Geometric Mechanics

Homework 8

Due on November 16

1. Let M be an n-dimensional differentiable manifold and $\omega \in \Omega^1(M)$. Show that:

   (a) If $X, Y \in \mathfrak{X}(M)$ then

   $$d\omega(X, Y) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X, Y]).$$

   (b) If ω does not vanish on M then the $(n-1)$-dimensional distribution determined by the kernel of ω is integrable if and only if

   $$d\omega \wedge \omega = 0.$$

   (**Hint:** Write $d\omega$ in a local basis of one-forms of the type $\{\theta^1, \ldots, \theta^{n-1}, \omega\}$).

2. Recall that our model for an ice skate is given by the non-holonomic constraint Σ determined on $\mathbb{R}^2 \times S^1$ by the kernel of the 1-form $\omega = -\sin\theta dx + \cos\theta dy$.

   (a) Show that the ice skate can access all points in the configuration space: given two points $p, q \in \mathbb{R}^2 \times S^1$, there exists a piecewise smooth curve $c : [0, 1] \to \mathbb{R}^2 \times S^1$, compatible with Σ, such that $c(0) = p$ and $c(1) = q$. Why does this show that Σ is non-integrable?

   (b) Assuming that the kinetic energy of the skate is

   $$K = \frac{M}{2}\left((v^x)^2 + (v^y)^2\right) + \frac{I}{2}\left(v^\theta\right)^2$$

   and that the reaction force is perfect, show that the skate moves with constant speed along straight lines or circles. What is the physical interpretation of the reaction force?