Geometric Mechanics

Homework 4

Due on October 19

1. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold with Levi-Civita connection ∇, and let

$$\langle\langle \cdot, \cdot \rangle\rangle = e^{2\rho} \langle \cdot, \cdot \rangle$$

be a metric conformally related to $\langle \cdot, \cdot \rangle$, where $\rho \in C^\infty(M)$. Show that the Levi-Civita connection $\widetilde{\nabla}$ of $\langle\langle \cdot, \cdot \rangle\rangle$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + d\rho(X)Y + d\rho(Y)X - \langle X, Y \rangle \operatorname{grad} \rho$$

for all $X, Y \in \mathfrak{X}(M)$, where the gradient is taken with respect to $\langle \cdot, \cdot \rangle$, that is, $\operatorname{grad} \rho$ is the vector field defined by

$$\langle \operatorname{grad} \rho, X \rangle = d\rho(X)$$

for all $X \in \mathfrak{X}(M)$. (**Hint:** Use the Koszul formula).

2. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. A curve $c : I \subset \mathbb{R} \to M$ is said to be a **reparameterized geodesic** if $c(t) = \gamma(s(t))$ for all $t \in I$, where $\gamma : J \subset \mathbb{R} \to M$ is a geodesic and $s : I \to J$ is a diffeomorphism. Show that c is a reparameterized geodesic if and only if it satisfies

$$\frac{D\dot{c}}{dt} = f(t)\, \dot{c}$$

for some differentiable function $f : I \to \mathbb{R}$.

3. The **hyperbolic plane** is the upper half plane

$$H = \left\{ (x, y) \in \mathbb{R}^2 \mid y > 0 \right\}$$

with the Riemannian metric

$$\langle \cdot, \cdot \rangle = \frac{1}{y^2} \left(dx \otimes dx + dy \otimes dy \right).$$

   (a) Use the local coordinate expression of Newton's law to write the geodesic equations.

   (b) Determine the Christoffel symbols for the Levi-Civita connection of $(H, \langle \cdot, \cdot \rangle)$ in the coordinates (x, y).

   (c) Show that vertical half-lines and half-circles centered on the x-axis are the images of geodesics. (**Hint:** Use the result in question 2).