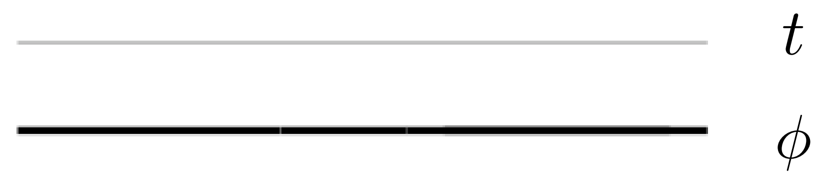

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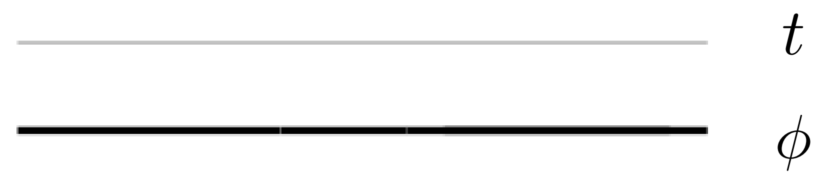

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4D Double-Schwarzschild

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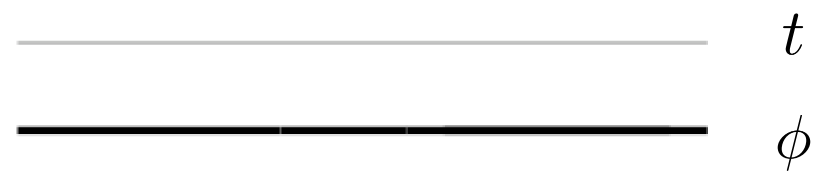

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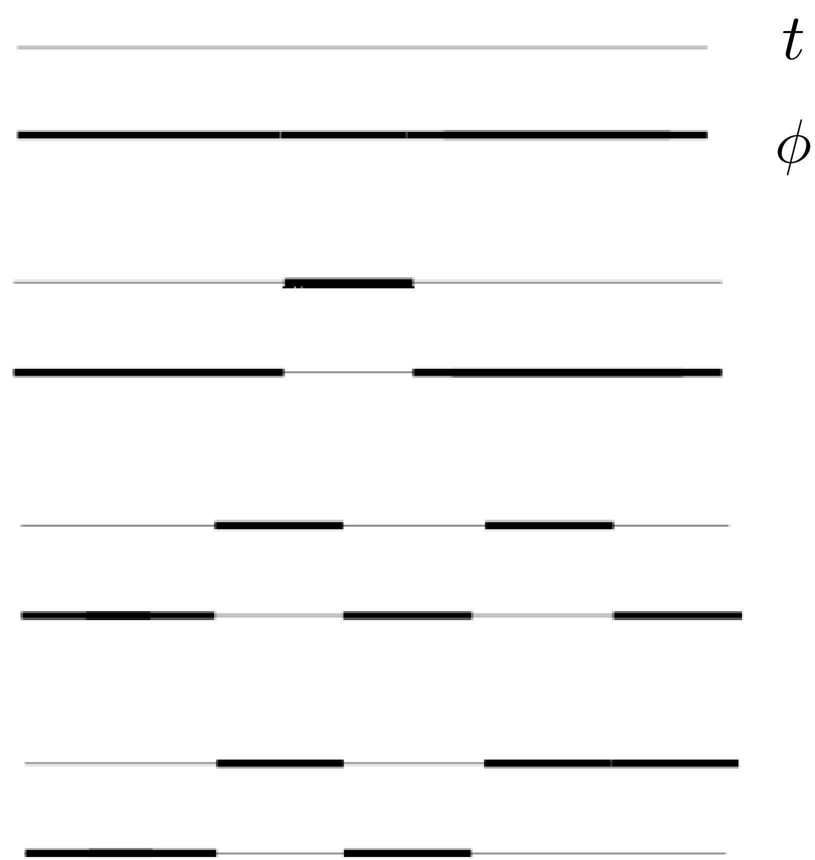

z axis for each case

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A point-like source gives rise to a naked singularity Chazy, Curzon '24

Static case: 5D examples of Rod Structure

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z axis for each case

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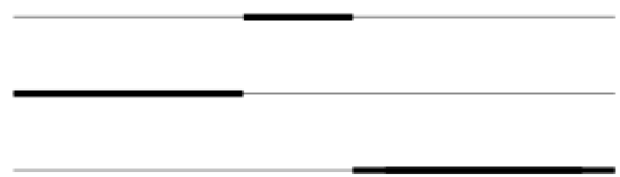
5D Schwarzschild-Tangherlini



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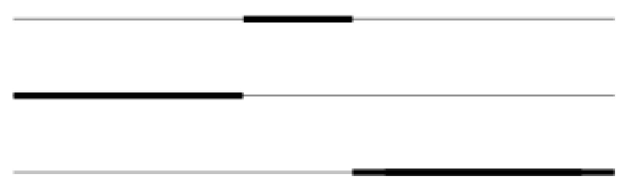

5D static black ring

Emparan and Reall '01

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5D Minkowski space

z axis for each case

5D Schwarzschild-Tangherlini

5D static black ring

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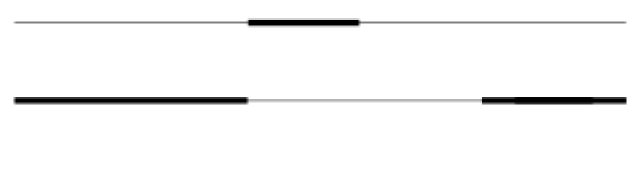
5D Black Hole plus KK bubble

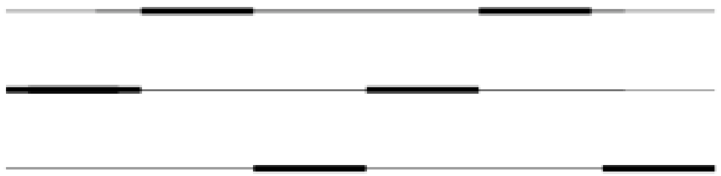


Static case: 5D multi-black holes examples of Rod Structure

5D Multi-Schwarzschild-Tangherlini

Tan and Teo '03

has conical singularities



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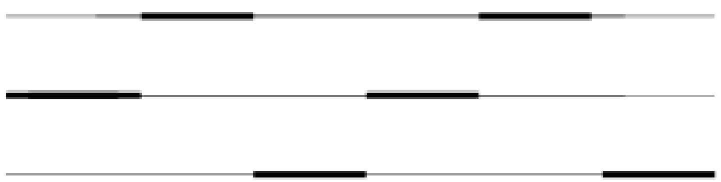

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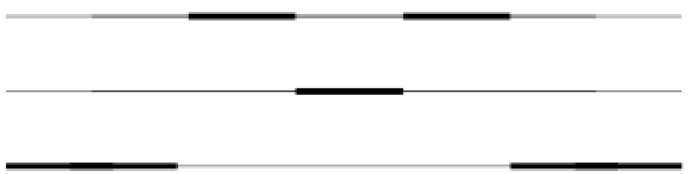

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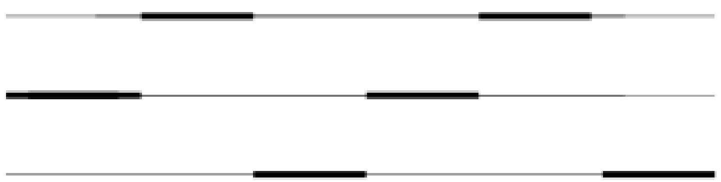

5D Multi-black holes plus KK bubbles
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5D Static Black Saturn
Elvang and Figueras '07
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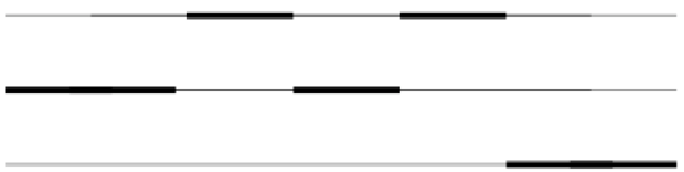

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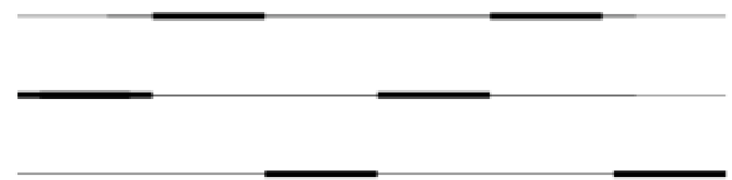

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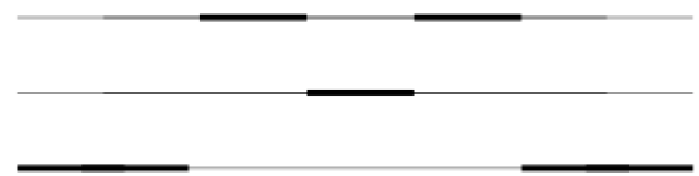

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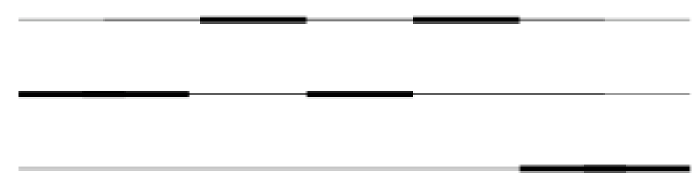

Could one have a **regular** Multi-Myers-Perry black hole?

Inverse Scattering Method: Generating Stationary Solutions

Recall the general case with D-2 commuting Killing vector fields,

$$ds^2 = g_{ab}(\rho, z)dx^a dx^b + e^{2\nu(\rho,z)}(d\rho^2 + dz^2) \quad a, b = 1 \ldots D-3$$

The "fundamental" Einstein vacuum equations are

$$\mathbf{U}_{,\rho} + \mathbf{V}_{,z} = 0 \qquad \mathbf{U} = \rho \mathbf{g}_{,\rho}\mathbf{g}^{-1}\,, \quad \mathbf{V} = \rho \mathbf{g}_{,z}\mathbf{g}^{-1} \tag{1}$$

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These are the equations for a non-linear sigma model with group GL(D-2,R), a completely integrable system; they admit a Lax pair of linear (spectral) equations

$$D_1\Psi = \frac{\rho\mathbf{V} - \lambda\mathbf{U}}{\lambda^2 + \rho^2}\,\Psi\,, \qquad D_2\Psi = \frac{\rho\mathbf{U} + \lambda\mathbf{V}}{\lambda^2 + \rho^2}\,\Psi\,,$$

λ is the (in general complex) spectral parameter, independent of ρ and z, and D_1, D_2 are two commuting differential operators

$$D_1 = \partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2}\,\partial_\lambda\,, \qquad D_2 = \partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2}\,\partial_\lambda\,. \qquad (2)$$

The $(D-2) \times (D-2)$ dim. **generating matrix** $\Psi(\lambda, \rho, z)$ obeys $\Psi(0, \rho, z) = g(\rho, z)$ where g is a solution of (1). Compatibility of equations (2) then implies equations (1)

Strategy

Construct new solutions by dressing a **seed** solution

$$\Psi = \chi \Psi_0$$

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One finds that

$$\tilde{\mu}_k = \begin{cases} \mu_k & \text{for a } \textit{soliton}, \\ \bar{\mu}_k & \text{for an } \textit{antisoliton}, \end{cases} \qquad (R_k)_{ab} = m_a^{(k)} \sum_{l=1}^{n} \frac{(\Gamma^{-1})_{lk} m_c^{(l)} (g_0)_{cb}}{\tilde{\mu}_l}$$

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One chooses: 1) the soliton positions; 2) the constant vectors. The latter determine the rod orientations in the new solution (**BZ vectors**). Then

$$g(\rho, z) = \Psi(0, \rho, z) = \chi(0, \rho, z) \Psi_0(0, \rho, z) , \qquad g_{ab} = (g_0)_{ab} - \sum_{k,l=1}^{n} \frac{(g_0)_{ac} \, m_c^{(k)} \, (\Gamma^{-1})_{kl} \, m_d^{(l)} \, (g_0)_{db}}{\tilde{\mu}_k \tilde{\mu}_l}$$

Problem and Solution

The introduction of n (anti-)solitons gives a determinant for the new metric

$$\det g = (-1)^n \rho^{2n} \left(\prod_{k=1}^{n} \tilde{\mu}_k^{-2} \right) \det g_0 \qquad (3)$$

Thus, in general, it does not obey $\det g = -\rho^2$ if $\det g_0 = -\rho^2$. A rescaling of g solves the problem in D=4, but not in D>4.

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General solution, Pomeransky '06; since (3) does not depend on the BZ vectors:
i) start from a solution with physical rod densities;

ii) **remove** a number of solitons from it (i.e. add solitons or antisolitons with negative densities -1/2);
iii) re-add these same solitons, but now with different BZ vectors, so that the rods affected by these solitons acquire in general new directions.

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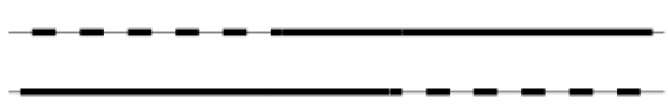
Recipe:

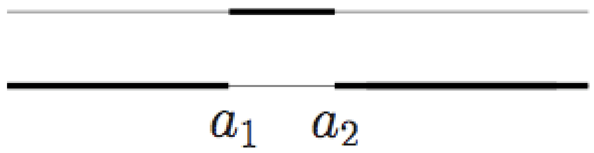
Removing a soliton (anti-soliton) at a_k with BZ vector along a given direction amounts to multiply that metric component by

$$\frac{\mu_k}{\bar{\mu}_k} \qquad \left(\frac{\mu_k}{\bar{\mu}_k} \right)$$



Game: Find the most convenient removal

Example: Generate Kerr from Schwarzschild.



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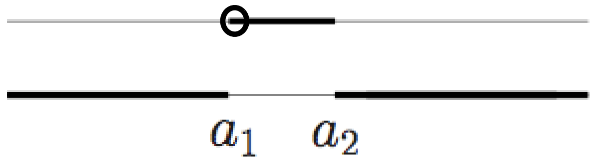

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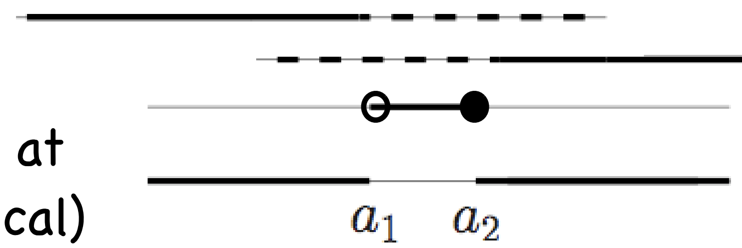

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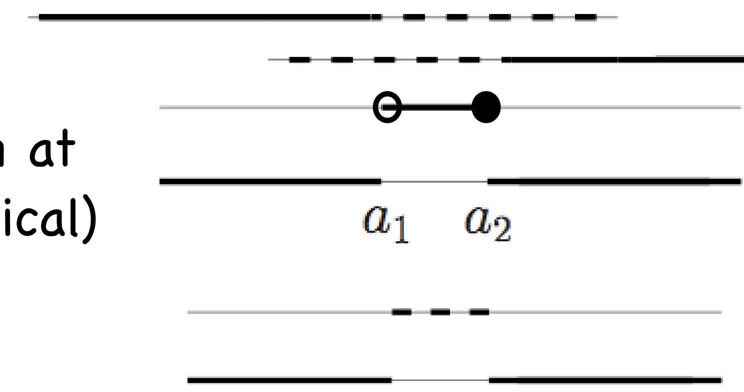

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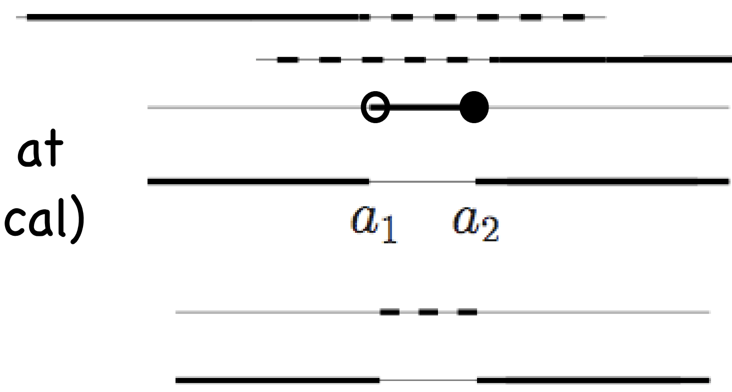

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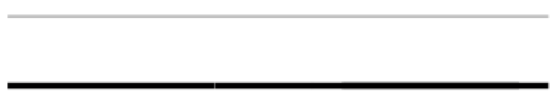

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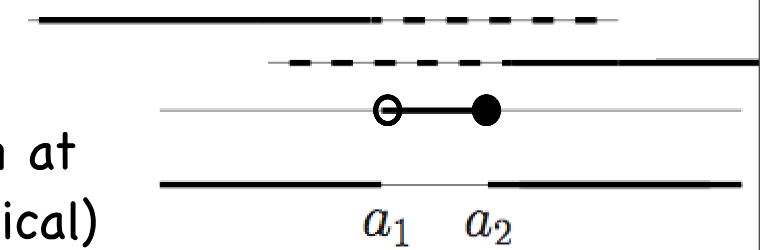

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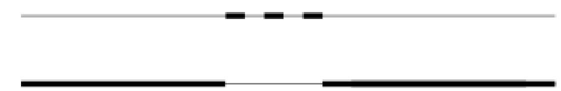

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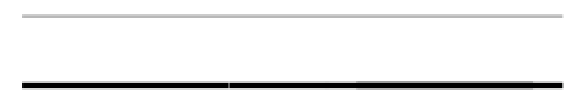

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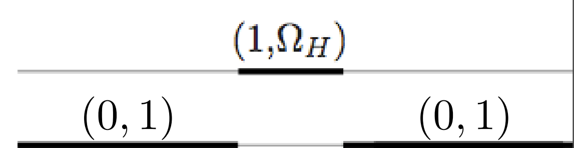

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Game: Find the most convenient removal

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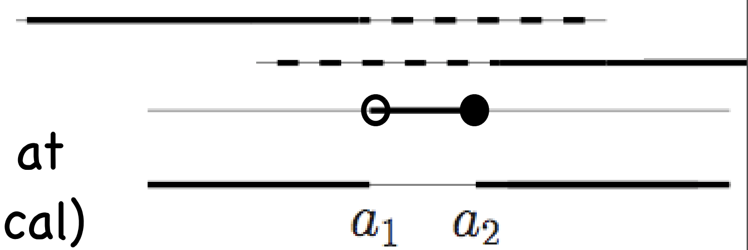

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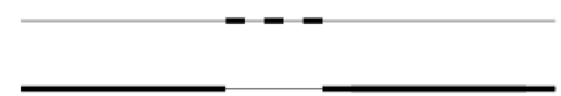

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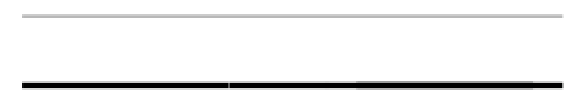

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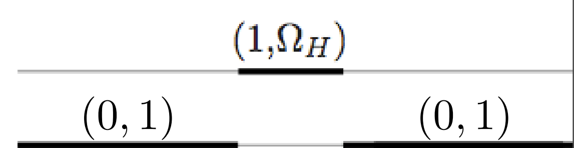

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Finally, the 2-dimensional conformal factor is obtained from the seed as

$$e^{2\nu} = e^{2\nu_0}\,\frac{\det\Gamma}{\det\Gamma_0}$$

Generating the double Kerr

Start from double Schwarzschild (Israel-Kahn):

Remove:
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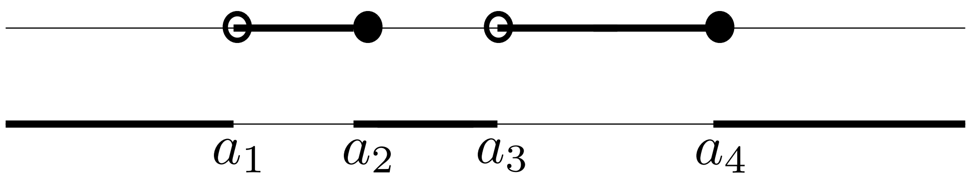
all with BZ vectors $(1,0)$



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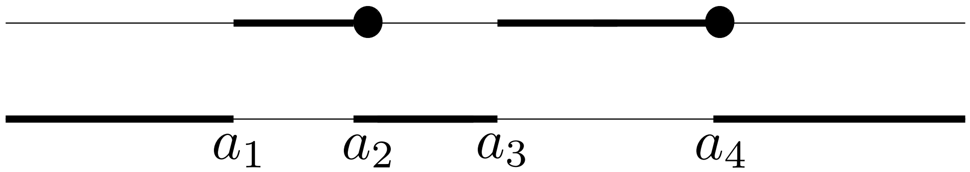

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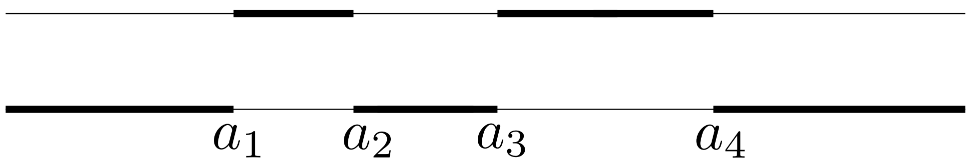

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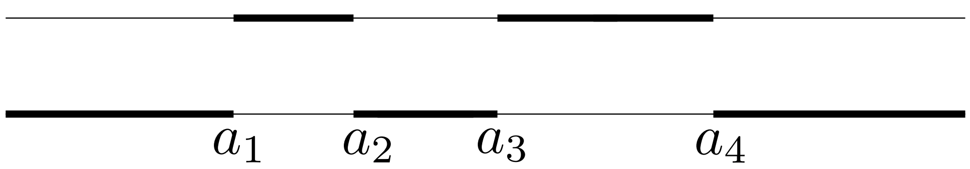

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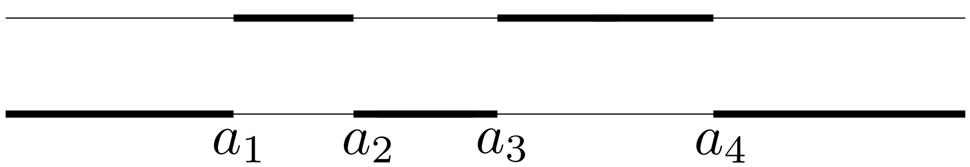

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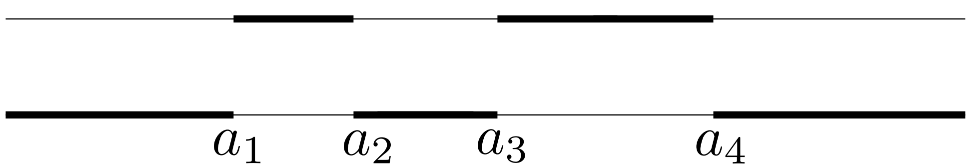

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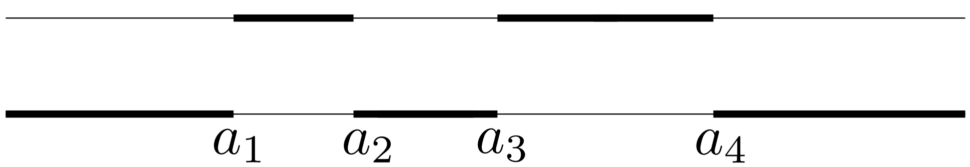

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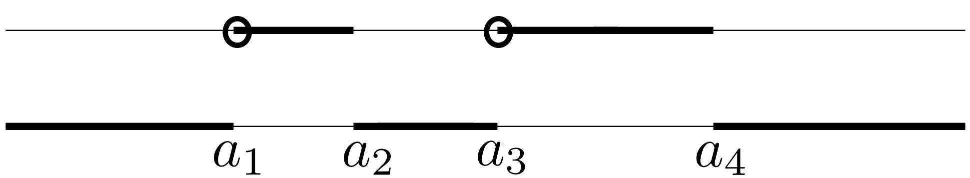

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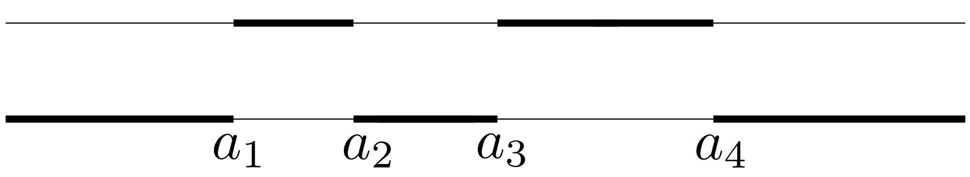

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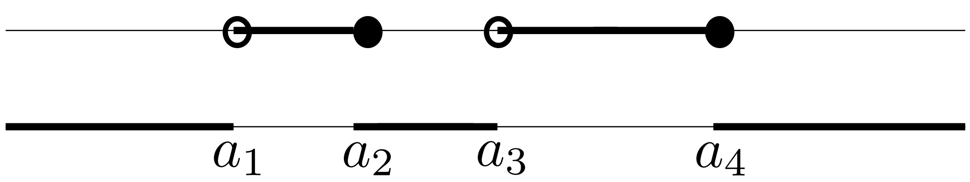

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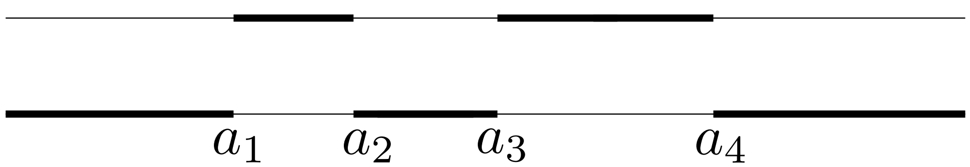

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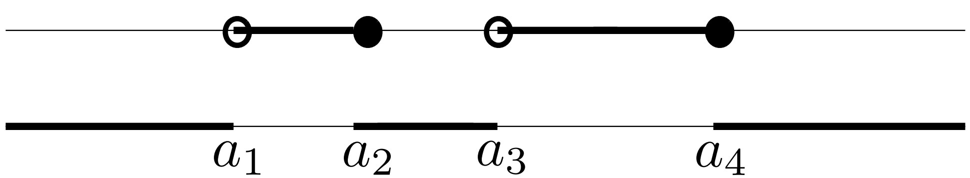
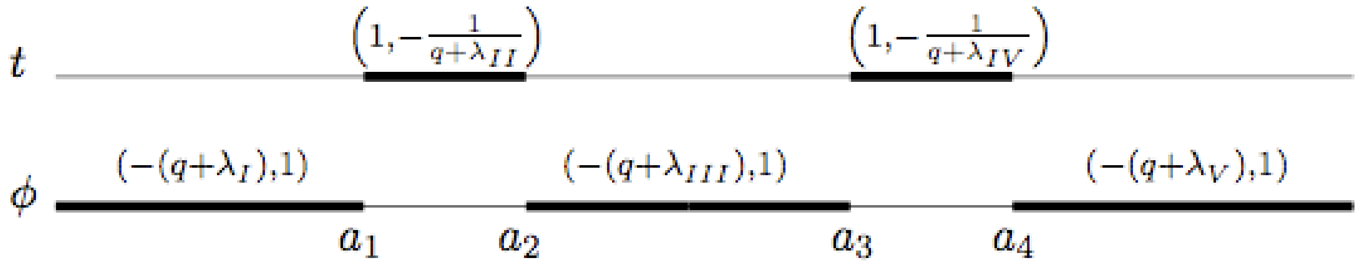


7 parameter family of solutions $a_{21}, a_{32}, a_{43}, b_1, c_2, b_3, c_4$

General solution: Extremely complicated!



$$\lambda_I = -2\frac{(a_{21}a_{41}b_1 + a_{32}a_{43}b_3)a_{42} + (a_{41}a_{43}c_2 + a_{21}a_{32}c_4)a_{31}b_1b_3}{a_{31}a_{42}(1+b_1c_2b_3c_4) + a_{32}a_{41}(b_1c_2 + b_3c_4) + a_{21}a_{43}(b_3c_2 + b_1c_4)}\,,$$

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$$\lambda_{III} = 2\frac{(a_{21}a_{42}c_2 - a_{31}a_{43}b_3)a_{41} + (a_{21}a_{31}c_4 - a_{42}a_{43}b_1)a_{32}b_3c_2}{a_{32}a_{41}(1+b_1c_2b_3c_4) - a_{21}a_{43}(b_1b_3 + c_2c_4) + a_{31}a_{42}(b_1c_2 + b_3c_4)}\,, \qquad q = -\frac{\lambda_I + \lambda_V}{2}\,.$$

$$\lambda_{IV} = 2\frac{(a_{31}a_{41} + a_{32}a_{42}b_1c_2)a_{43} + (a_{41}a_{42}b_3 - a_{31}a_{32}c_4)a_{21}c_2}{a_{21}a_{43}(b_1 - b_3c_2c_4) + a_{32}a_{41}(b_3 - b_1c_2c_4) + a_{31}a_{42}(b_1b_3c_2 - c_4)}\,,$$

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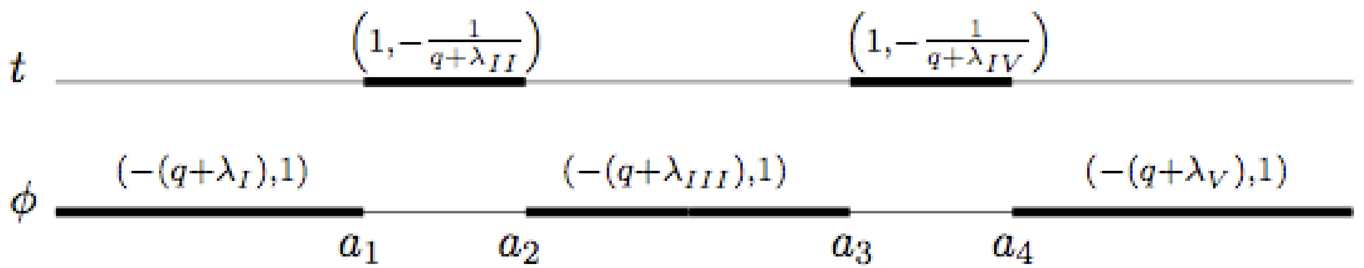

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Parameters: 2 masses, 2 angular momenta, distance
NUT charge, Axis condition

Matches original solution obtained by different generating technique
Kramer and Neugebauer '80

First obtained using inverse scattering by Letelier and Oliveira '98

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Symmetric rod structure:

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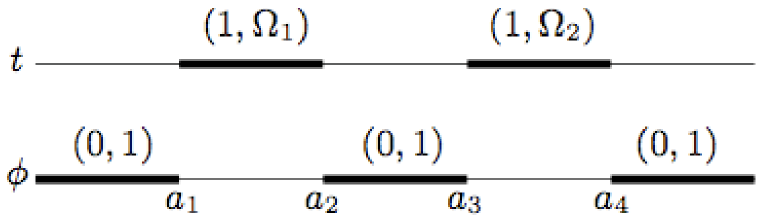

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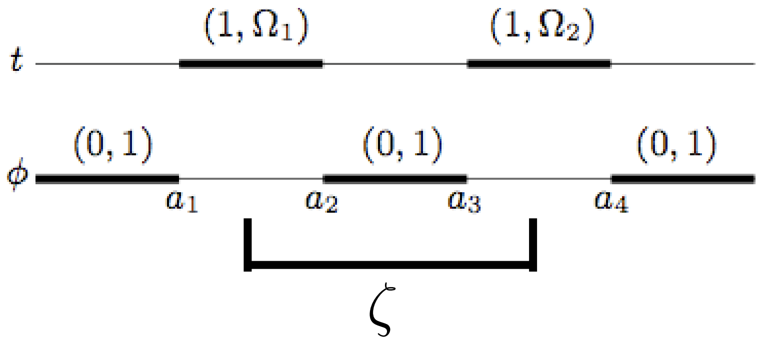

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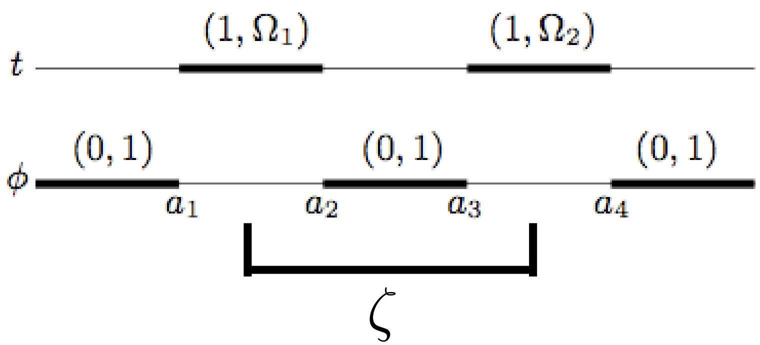

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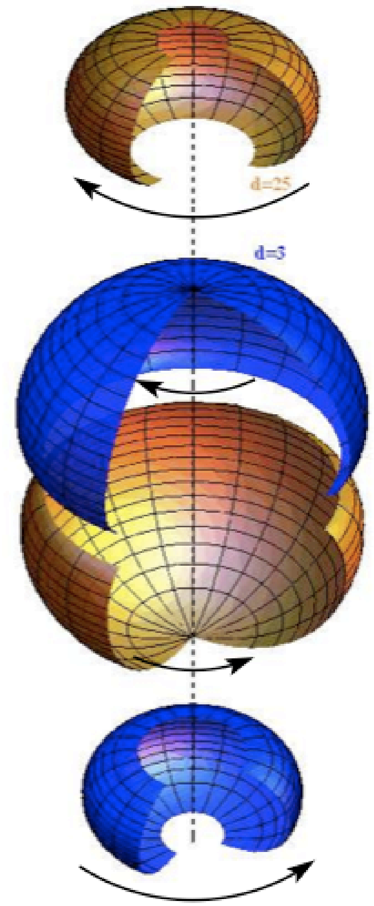

counter-rotating case

$$J \equiv J_1 = -J_2$$

$$\Omega_1 = -\Omega_2$$

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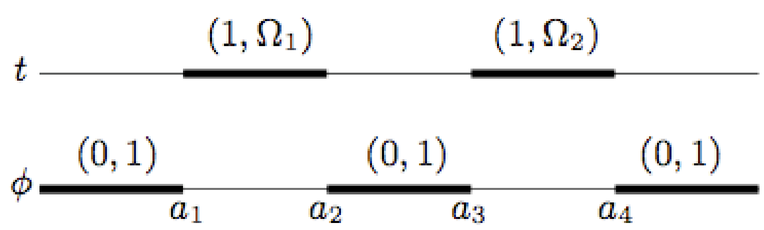

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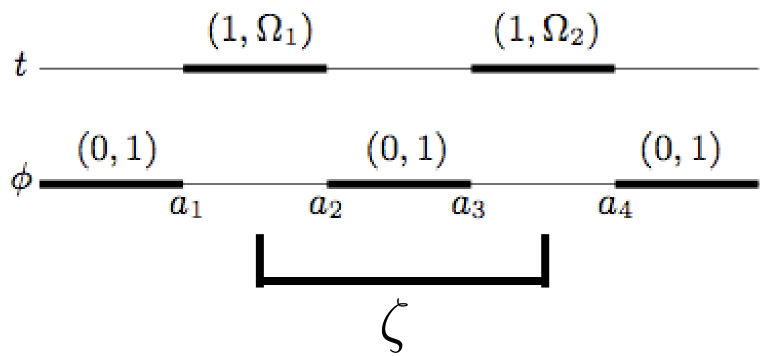


NUT charge vanishes automatically.

Axis condition is obeyed if one takes:

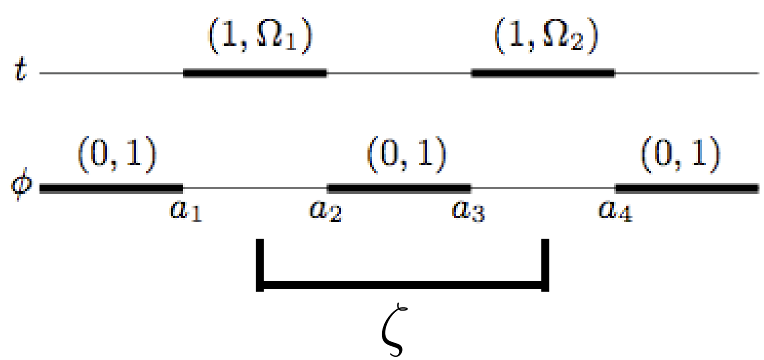
$$(1+2c^2+b^2c^2)\,\frac{a_{21}^2}{\zeta^2}+2c\,\frac{1-b^2c^2}{b-c}\,\frac{a_{21}}{\zeta}=(-1+bc)^2$$

Since general solution is extremely complicated we shall focus on two special `treatable' cases:

"Anti-symmetric" rod structure:

$$b \equiv b_1 = -c_4 \ , b_3 = -c_2 \equiv c$$

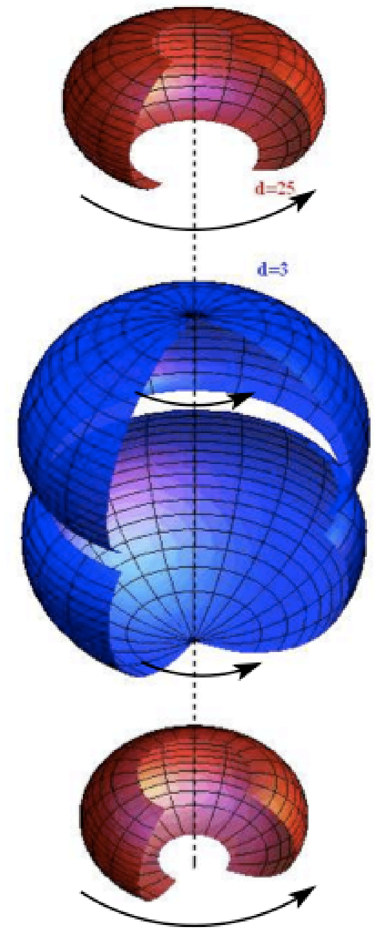
$$a_{21} = a_{43}$$



co-rotating case

$$J \equiv J_1 = J_2$$

$$\Omega_1 = \Omega_2$$



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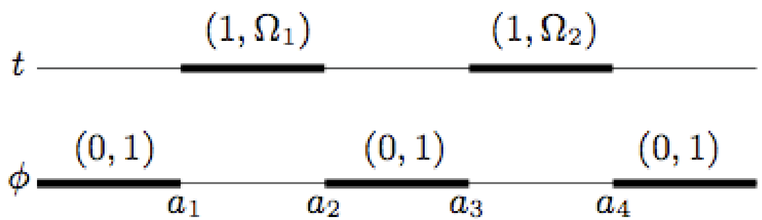
Two special cases of the double Kerr solution we will consider:



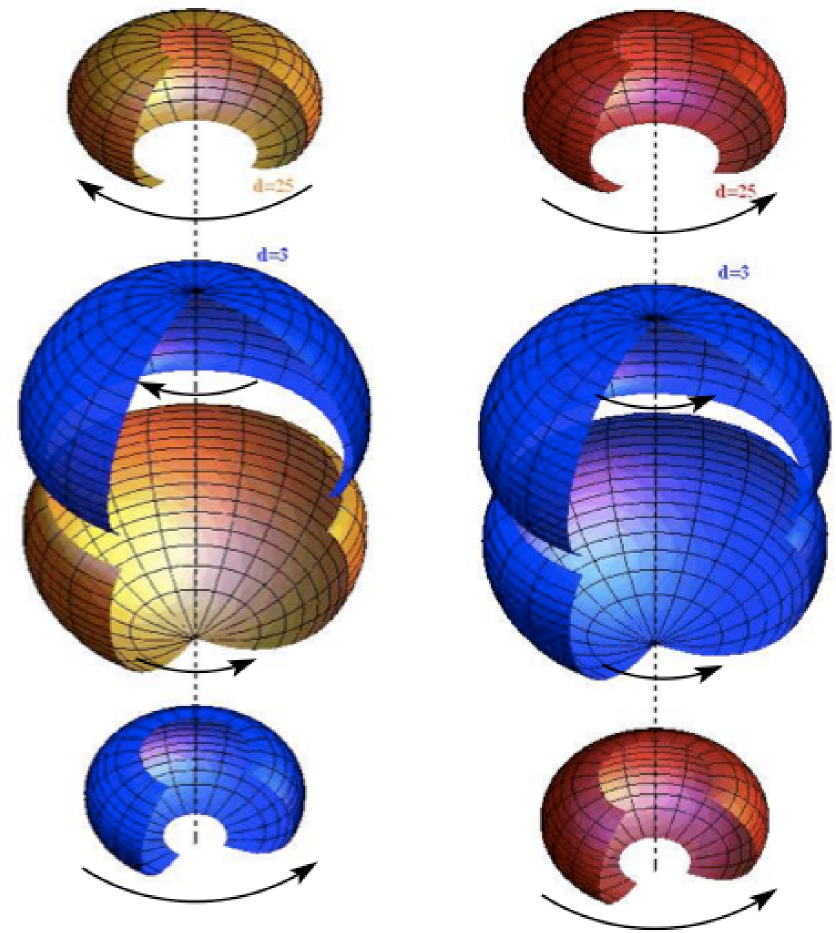
These are 3 parameter families, (M, J, ζ) corresponding to:

- mass $M \equiv M_1 = M_2$
- spin $J = |J_1| = |J_2|$
- distance

They are:

i) asymptotically flat;

ii) obey the axis condition;



counter-rotating case

$$\Omega_1 = -\Omega_2$$

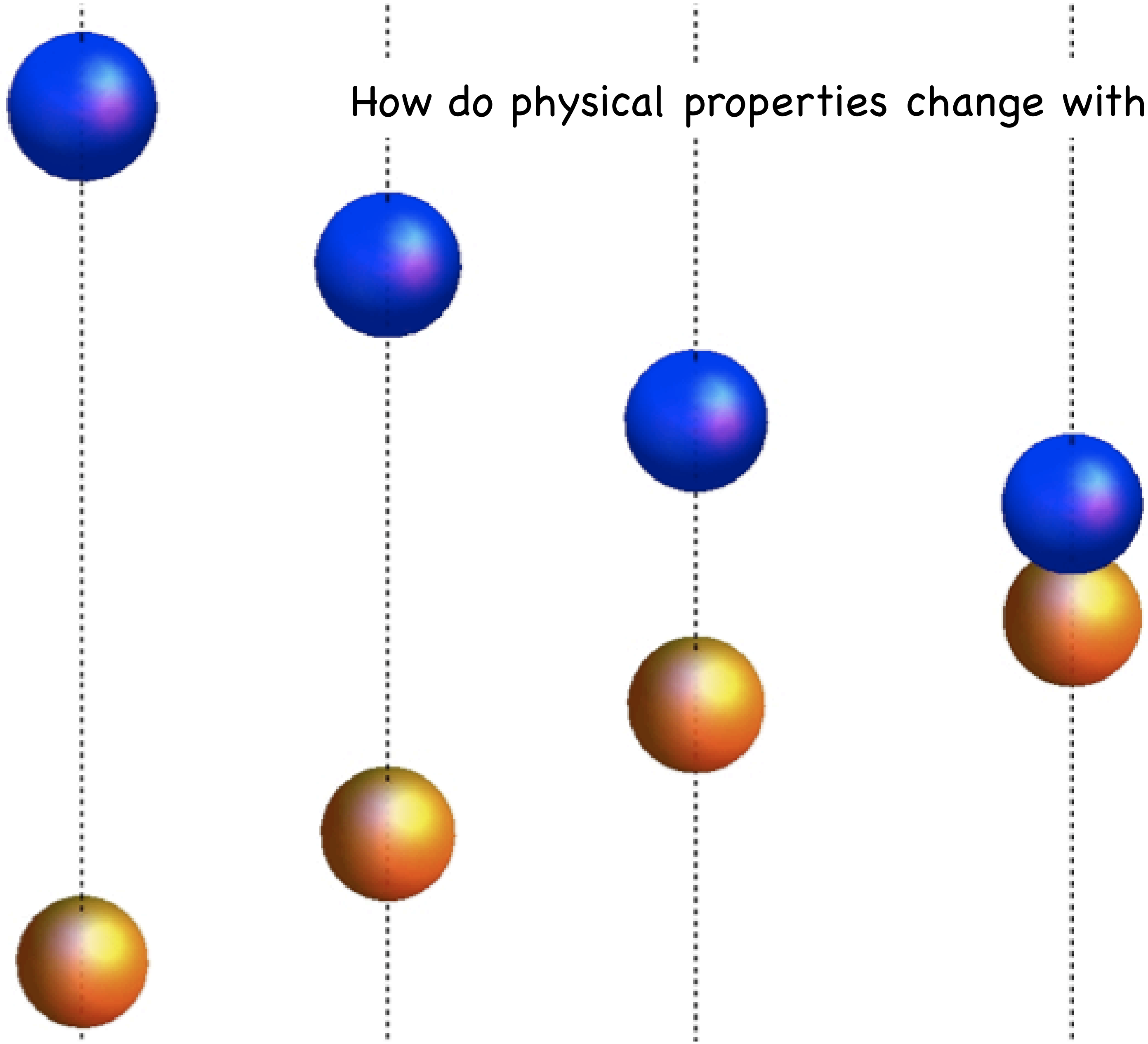
$$J \equiv J_1 = -J_2$$

co-rotating case

$$\Omega_1 = \Omega_2$$

$$J \equiv J_1 = J_2$$

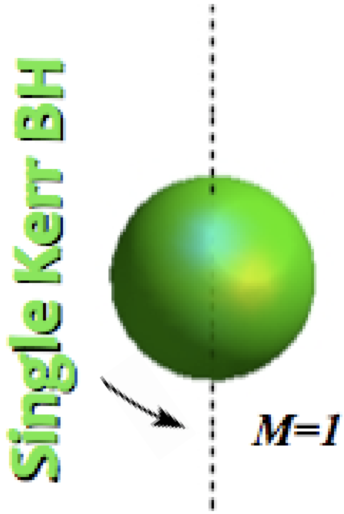
2) Physical properties

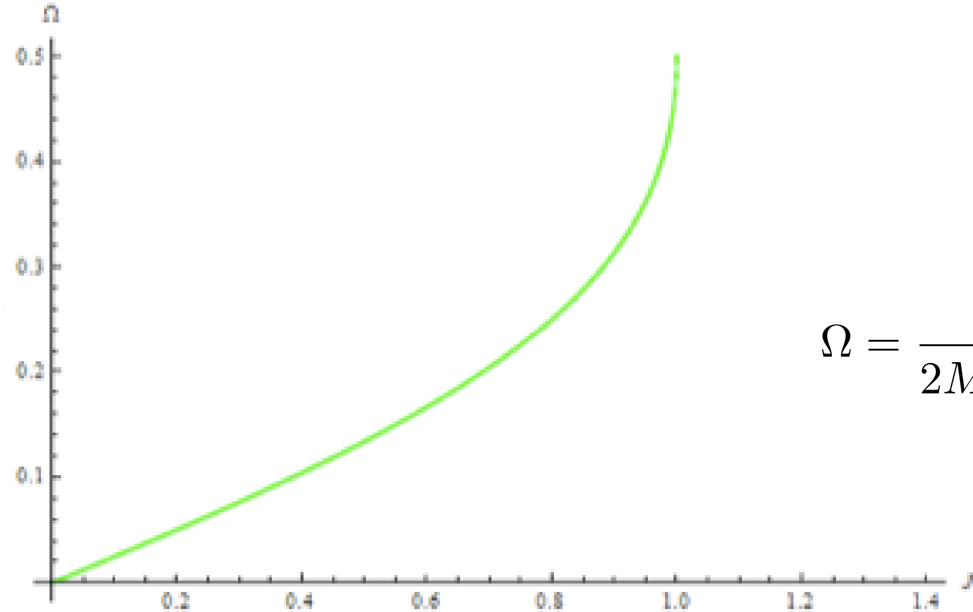


How do physical properties change with distance?

Angular velocity of the horizon (computed from rod structure):

Keep M fixed:





$$\Omega = \frac{J}{2M\left(M^2 + \sqrt{M^4 - J^2}\right)}$$

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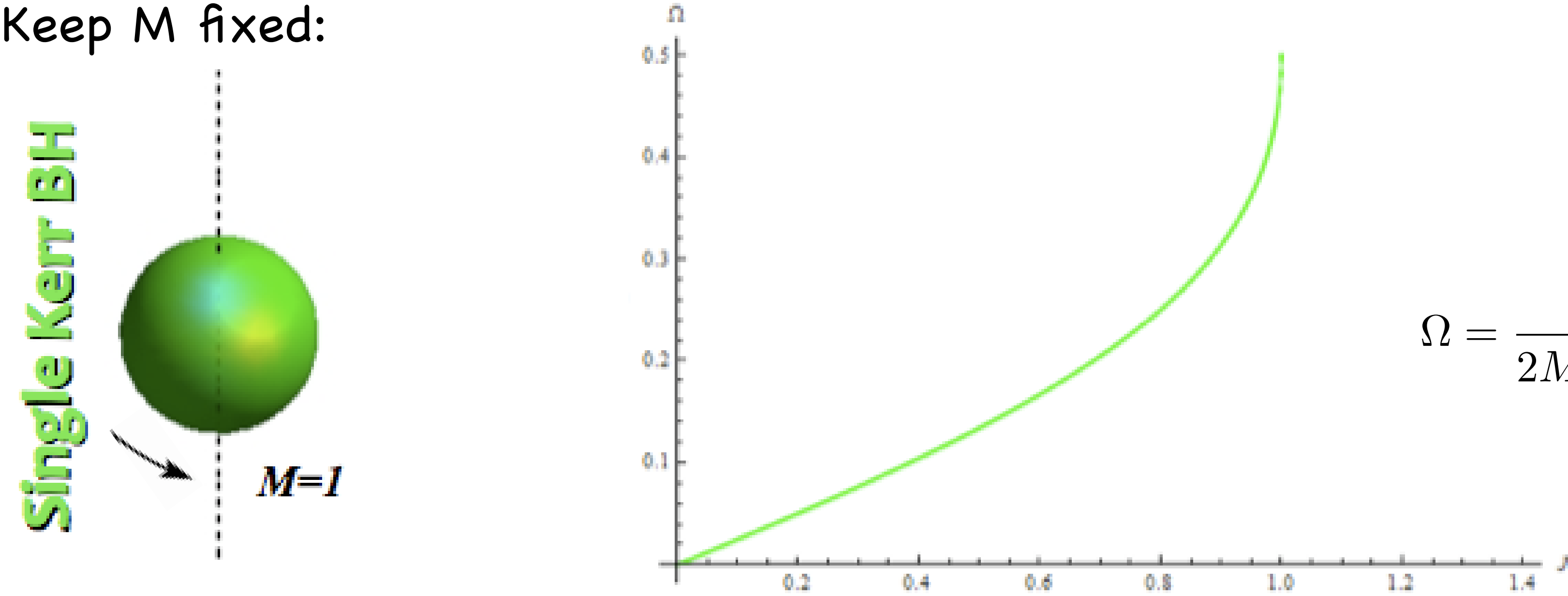

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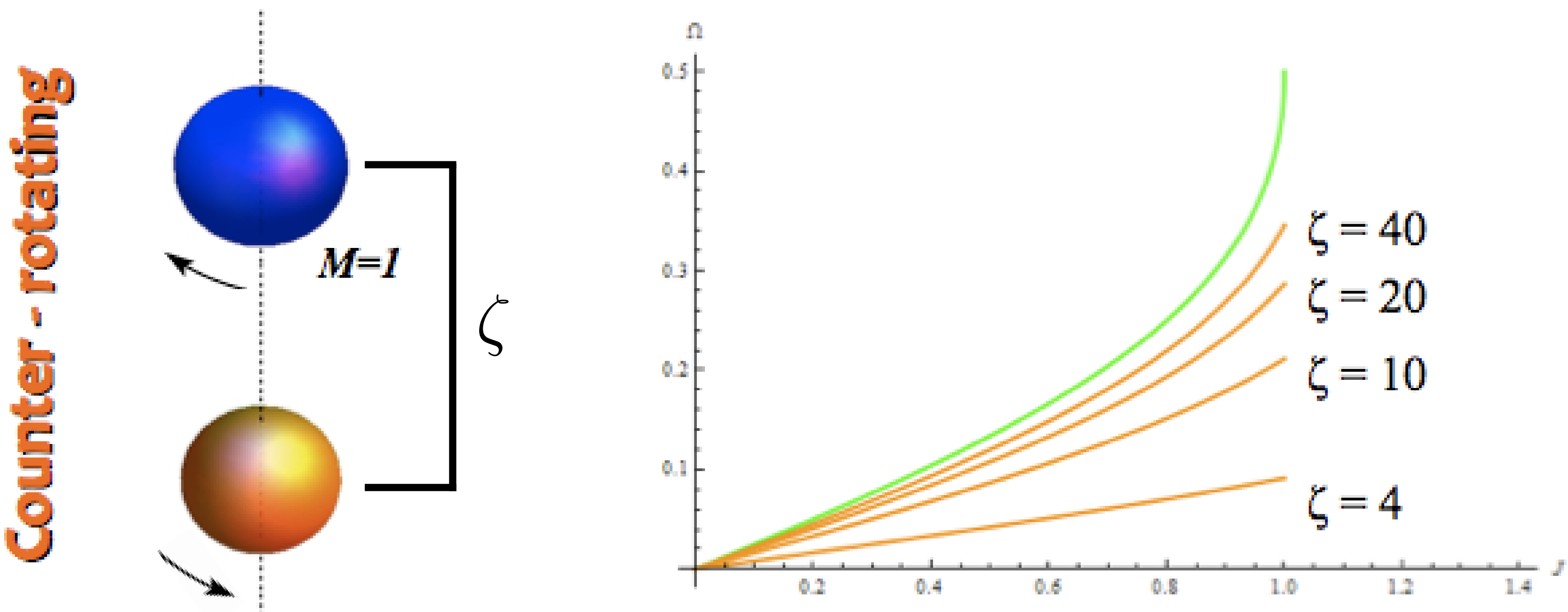
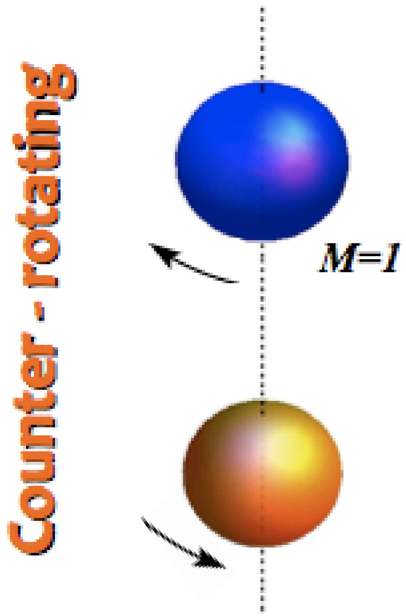
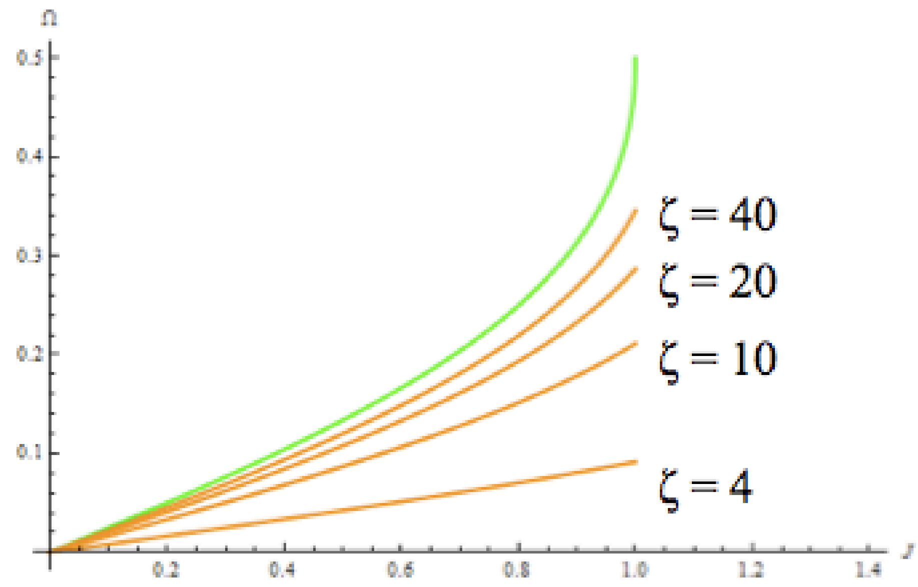


$$\Omega_1 = \frac{J}{2M\left(M^2+\sqrt{M^4-J^2\frac{\zeta-2M}{\zeta+2M}}\right)}\frac{\zeta-2M}{\zeta+2M} = -\Omega_2$$

Due to the (counter) rotational dragging of the other black hole each black hole has a **smaller** angular velocity than if isolated.





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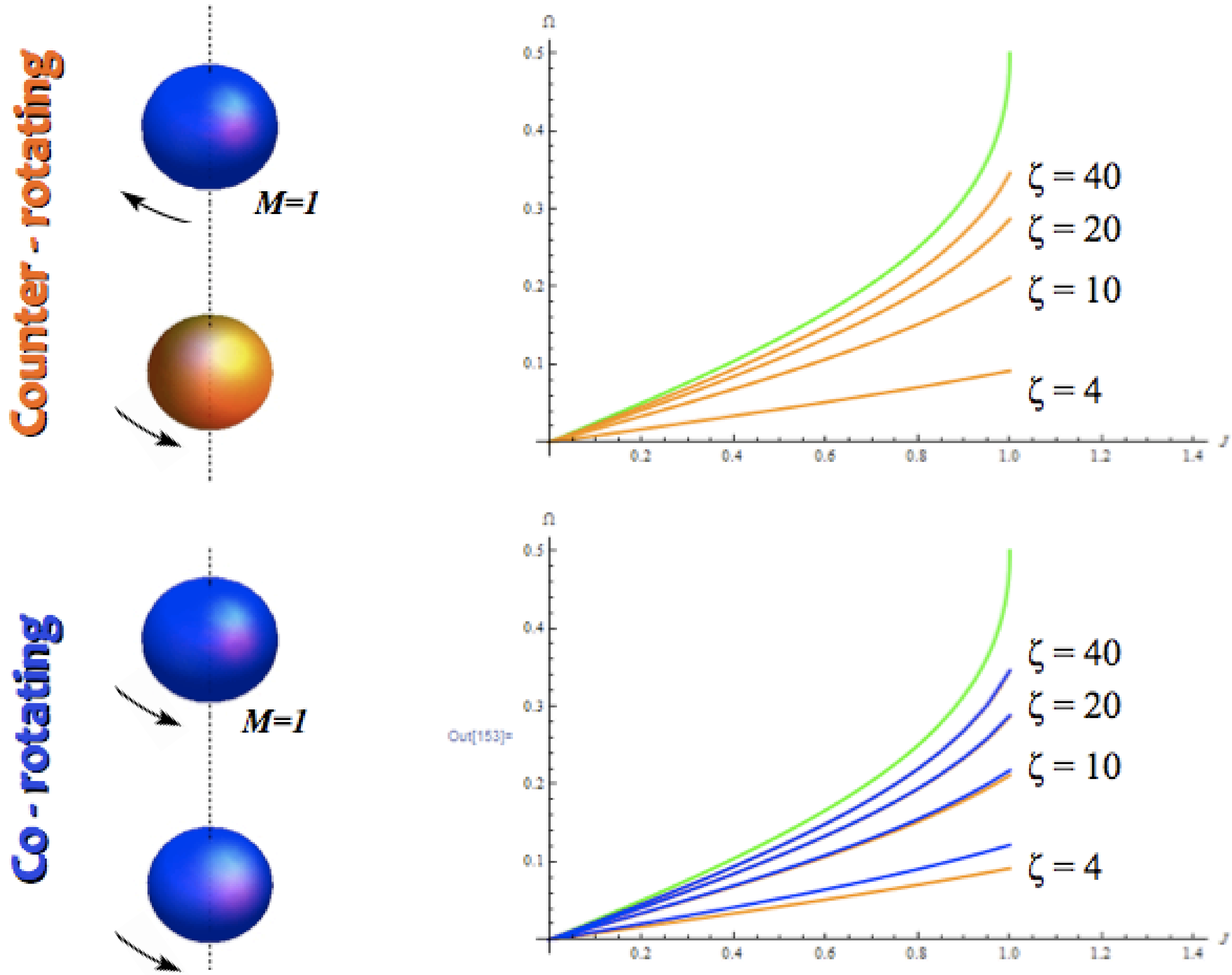
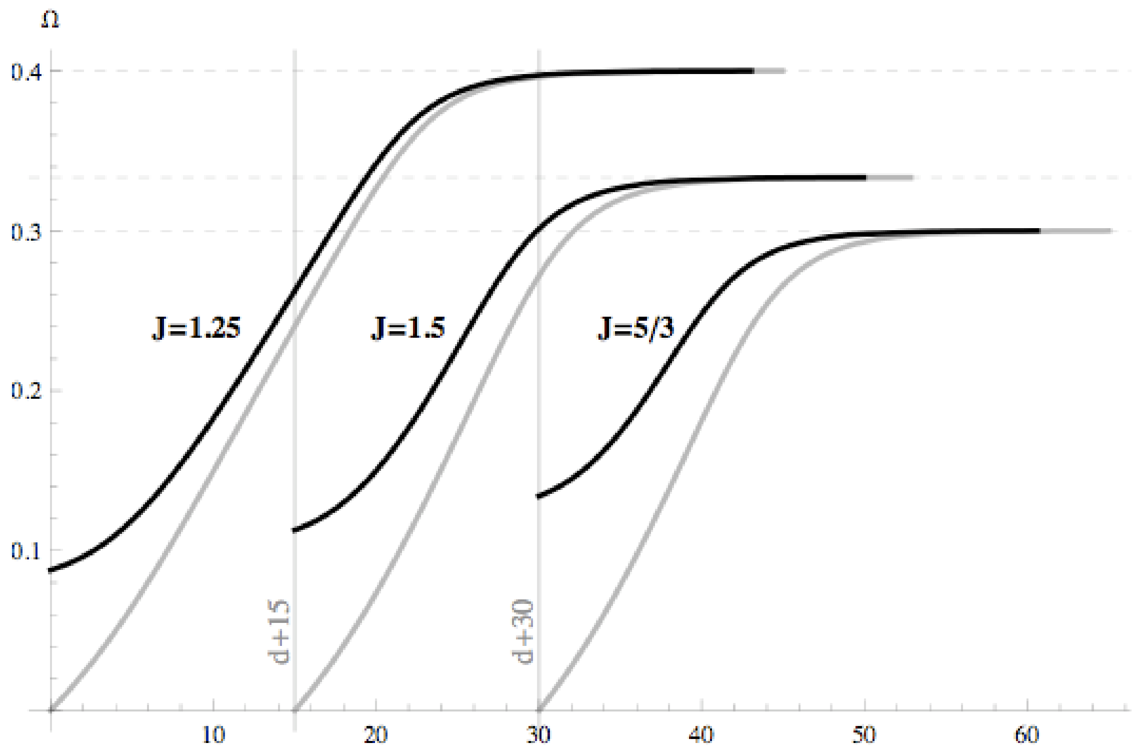


Comparison of co-rotating with counter rotating case:

Exact angular velocity of the horizon in terms of physical distance:



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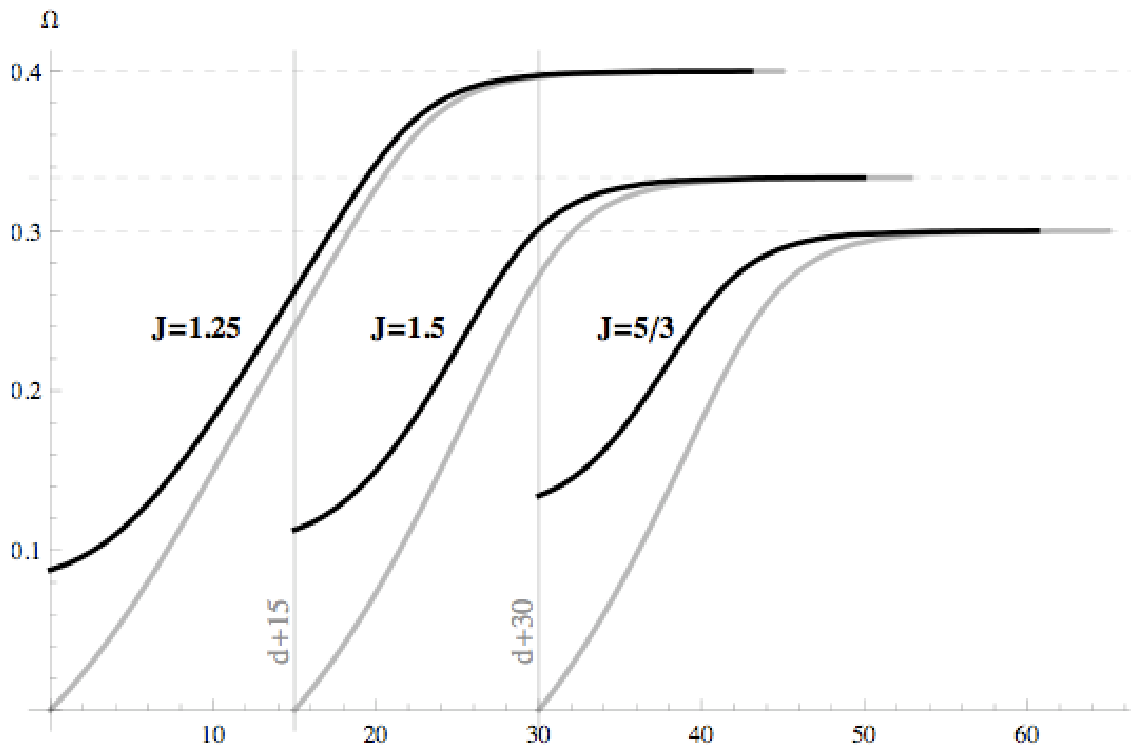

Two competing effects:
i) rotational co-dragging by other BH;
ii) rotational counter-dragging due to the dragging of the other BH;

Second dominates; each BH has a **smaller** angular velocity than if it would be isolated but **higher** in the co-rotating than in the counter-rotating case

If BHs have smaller angular velocity (than if isolated) can we give them more angular momentum for the same mass (than if isolated)? In other words, do they have the same extremal limit?

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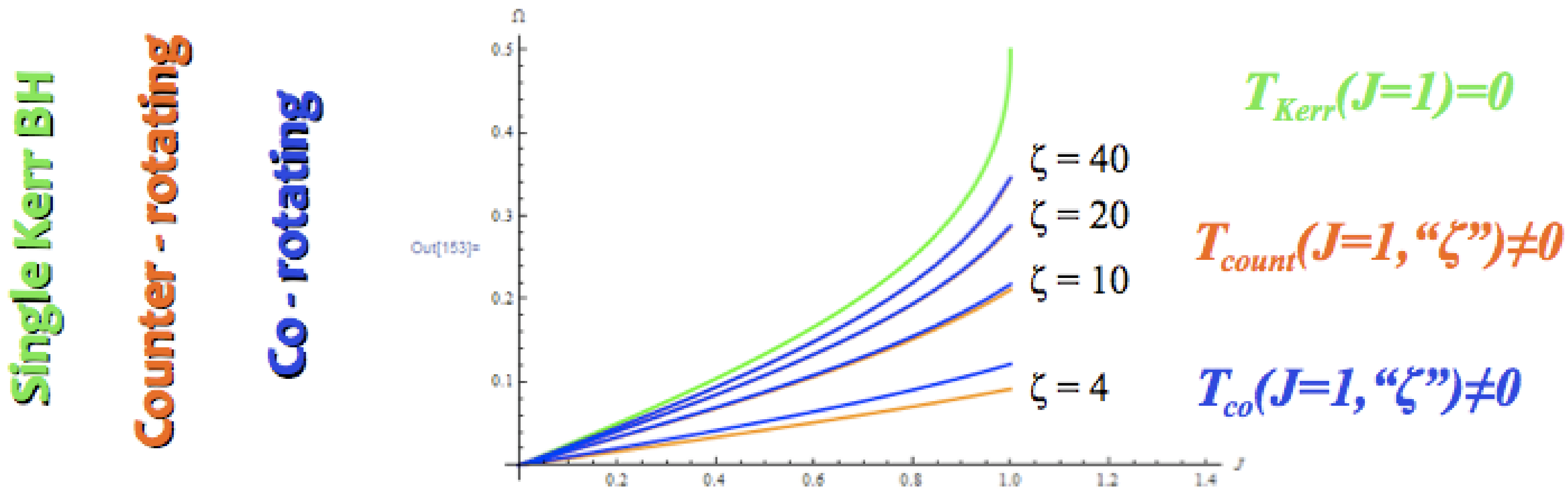

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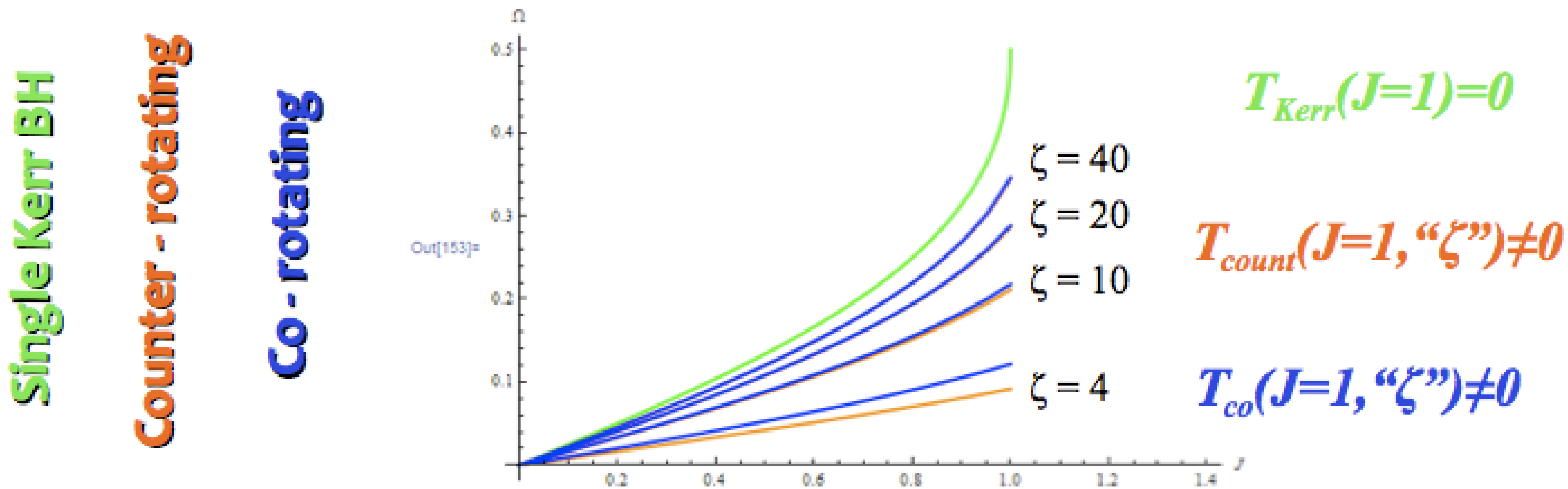
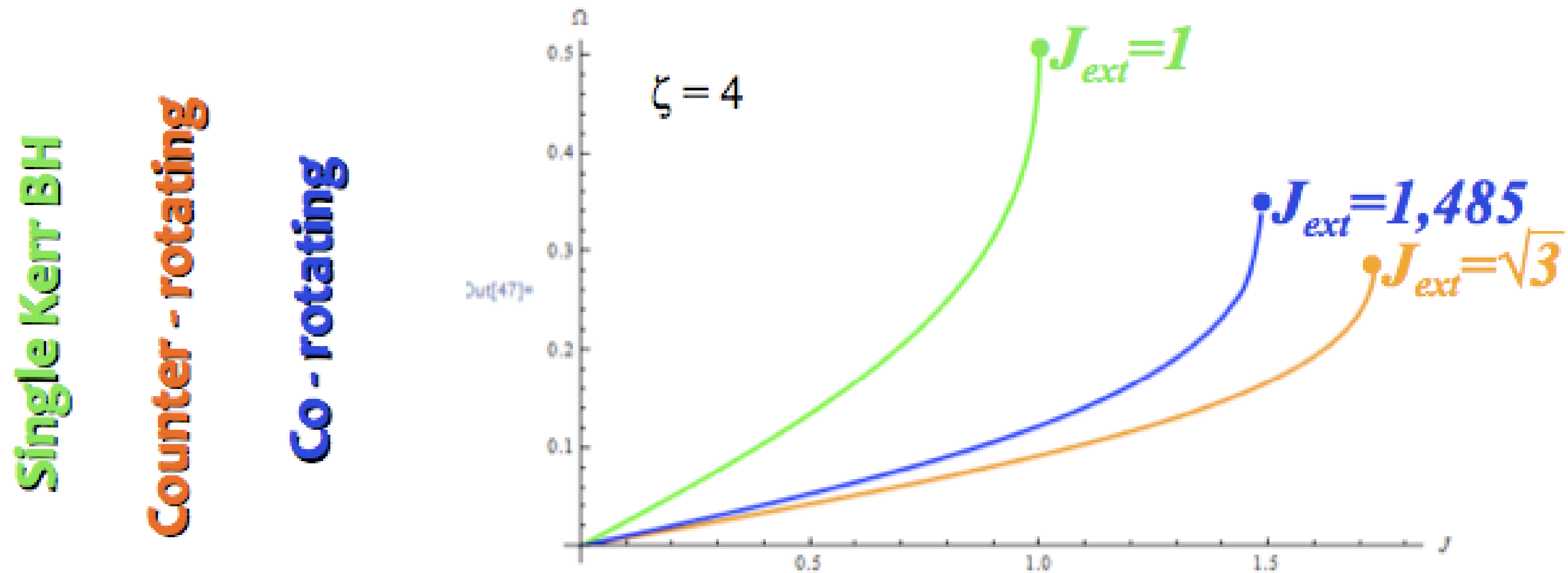


How much more angular momentum can we give them?

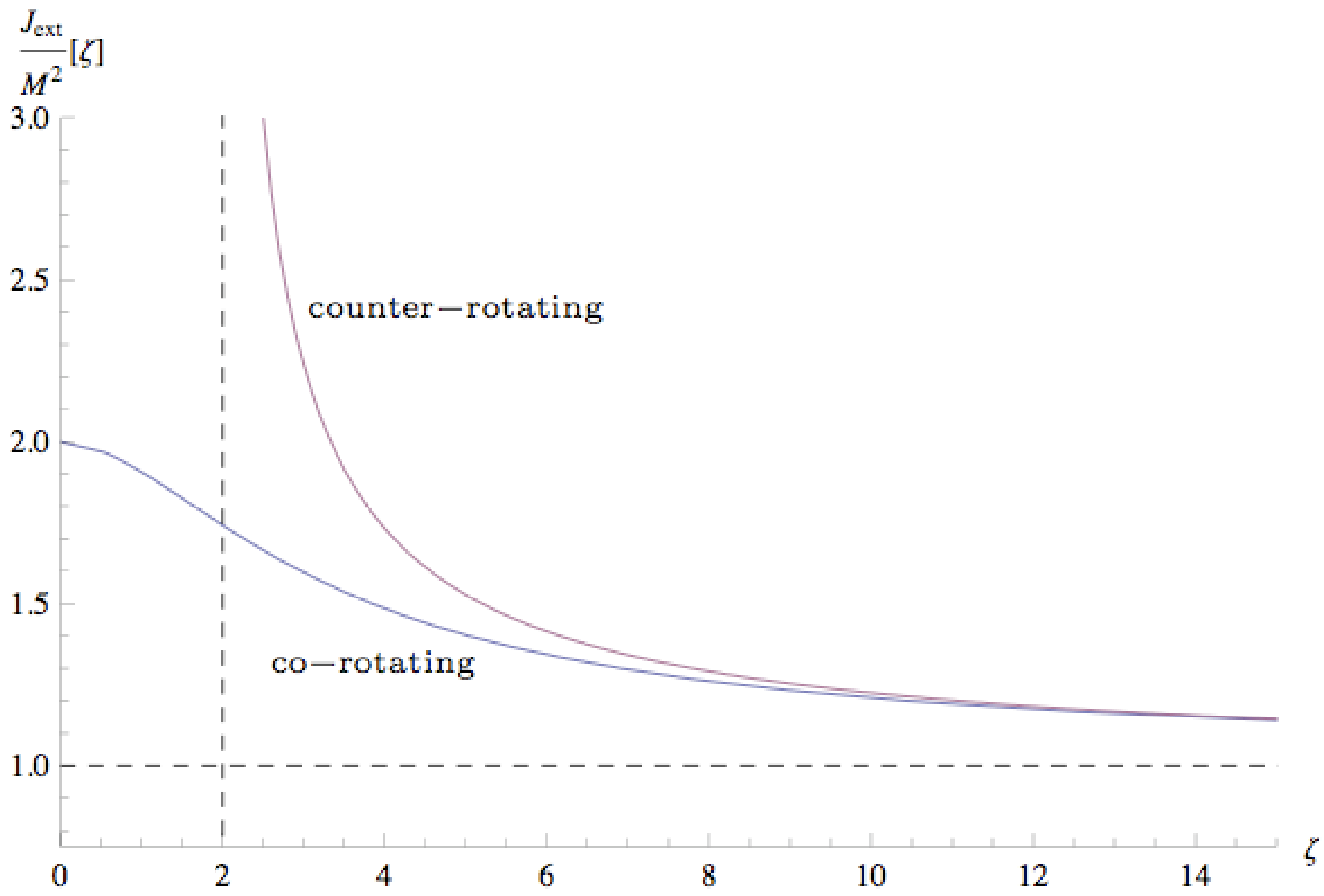
Example for a particular (coordinate) distance:



Extremality condition

Counter rotating case:

$$\frac{|J|}{M^2} = \sqrt{\frac{\zeta + 2M}{\zeta - 2M}}$$



Fixed M=1

Can exceed unity!

Novel feature in four dimensional black holes in general relativity.

The slow down effect can be seen at the level of the horizon geometry:

Single Kerr black hole: In Weyl canonical coordinates, horizon is located at:

$$\rho = 0 \ , a_1 \leq z \leq a_2$$

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Define new polar coordinate: $2z = a_1 + a_2 + a_{21} \cos\theta \ ,$

Induced metric on the horizon becomes:

$$ds_H^2 = A\left[g(u)d\phi^2 + \frac{du^2}{g(u)}\right] \ , \qquad u \equiv \cos\theta \ .$$

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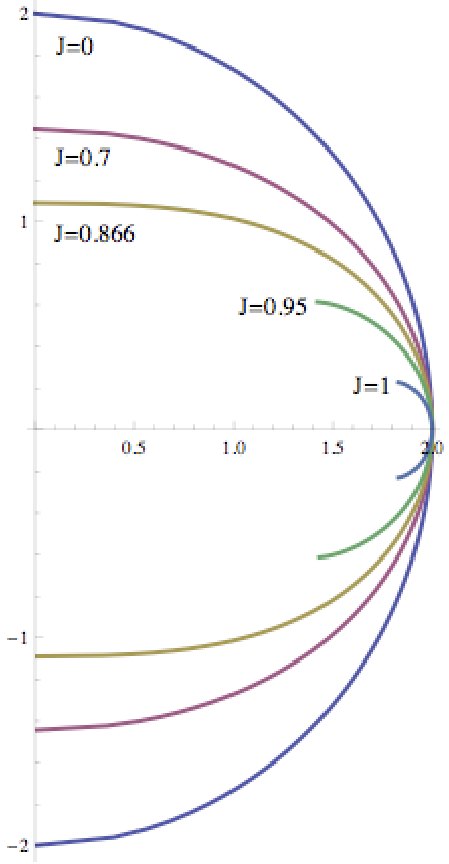
The spatial section of the horizon can then be embedded in Euclidean 3-space by using the embedding functions:

$$X + iY = \sqrt{Ag(u)}e^{i\phi} \ , \qquad Z' = \sqrt{\frac{A}{g(u)}\left(1 - \frac{g'(u)^2}{4}\right)} \ ,$$

For the single Kerr with M=1
and different angular momenta

Smarr '73:



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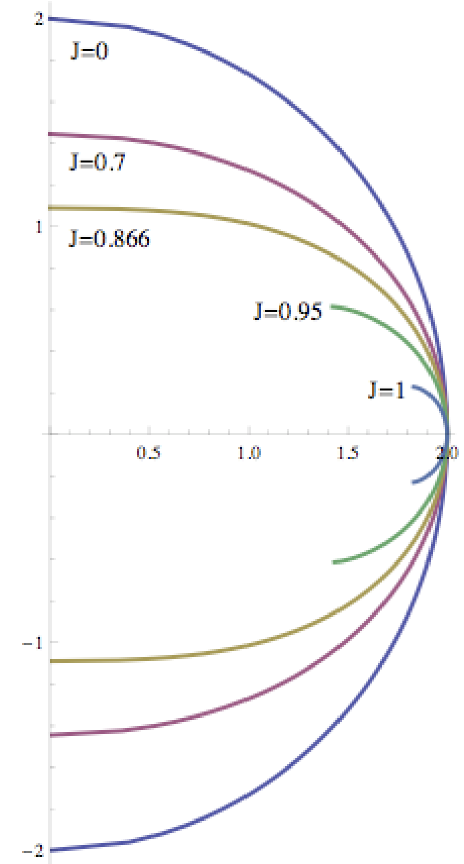
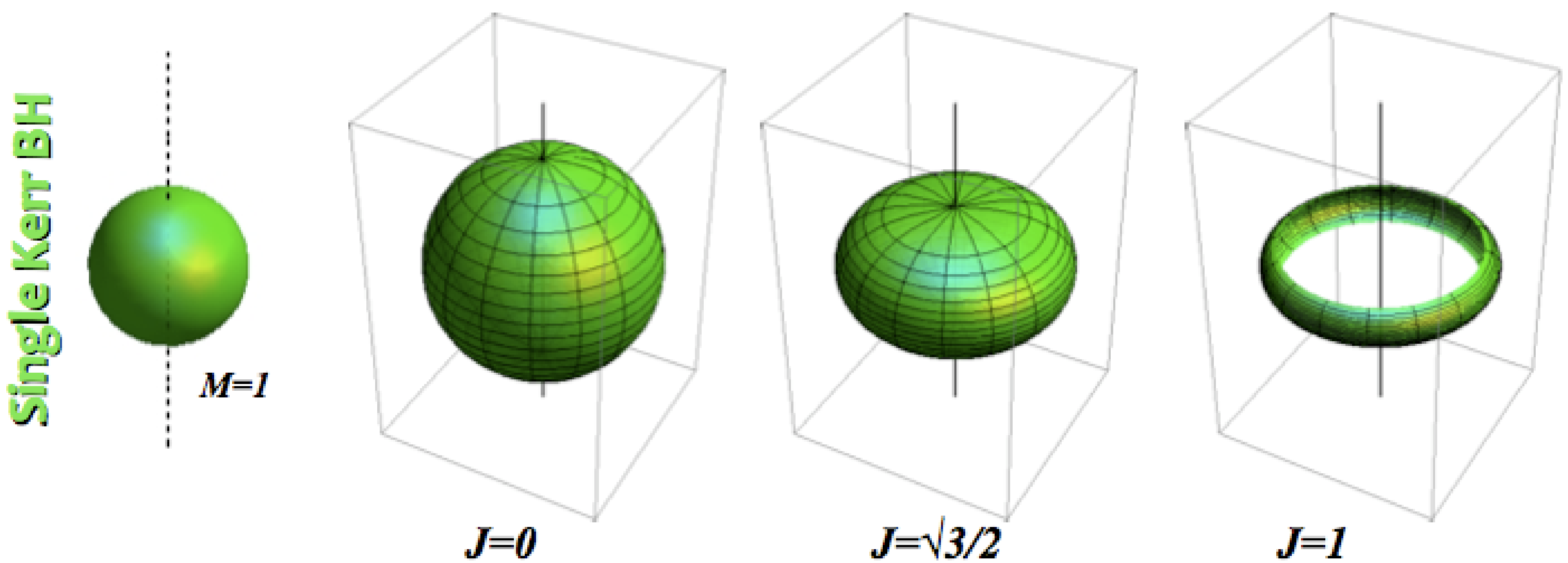
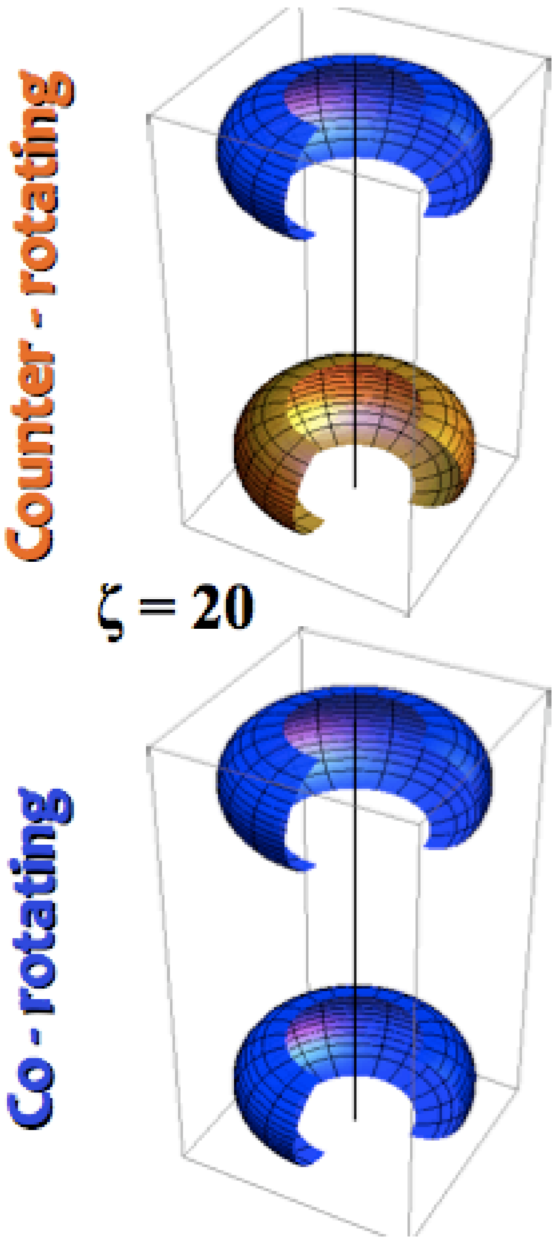




For the double Kerr with mass and angular momentum fixed:



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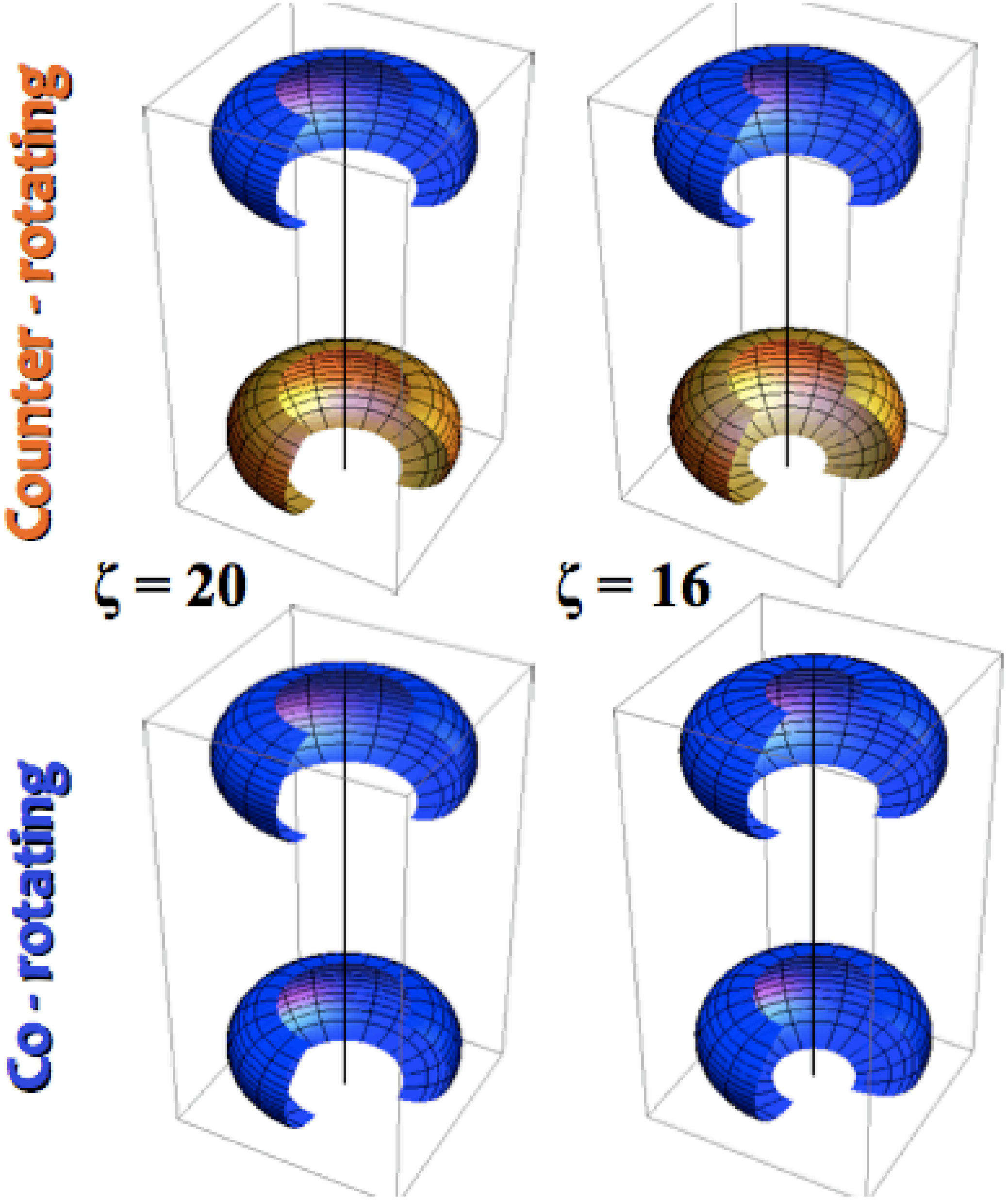

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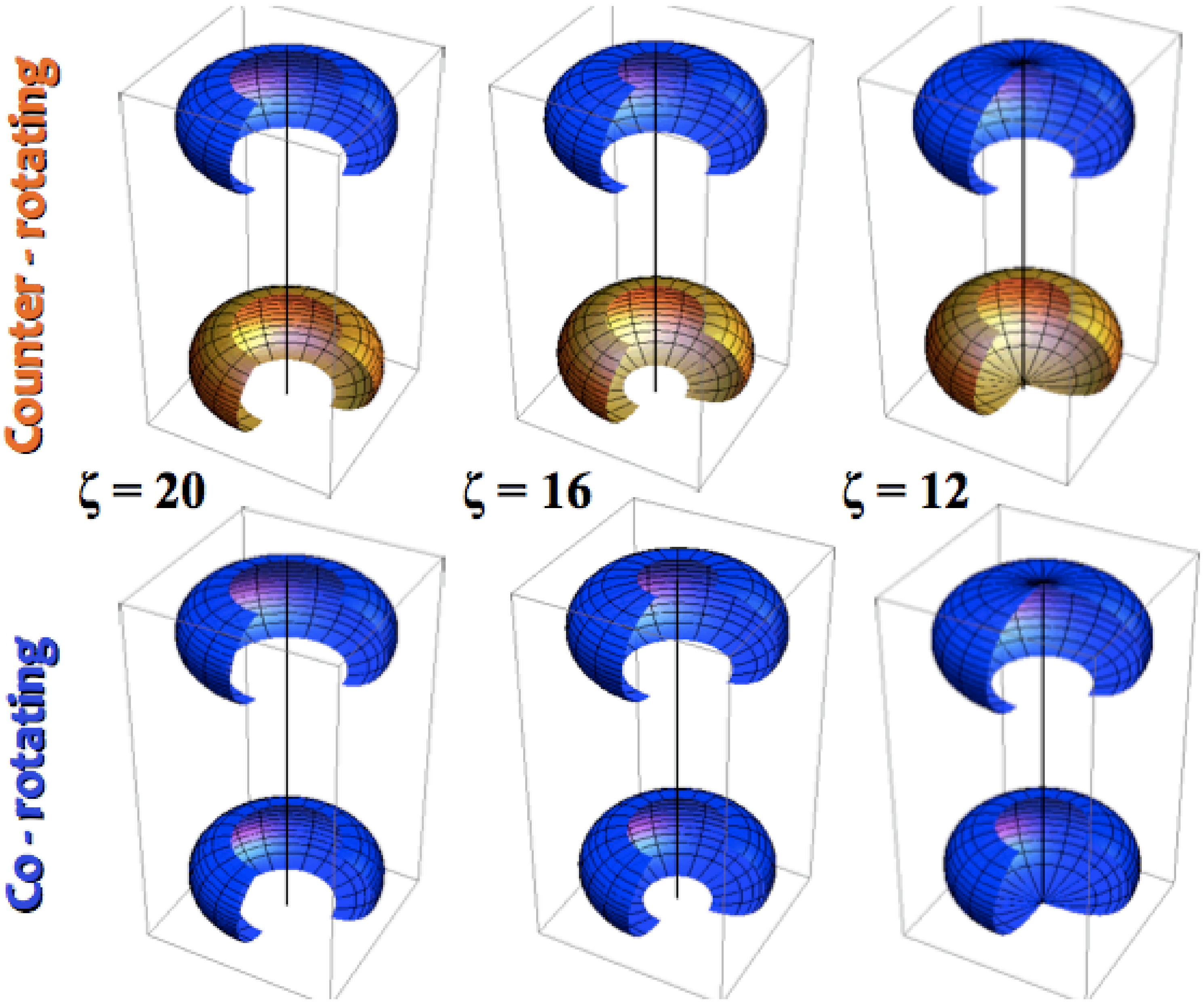

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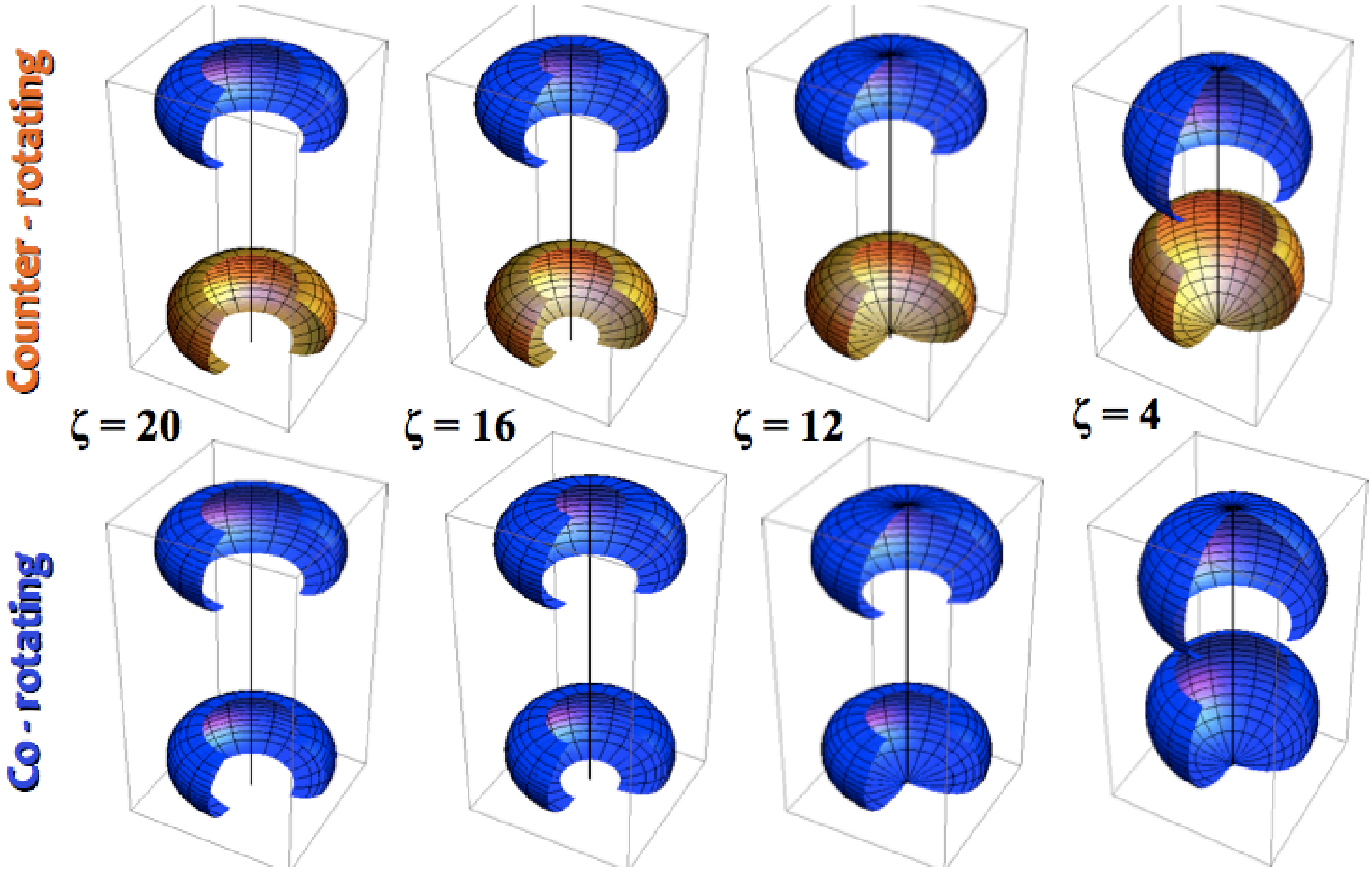
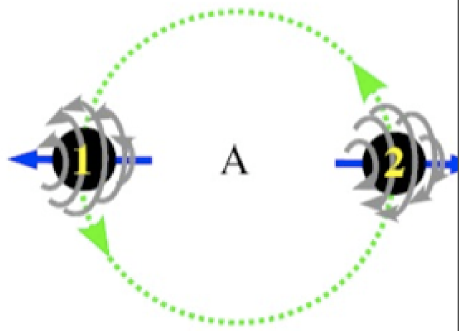


Dragging effects are seen in binary systems studied in numerical relativity.
Consider equal mass binary with anti-aligned spins; black hole go “up and down” due to frame dragging:



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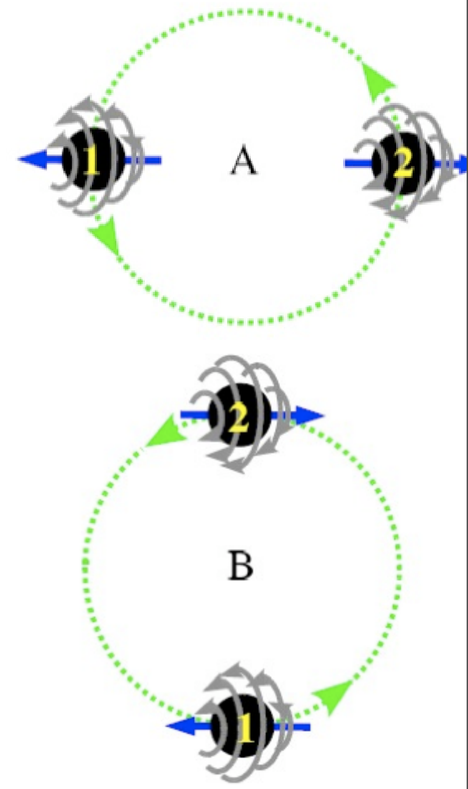
Down!



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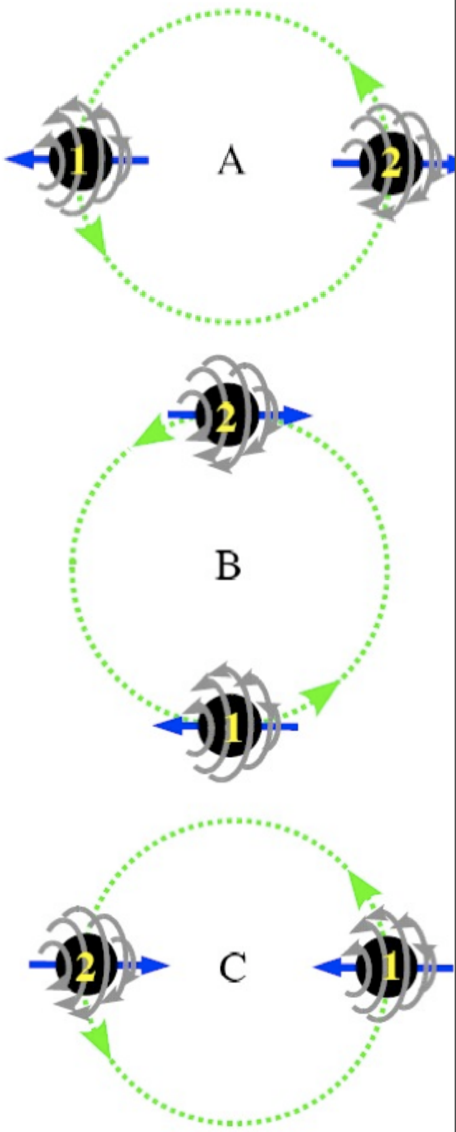

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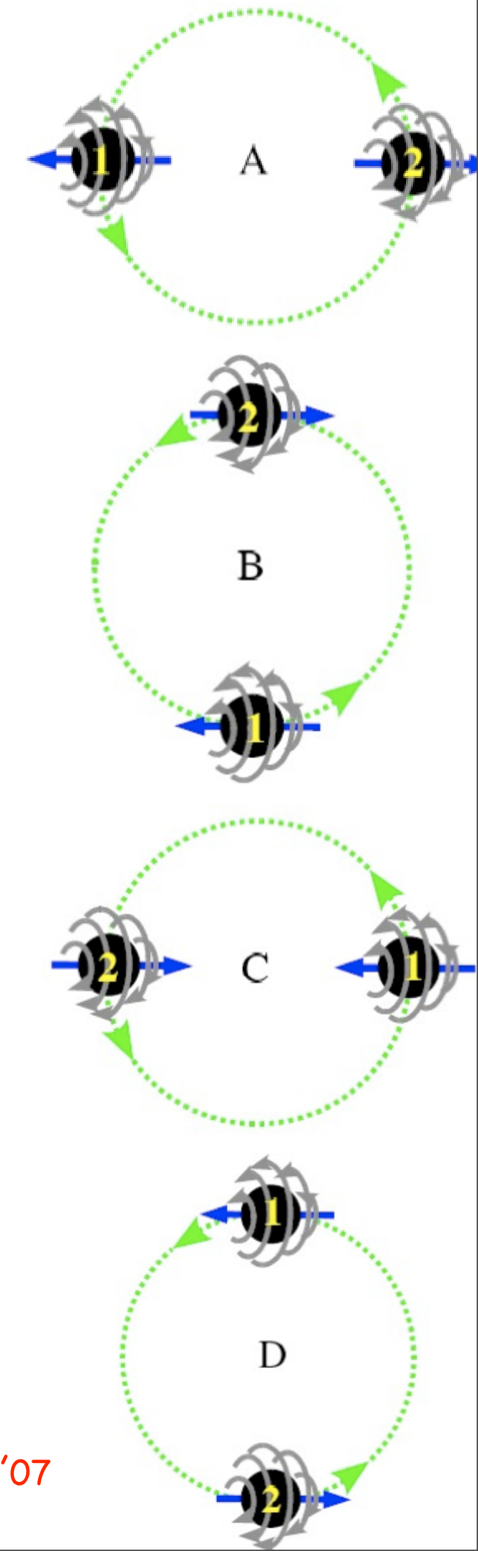

Down!

Up!

Pretorius '07

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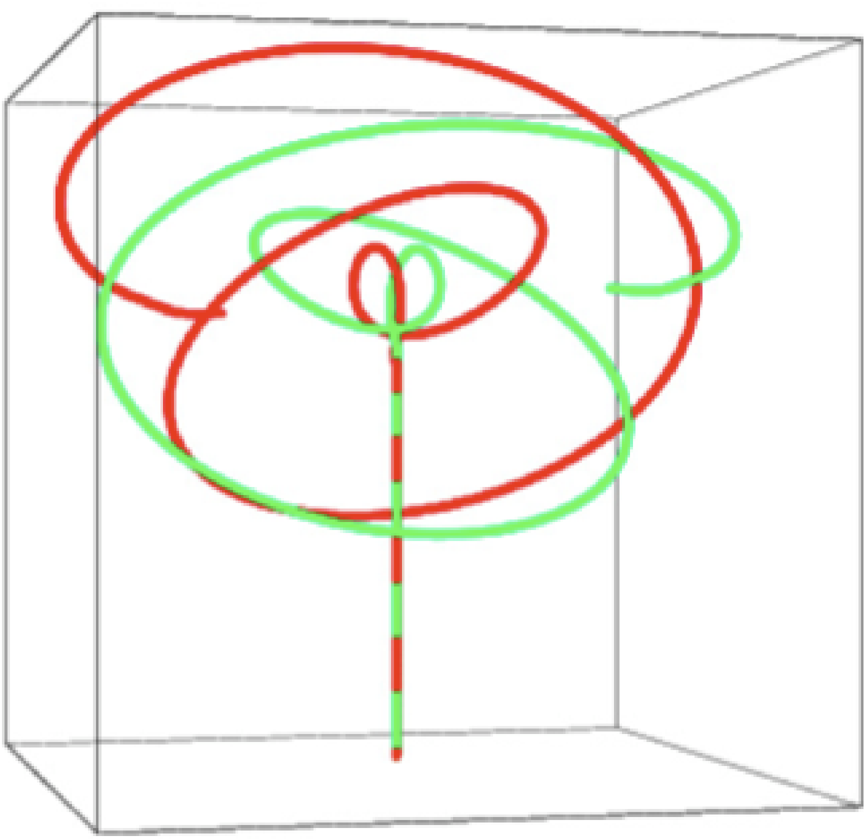
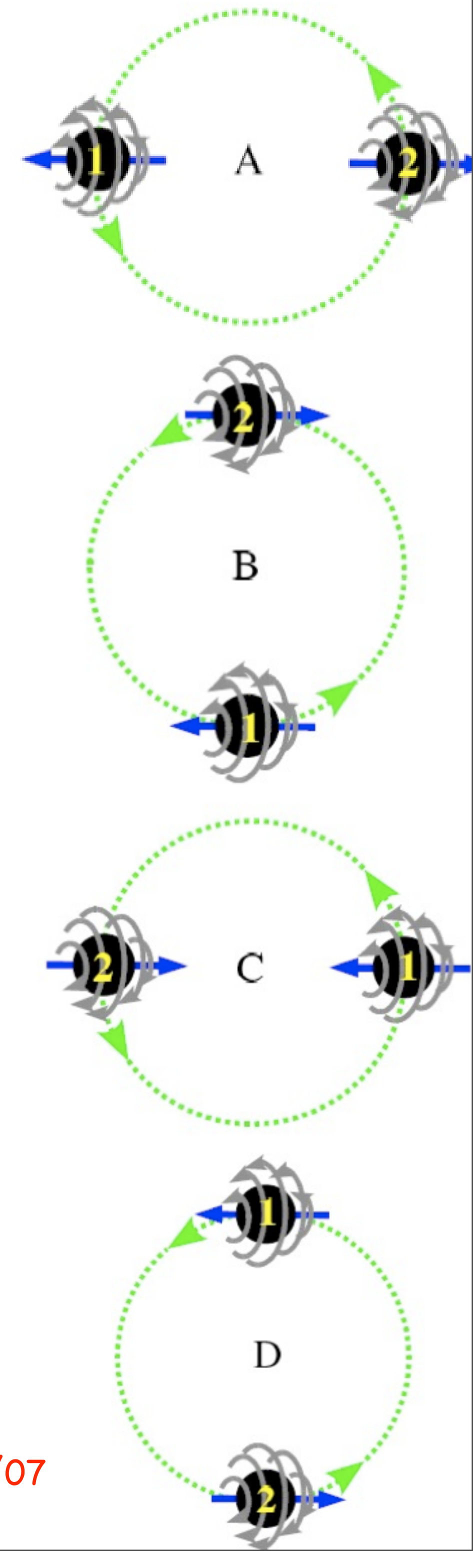


Gonzalez, Hannam, Sperhake, Bruegmann and Husa, '07

Down!

Up!

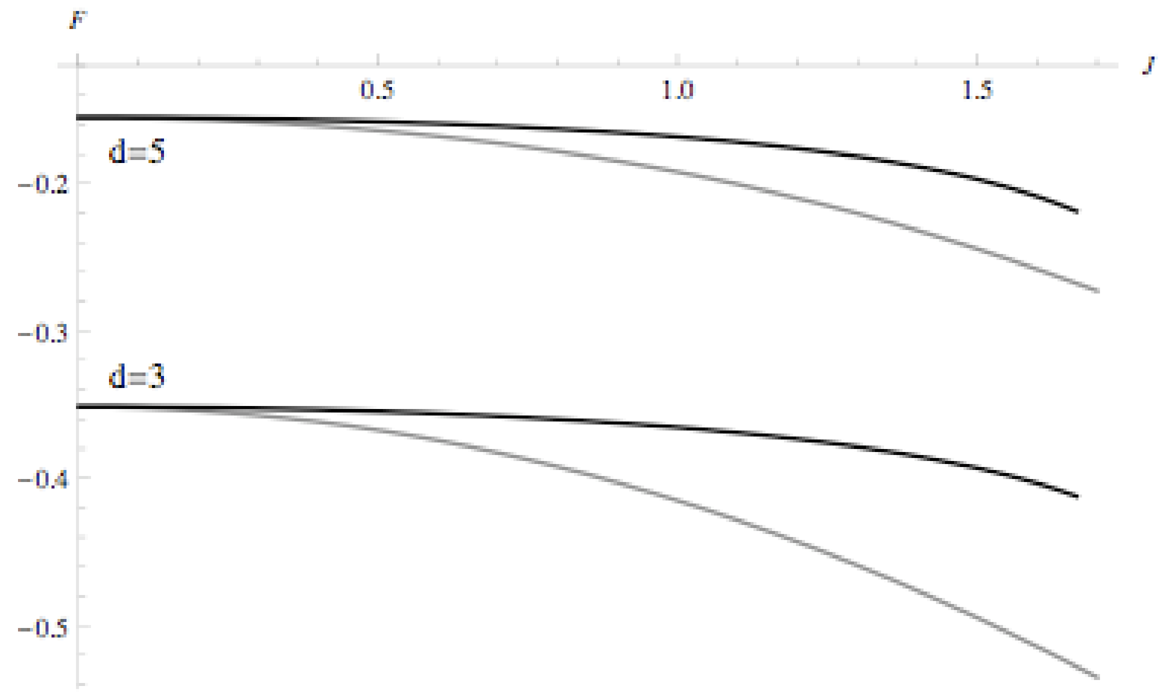


Pretorius '07

Force in terms of physical distance:

$$d(\zeta, M, J) = \int_{a_2}^{a_3} dz \sqrt{g_{zz}}|_{\rho=0} \ ,$$

Force, computed from the conical singularity, versus physical distance:



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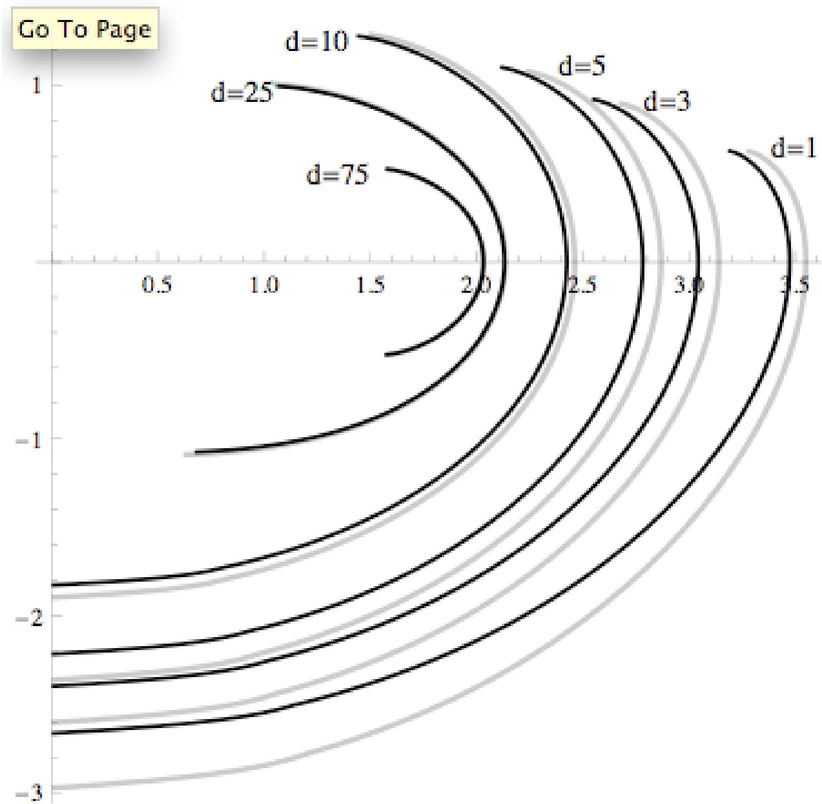

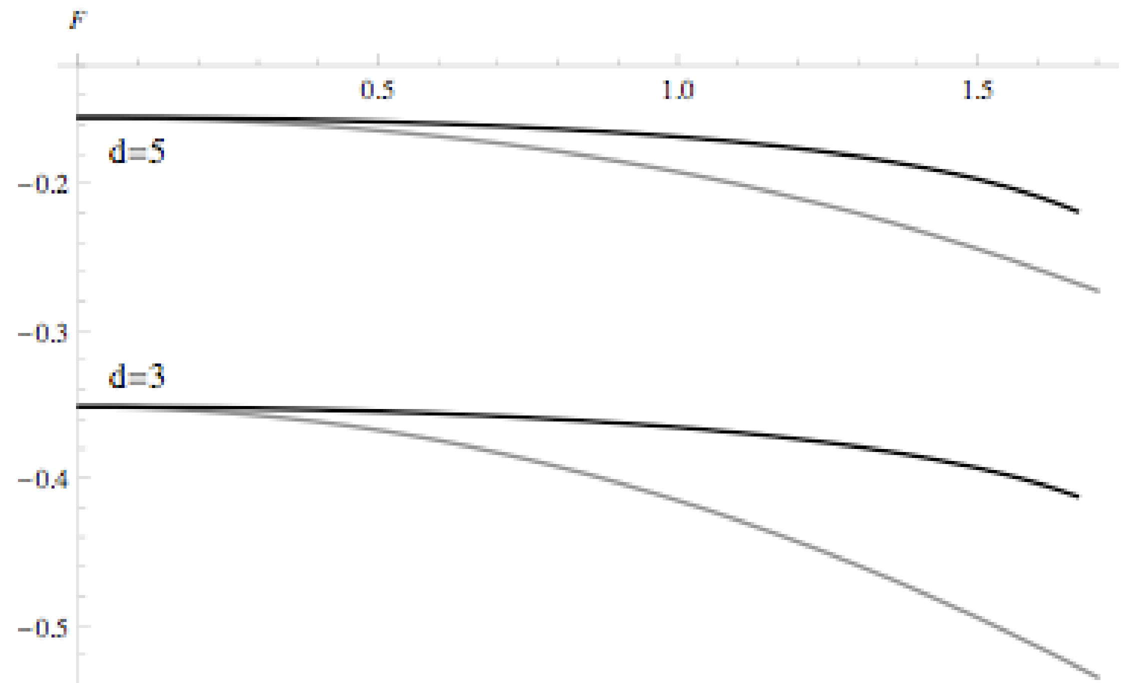

Quadrupole moment interaction is attractive and masks spin-spin interaction?
Lesson for higher dimensions?

Merging of ergo-regions

Universal behaviour Elvang, Figueras, Horowitz, Hubeny and Rangamani '08

For a d+1 dimensional vacuum solutions of the Weyl class with rotation in a single parameter, the merger angle is $\theta_m = 2\mathrm{arccot}\sqrt{\delta - 1}$

where $\delta = d - p$ and the merger is extended along p spatial dimensions.

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Thus the merger should occur at right angles.

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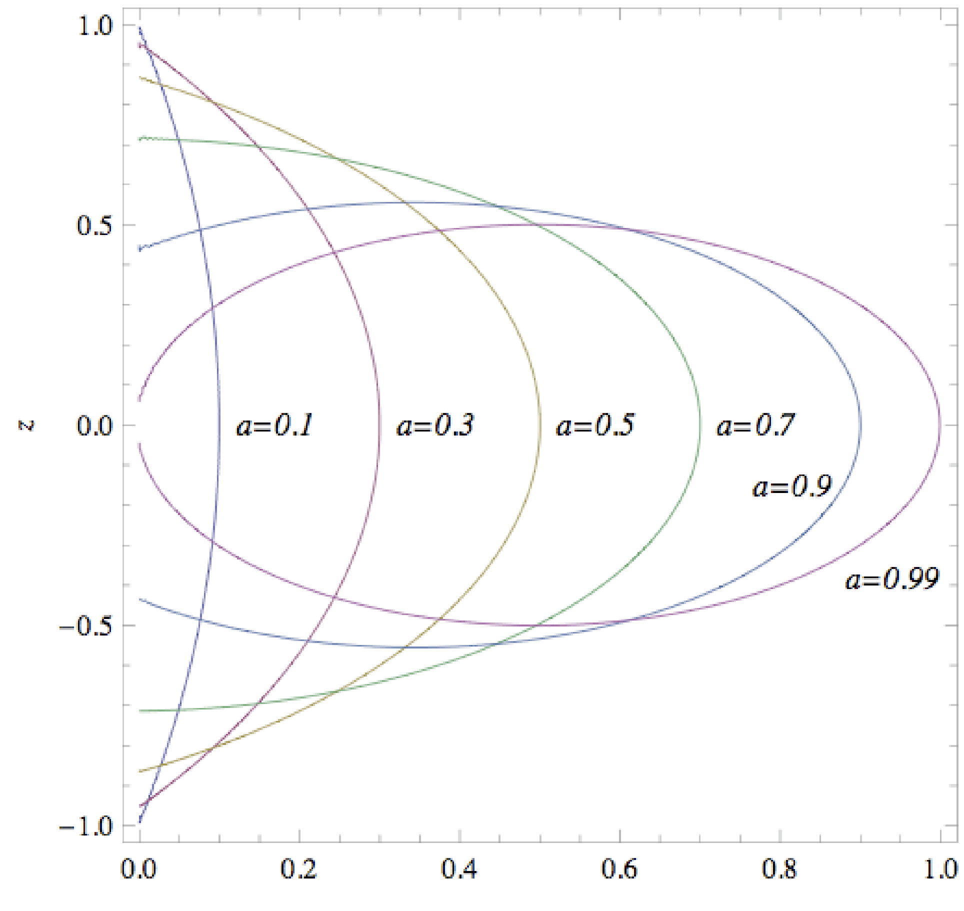
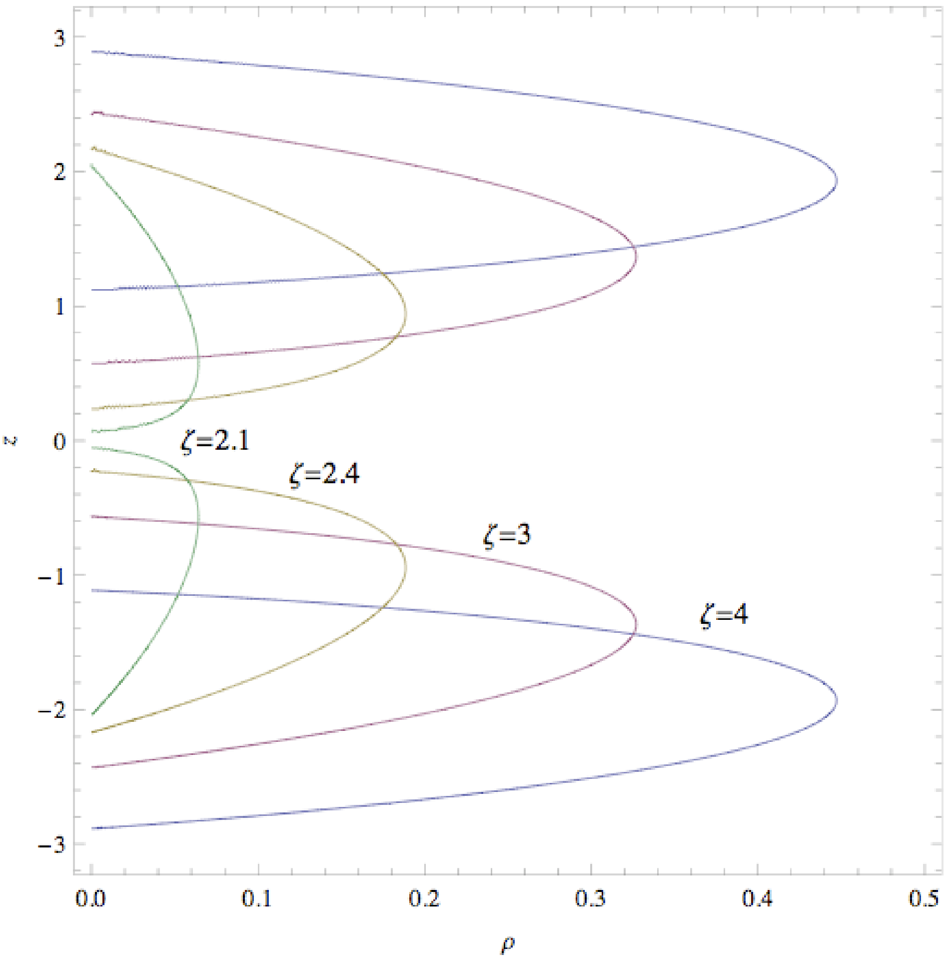
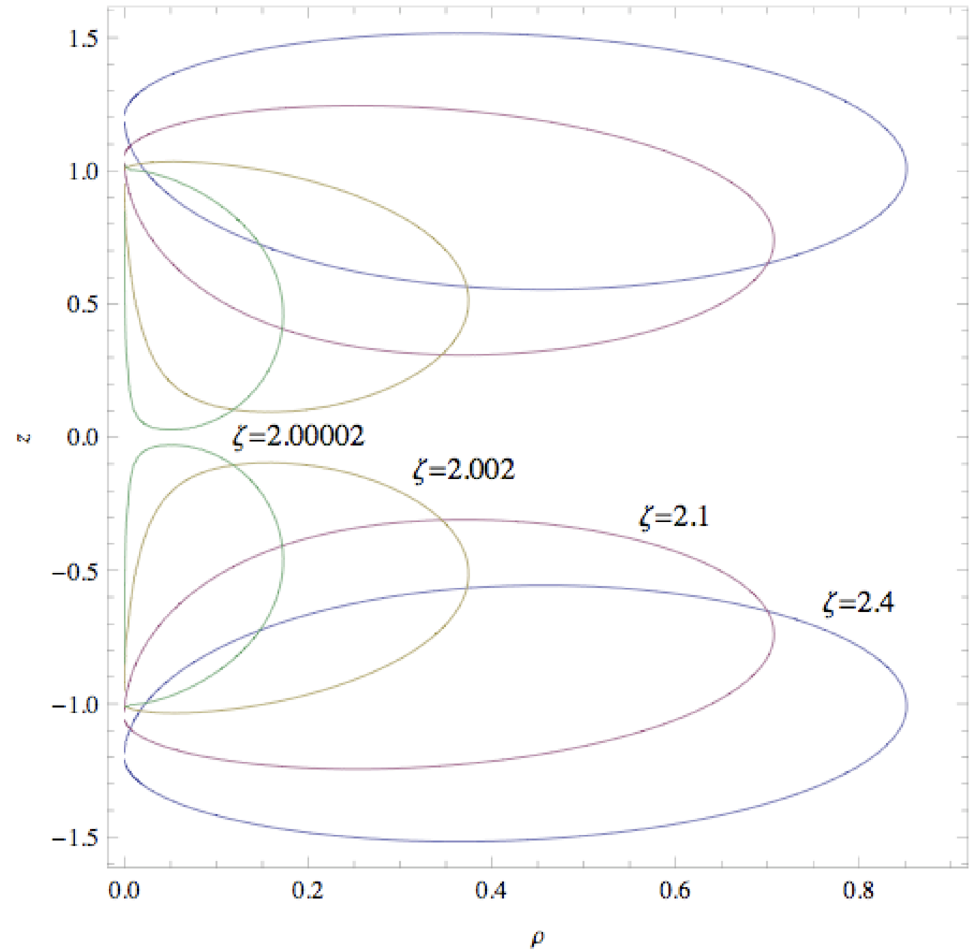
The single Kerr ergo-surface
in Weyl coordinates:



Counter-rotating case





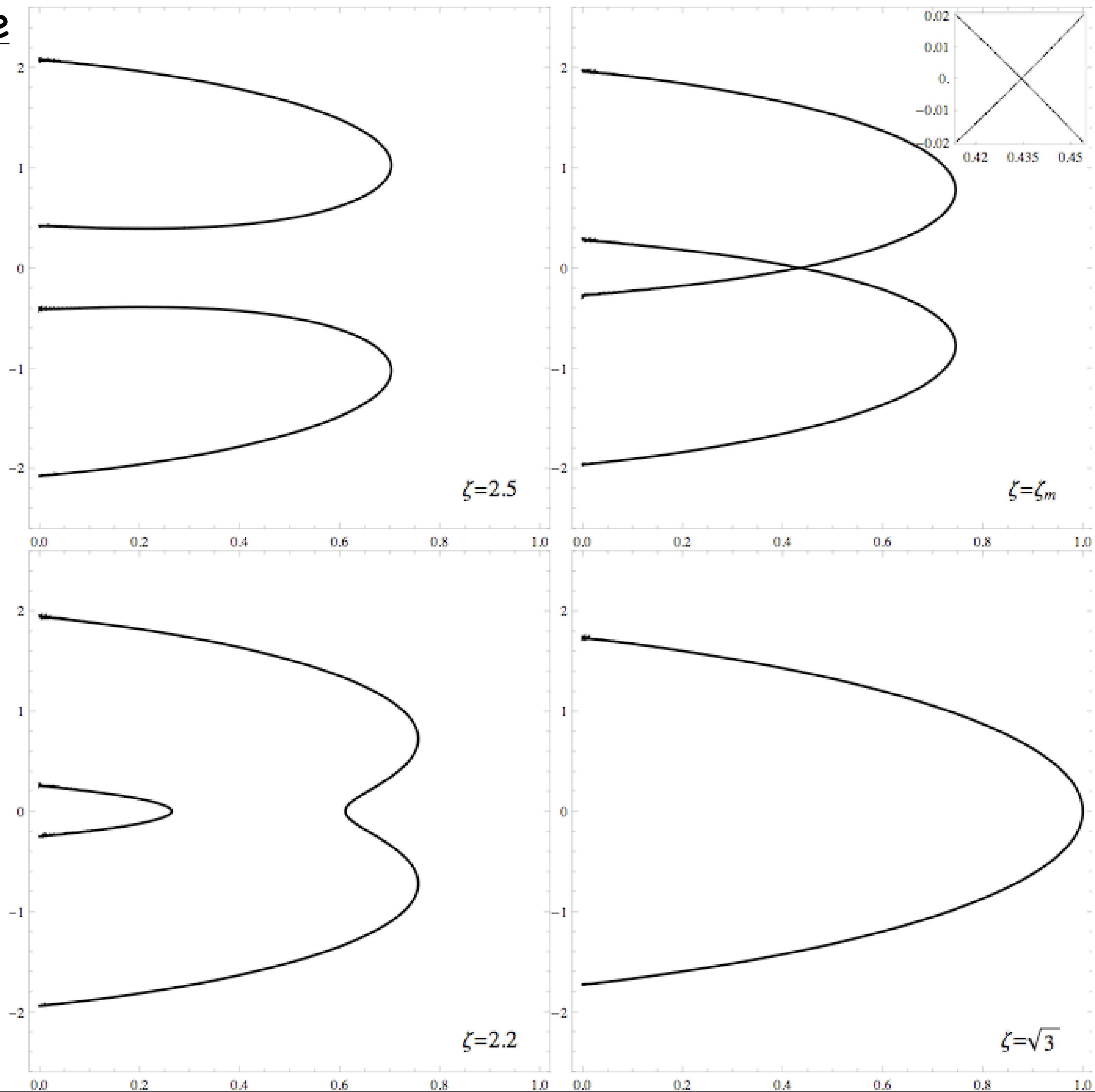
Fixed M=1, J=0.8 and varied distance

Fixed M=1 and kept extremality

Rotational dragging slow down is sufficiently fast so that the ergoregions never merge.

Co-rotating case

Fixed M=1=J and varied distance



Merging at right angles as expected

3) Final remarks

Other applications of the double Kerr solution:

i) Comparison with black saturn

ii) Comparison with Bonnor dipoles Emparan '99, Emparan and Teo ´01;

iii) Geodesic integrability? Will '08

Equilibrium configurations of Myers Perry black holes; do they exist? Naively equilibrium may be possible in six or higher; quadrupole moments?

Could one see the consequences of the spin-spin interaction and dragging effects in head on collision of spinning black holes in numerical relativity?