

Bianchi Cosmologies with Anisotropic Matter

Simone Calogero

Department of Applied Mathematics
University of Granada, Spain

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Joint work with [Mark Heinzle](#) (University of Vienna)

Main Goal

Classify the asymptotic dynamics of spatially homogeneous spacetimes with a general class of matter models.

We study the Einstein-matter equations under the assumption of spatial homogeneity using dynamical systems techniques. The matter model is not given explicitly but only through a set of physically motivated assumptions.

So far we have analyzed the following spatially homogeneous spacetimes:

- ▶ Bianchi type I spacetimes
- ▶ Bianchi class A spacetimes with Local Rotational Symmetry

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- ▶ We propose a class of [anisotropic](#) matter models that naturally generalize perfect fluids and which is defined through a set of physically motivated assumptions.
- ▶ There exist important explicit examples of matter models included in our class (Magnetic fields, Vlasov matter, Elastic matter).

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- ▶ Each orbit of the reduced dynamical system identifies in a unique way a family of solutions to the Einstein-matter equations.
- ▶ We use tools from dynamical systems theory (stability of fixed points, monotonicity principle, Poincaré-Bendixson theorem, etc.) to identify the α/ω -limit sets of the orbits to the reduced dynamical system \longrightarrow **asymptotic behavior of the spacetime**.

Spatially Homogeneous Spacetimes

We consider a spacetime (M, \bar{g}) with a 3-dimensional group G of isometries whose orbits are spacelike hypersurface that foliate M .

Let t denote the proper time of the geodesic congruence orthogonal to the orbits of G .

Let ω^i , $i = 1, 2, 3$ denote a **time independent** group invariant frame on the orbits of G . The metric can be written in the form

$$\bar{g} = -dt^2 + g_{ij}(t)\omega^i\omega^j$$

$g_{ij}(t_0)$ is the Riemannian metric induced on the orbit $t = t_0$ of the group G .

Class A Bianchi models

For a spatially homogeneous model of Bianchi Class A the dual forms ω^i can be chosen to satisfy

$$d\omega^1 = -n_1 \omega^2 \wedge \omega^3, \quad d\omega^2 = -n_2 \omega^3 \wedge \omega^1, \quad d\omega^3 = -n_3 \omega^1 \wedge \omega^2$$

Bianchi	n_1	n_2	n_3
I	0	0	0
II	1	0	0
VI ₀	0	1	-1
VII ₀	0	1	1
VIII	-1	1	1
IX	1	1	1

The Ricci curvature R_{ij} of the group orbits is determined by the structure constants (n_1, n_2, n_3) .

Einstein equations

The Einstein equations (in units $8\pi G = c = 1$) split into **evolution equations**

$$\partial_t g_{ij} = -2k_{ij}, \quad \partial_t k^i_j = R^i_j + (g^{lm} k_{lm}) k^i_j - T^i_j + \frac{1}{2} \delta^i_j (g^{lm} T_{lm} - \rho).$$

and **constraint equations**

$$R - k^{ij} k_{ij} + (g^{ij} k_{ij})^2 = 2\rho, \quad \nabla_i k^i_j - \nabla_j (g^{lm} k_{lm}) = -T_{0j},$$

R_{ij} is the Ricci tensor of the Riemannian metric g_{ij} (which depends on n_1, n_2, n_3)

$T_{\mu\nu}$ is the matter energy-momentum tensor, with $T_{00} = \rho$.

Assumptions on the matter model

The most commonly used energy-momentum tensor in cosmology is that of a (non-tilted) perfect fluid:

$$T_{\mu\nu} = \rho(dt \otimes dt)_{\mu\nu} + p[\bar{g}_{\mu\nu} + (dt \otimes dt)_{\mu\nu}] ,$$

where the **pressure** p and the **energy density** ρ are usually required to obey a **linear equation of state**,

$$p = w\rho , \quad w = \text{const.}$$

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The Bianchi identities $\nabla_{\mu} T^{\mu\nu} = 0$ can be solved exactly to express ρ , and therefore the entire $T_{\mu\nu}$, in terms of the spatial metric:

$$\rho = \rho_0 n^{1+w} , \quad \rho_0 = \text{constant} , \quad n = (\det g)^{-\frac{1}{2}} .$$

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The components of the energy-momentum tensor are represented by smooth (at least C^1) functions of the metric g_{ij} . We assume that $\rho = \rho(g_{ij})$ is positive (as long as g_{ij} is non-degenerate).

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Remark. In the previous assumption it is understood that also the **matter fields** which (possibly) enter the definition of the energy-momentum tensor can be written in terms of the metric g_{ij} by solving the Bianchi identities $\nabla_\mu T^{\mu\nu} = 0$ (this might be possible only under certain restrictions on the initial data for the matter fields).

Lemma

Under Assumption 1, the spatial component T^i_j of the energy momentum tensor w.r.t. the frame ω^i satisfy

$$T^i_j = -2 \frac{\partial \rho}{\partial g_{il}} g_{jl} - \delta^i_j \rho .$$

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Assumption (diagonal models)

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By the Einstein equations it follows that (g_{ij}, k^i_j, T^i_j) remain diagonal for all times.

We set

$$T_j^i = \text{diag}(p_1, p_2, p_3).$$

where p_i are the **principal pressures** and

$$p = \frac{1}{3}(p_1 + p_2 + p_3).$$

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Assumption

We suppose that the isotropic pressure and the density are proportional, i.e., we assume $p = w\rho$, $w = \text{const}$, where

$$w \in \left(-\frac{1}{3}, 1\right).$$

Accordingly, the density and the isotropic pressure behave like those of a perfect fluid with a linear equation of state satisfying **the strong and the dominant energy condition**.

Lemma

For $i = 1, 2, 3$ (no sum over i) let $n = \sqrt{g^{11}g^{22}g^{33}}$ and

$$s_i = (g^{11} + g^{22} + g^{33})^{-1} g^{ii} \quad \Rightarrow \quad s_1 + s_2 + s_3 = 1.$$

There exists a function $\psi = \psi(s_1, s_2, s_3)$ such that the components of the energy-momentum tensor can be written as

$$\rho = n^{1+w} \psi(s_1, s_2, s_3), \quad p_i = w_i \rho,$$

where the *normalized principal pressures* w^i are given by

$$w_i(s_1, s_2, s_3) = w + 2 \left(\frac{\partial \log \psi}{\partial \log s_i} - s_i \sum_l \frac{\partial \log \psi}{\partial \log s_l} \right).$$

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For $\psi = \text{constant}$ we obtain the energy-momentum tensor of a perfect fluid with linear equation of state $p = w\rho$.

Example (John's Elastic Materials¹.)

The function ψ is given by

$$\psi(s_1, s_2, s_3) = 1 + \mu \mathfrak{s}$$

where $\mu = \text{const} > 0$ (**modulus of rigidity**) and

$$\mathfrak{s} = (s_1 s_2 s_3)^{-1/3} - 3 \text{ is the } \text{shear scalar} .$$

For this matter model we obtain

$$w_i(s_1, s_2, s_3) = w + 2 \left(s_i - \frac{1}{3} \right) \frac{\mu}{1 + \mu \mathfrak{s}} .$$

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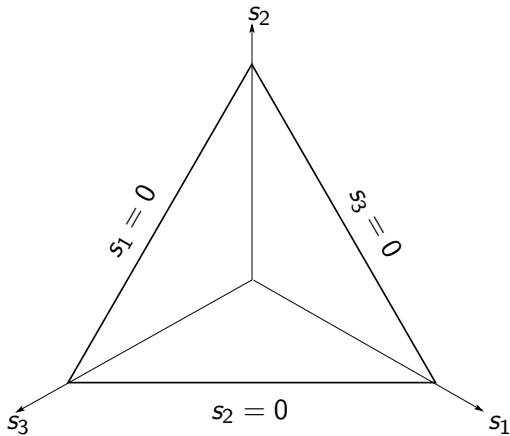
$$w_i(s_1, s_2, s_3) = w + 2 \left(s_i - \frac{1}{3} \right) \frac{\mu}{1 + \mu \mathfrak{s}} .$$

Remark. For elastic materials, the choice of the function ψ is equivalent to the choice of a **constitutive equation**.

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Since $s_1 + s_2 + s_3 = 1$, the functions $\psi(s_1, s_2, s_3)$ and $w^i(s_1, s_2, s_3)$ are defined in the interior of a triangle, which we call \mathcal{T} .

The boundary of the triangle \mathcal{T} corresponds to a singularity of the metric.



Assumption

We assume that the function ψ satisfies the following properties:

(i) It is invariant under permutations of the axes, i.e.,

$$\psi(s_1, s_2, s_3) = \psi(s_2, s_3, s_1) = \psi(s_3, s_1, s_2).$$

(ii) It has a smooth (at least C^1) extension on the boundary of \mathcal{T} .

(iii) It has only one critical point in the interior of \mathcal{T} and it is either a maximum or a minimum (or $\psi = \text{const.}$).

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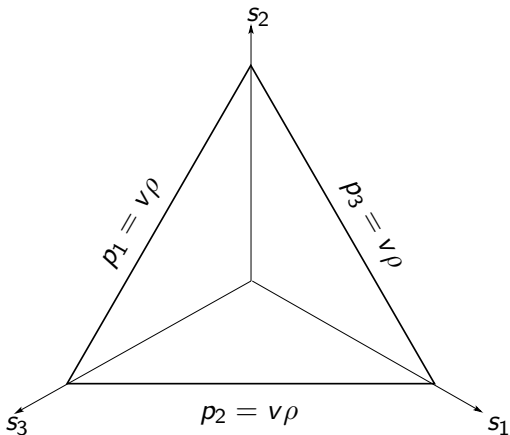
Remark. This assumption has the following physical interpretation:

- (i) means that the energy density is independent of the axes orientation
- (ii) is a regularity condition on the variables s_i
- (iii) is equivalent to require the existence of one and only one isotropic state of the matter ($w_i = w$, for all i) and that this state is either a maximum or a minimum state of energy.

The next assumption restricts the possible behavior of the matter source 'at the singularity' (and implies [asymptotic self-similarity](#)).

Assumption

We assume that there exists a constant v such that $w_i(s_1, s_2, s_3) = v$ for $s_i = 0$.



Definition

We define the **anisotropy parameter** β as

$$\beta = 2 \frac{w - v}{1 - w}$$

and classify the matter models in different types according to the values of β as shown in the Table.

Type	β values	Energy cond.	Example
D₋	$\beta \leq -2$	compatible	Electromagnetic fields
C₋	$\beta \in (-2, -1)$	compatible	?
B₋	$\beta = -1$	compatible	?
A₋	$\beta \in (-1, 0)$	compatible	?
A₀	$\beta = 0$	compatible	Elastic/ perfect fluid matter
A₊	$\beta \in (0, 1)$	compatible	Elastic matter
B₊	$\beta = 1$	compatible	Elastic and Vlasov matter
C₊	$\beta \in (1, 2)$	violated	Elastic matter
D₊	$\beta \geq 2$	violated	Elastic matter

(some) Results for Bianchi type I

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The larger is $|\beta|$, the more the asymptotic dynamics differs from the perfect fluid case (both in the past and the future direction).

Locally Rotationally Symmetric Solutions Bianchi type IX

In the perfect fluid case (with $p = w\rho$, $-\frac{1}{3} < w < 1$) the spacetime collapses in both time directions (except for a zero measure set of initial data).

- ▶ In one time direction the solution blows-up like the vacuum Taub solution

$$T : \quad g^{11} = t^2, \quad g^{22} = g^{33} = 1.$$

- ▶ In the opposite time direction the solution blows-up like the vacuum non-flat LRS Kasner solution

$$Q : \quad g^{11} = t^{2/3}, \quad g^{22} = g^{33} = t^{-4/3}.$$

For anisotropic matter models we have the following result:

Theorem

Suppose that the matter model satisfies the following inequalities:

$$-\frac{1}{3} < w < \frac{1 - \sqrt{3}}{3} \approx -0.244 \quad \text{and} \quad \beta_- < \beta < \beta_+$$

where

$$\beta_{\pm} = \frac{-1 \pm \sqrt{(1 - 3w)^2 - 3}}{3(1 - w)}$$

(since $\beta_{\pm} \in (-\frac{1}{2}, 0)$ this is a special subcase of \mathbf{A}_-). Then

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- (i) The spacetime becomes singular in one time direction (say, the past) and toward the singularity the solution blows-up like the vacuum Taub solution \mathbb{T}
- (ii) For an open set of initial data, the spacetime is forever expanding toward the future (*no singularity!*) and is asymptotic to a non-vacuum, anisotropic solution \mathbb{R}_{\sharp} .

Remark. The solution R_{\sharp} of the Einstein equations is of Bianchi type I. The metric is given explicitly by

$$g^{11} = t^{-2\gamma_1}, \quad g^{22} = g^{33} = t^{-2\gamma_2},$$

where

$$\gamma_1 = \frac{2(1 + 2\beta)}{3[\beta^2(1 - w) + 1 + w]}, \quad \gamma_2 = \frac{2(1 - \beta)}{3[\beta^2(1 - w) + 1 + w]},$$

For $\beta \rightarrow 0$ this solution reduces to the FLRW (isotropic) perfect fluid solution:

$$g^{ii} \sim t^{-\frac{4}{3(w+1)}} \quad \forall i = 1, 2, 3.$$

Proof of (ii)

The Einstein equations for the spatial metric g^{11} , $g^{22} = g^{33}$ are

$$\partial_t g^{11} = 2g^{11}k^1_1, \quad \partial_t g^{22} = 2g^{22}k^2_2, \quad m_1 = \frac{g^{22}}{\sqrt{g^{11}}}, \quad m_2 = \sqrt{g^{11}},$$

$$\partial_t k^1_1 = \frac{1}{2}m_1^2 - 3Hk^1_1 - p_1 + \frac{1}{2}(p_1 + 2p_2 - \rho),$$

$$\partial_t k^2_2 = \frac{1}{2}[2m_1m_2 - m_1^2] - 3Hk^2_2 - p_2 + \frac{1}{2}(p_1 + 2p_2 - \rho).$$

whereas the Hamiltonian constraint equation becomes

$$H^2 + \frac{1}{3}m_1m_2 = \frac{1}{3}\rho + \frac{1}{12}m_1^2 + \sigma_+^2,$$

where as usual we introduced the Hubble scalar and the shear variable

$$H = -\frac{\text{tr } k}{3} = -\frac{k^1_1 + 2k^2_2}{3}, \quad \sigma_+ = \frac{k^1_1 - k^2_2}{3}.$$

- ▶ For LRS solutions, the Einstein-matter equations form an autonomous dynamical system in the variables $(g^{11}, g^{22}, k^1, k^2) \in (0, \infty)^2 \times \mathbb{R}^2$ subject to a constraint.

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- ▶ **Dynamical system approach:** We replace $g^{11}, g^{22}, k_1^1, k_2^2$ with a new set of dynamical variables such that the relevant state space becomes **3-dimensional** and **bounded**. In the new state space, the solution R_{\sharp} is represented by a fixed point on the boundary.

New dynamical variables. Define

$$s = \frac{g^{22}}{g^{11} + 2g^{22}} \in (0, \frac{1}{2})$$

In LRS symmetry the variable s is equivalent to the variables (s_1, s_2, s_3) defined before, since $s_2, s_3 = s$, $s_1 = 1 - 2s$.

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Define the **dominant variable** D as

$$D = \sqrt{H^2 + \frac{m_1 m_2}{3}} \quad (m_1 = \frac{g^{22}}{\sqrt{g^{11}}}, m_2 = \sqrt{g^{11}})$$

The Hamiltonian constraint

$$H^2 + \frac{1}{3}m_1 m_2 = \sigma_+^2 + \frac{1}{12}m_1^2 + \frac{1}{3}\rho,$$

guarantees that the square root is real.

We employ D to introduce **normalized variables** according to

$$H_D = \frac{H}{D}, \quad \Sigma_+ = \frac{\sigma_+}{D}, \quad M_1 = \frac{m_1}{D} > 0, \quad M_2 = \frac{m_2}{D} > 0.$$

In addition we define a normalized energy density Ω by

$$\Omega = \frac{\rho}{3D^2} \geq 0,$$

and we replace the cosmological time t by a rescaled time variable τ via

$$\frac{d}{d\tau} = \frac{1}{D} \frac{d}{dt}.$$

Henceforth, a prime denotes differentiation w.r.t. τ .

In the new variables the Einstein eqs split into a **decoupled equation** for D and a system of equations for $(H_D, \Sigma_+, M_1, M_2)$

$$H'_D = -(1 - H_D^2)(q - H_D \Sigma_+)$$

$$\Sigma'_+ = (q - 2)H_D \Sigma_+ + (1 - H_D^2)\Sigma_+^2 + \frac{1}{3}M_1^2 + 3\Omega(w_2(s) - w)$$

$$M'_1 = M_1(qH_D - 4\Sigma_+ + (1 - H_D^2)\Sigma_+)$$

$$M'_2 = M_2(qH_D + 2\Sigma_+ + (1 - H_D^2)\Sigma_+)$$

where $q = 2\Sigma_+^2 + \frac{1}{2}(1 + 3w)\Omega$. The constraint eq. implies

$$s = \left(2 + \frac{3(H_D^2 - 1)}{M_1^2}\right)^{-1}, \quad \Omega = 1 - \Sigma_+^2 - \frac{1}{12}M_1^2.$$

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The equation on M_2 decouples \rightarrow **3-dimensional state space**.

The **reduced dynamical system** consists of the equations on H_D, Σ_+, M_1 and is defined over the state space

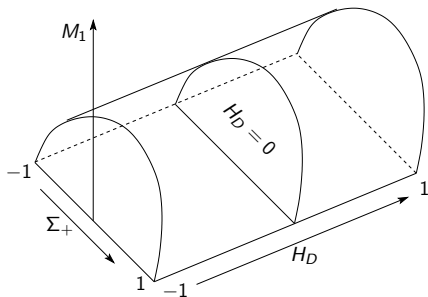
$$\mathcal{X}_{\text{IX}} = \left\{ (H_D, \Sigma_+, M_1) \mid H_D \in (-1, 1), M_1 > 0, \Sigma_+^2 + \frac{1}{12}M_1^2 < 1 \right\}$$

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- ▶ The plane $H_D = 0$ acts as a **semipermeable membrane** for the flow of the reduced dynamical system, since $H'_D|_{H_D=0} < 0$. This makes $H_D > 0$ a subset that is past invariant under the flow, while $H_D < 0$ is future-invariant.

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- ▶ The physical interpretation is the following: A type IX model that is expanding, i.e., $H > 0$ ($\Leftrightarrow H_D > 0$), at $t = t_0$ must have been expanding up to that time, i.e., for $0 < t < t_0$; conversely, a model that is contracting (i.e., $H < 0$) at $t = t_1$ must continue to contract $\forall t > t_1$ (which eventually leads to a big crunch); finally, a model that satisfies $H = 0$ at some time, is a model that starts from an initial singularity, expands to a state of maximum expansion ($H = 0$), from which it then recontracts to a final singularity.

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the flow in $H_D < 0$ is the image of the flow in $H_D > 0$ under this discrete symmetry. In particular, it suffices to analyze the asymptotic behavior of solutions in the half $H_D > 0$; the asymptotic behavior of solutions in $H_D < 0$, follows by applying the discrete symmetry.

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In particular we can restrict ourselves to study the flow on the half $H_D > 0$.

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- ▶ There exists a **monotone function** in the interior of the state space \Rightarrow the α/ω -limit sets are located at the boundary of the state space.
- ▶ The only problem left is to study the flow induced on the boundary of $\mathcal{X}_{IX} \cap \{H_D > 0\}$.
- ▶ Under the inequalities on (w, β) in the theorem, there exist 5 fixed points on the boundary, each of which corresponds to a self-similar solution of the Einstein equations.

However there is a problem: While the reduced dynamical system admits a regular extension on each of the boundary components $H_D = 1$ and $M_1 = 0$ separately, it does not extend regularly on the line

$$\mathcal{L} = \{H_D = 1\} \cap \{M_1 = 0\}$$

The problem lies in the equation

$$\Sigma'_+ = (q-2)H_D\Sigma_+ + (1-H_D^2)\Sigma_+^2 + \frac{1}{3}(M_1^2 - M_1M_2) + 3\Omega(w_2(s) - w)$$

In fact, the variable

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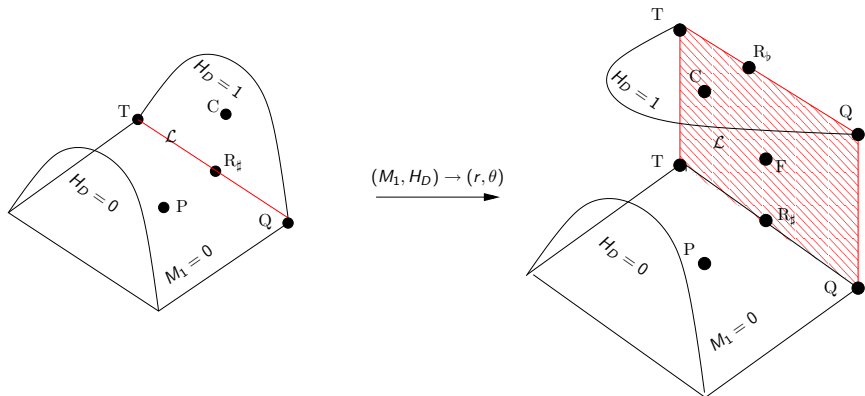
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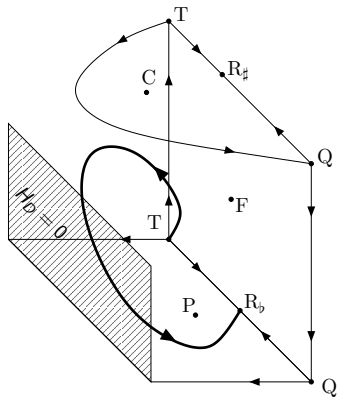
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We introduce polar coordinates (r, θ) centered on the line \mathcal{L} . This gives rise to a **blow-up** of the line \mathcal{L} into a rectangle, in which the flow is equivalent to the Bianchi type I flow.

In the new coordinates, the reduced dynamical system extends regularly on the entire boundary.

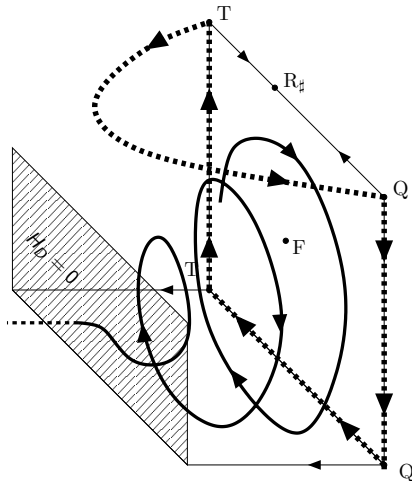
The structure of the flow on the boundary and an application of the monotonicity principle reveal that the α -limit set of the non-recollapsing orbits is the fixed point T (i.e., the Taub solution), as claimed in the Theorem.



This is the only case in which the solution does not cross the barriers $H_D = 0$, i.e., it does not recollapse. A two parameters family of interior orbits have the fixed point P as the ω -limit set.

Oscillations toward the singularity in the \mathbf{B}_+ case

For materials of type \mathbf{B}_+ ($\beta = 1$, e.g. [Vlasov matter](#)) the approach toward the singularities of LRS Bianchi type IX models is [oscillatory](#).



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- ▶ Considering anisotropic matter sources is physically relevant, as recent observations confirm the existence of small anisotropies in the matter distribution at large scales.
- ▶ An interesting open question: is there any relation between the classification of the matter models and the **geometric structure of the singularity**? (Curvature vs Coordinates sing.)