Geometric Mechanics

Homework 3

Due on October 12

1. Consider the usual local coordinates (θ, φ) on the sphere $S^2 \subset \mathbb{R}^3$, defined by the parameterization $\phi :]0, \pi[\times]0, 2\pi[\to \mathbb{R}^3$ given by

$$\phi(\theta, \varphi) = (\operatorname{sen}\theta \cos\varphi, \operatorname{sen}\theta \operatorname{sen}\varphi, \cos\theta).$$

   Recall that the Riemannian metric g induced in S^2 by the usual Euclidean metric in $\mathbb{R}^3$ is given in these coordinates by

$$g = d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi.$$

   (a) Determine the Christoffel symbols for the Levi-Civita connection associated to these local coordinates.

   (b) Show that the equator $\theta = \frac{\pi}{2}$ is the image of a geodesic. Are the parallels $\theta = \theta_0$ (with $\theta_0 \neq \frac{\pi}{2}$) images of geodesics?

   (c) Let $c : [0, 2\pi] \to S^2$ be the curve given in local coordinates by $(\theta(t), \varphi(t)) = (\theta_0, t)$, where $\theta_0 \in \left]0, \frac{\pi}{2}\right[$. Let V be a vector field parallel along c such that $V(0) = \frac{\partial}{\partial\theta}$. Compute the angle by which V is rotated when it returns to the point $c(2\pi) = c(0)$, that is, the angle between $V(0)$ and $V(2\pi)$.

   (**Remark:** This is precisely the angle by which the oscillation plane of the Foulcaut pedulum – which is just any pendulum sufficiently long and heavy to remain oscillating for a long time – rotates after 24 hours at latitude $\frac{\pi}{2} - \theta_0$; the reason for this is that the oscillation plane tries to remain fixed with respect to the distant stars as the Earth rotates about its axis).

   (d) Indicate a geodesic triangle (that is, a triangle whose sides are images of geodesics) with 3 right angles. Without solving any differential equation, compute the angle by which a vector is rotated as it is parallel propagated once around the triangle.

2. Let (M, g) be a Riemannian manifold and $p \in M$. Show that the map $\mu : T_pM \to T_p^*M$ given by

$$\mu(v)(w) = g(v, w) \qquad \forall v, w \in T_pM$$

   is a linear isomorphism, given in the bases associated to local coordinates $(x^1, \dots, x^n)$ by

$$\mu\left(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\right) = \sum_{i,j=1}^{n} g_{ij} v^j dx^i.$$