

Geometric Mechanics

Homework 12

Due on December 19

1. Consider the sequence formed by the first digit of the decimal expansion of each of the integers 2^n for $n \in \mathbb{N}_0$:

1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, ...

The purpose of this exercise is to answer the following question: is there a 7 in this sequence?

- (a) Show that if $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then

$$\lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n e^{2\pi i \nu k} = 0.$$

- (b) Prove the following discrete version of the Birkhoff Ergodic Theorem: if a smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$ is periodic with period 1 and $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then for all $x \in \mathbb{R}$

$$\lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n f(x + \nu k) = \int_0^1 f(x) dx.$$

- (c) Show that $\log 2$ is an irrational multiple of $\log 10$.
(d) Is there a 7 in the sequence above?

2. **Twin paradox:** Twins Alice and Bob part on their 20th anniversary: while Alice stays on the Earth (which is approximately an inertial frame), Bob leaves at 80% of the speed of light towards Planet X, 8 light-years away from the Earth, which he therefore reaches 10 years later (as measured in the Earth's frame). After a short stay, Bob returns to the Earth, again at 80% of the speed of light. Consequently, Alice is 40 years old when they meet again.

- (a) How old is Bob at this meeting?
(b) How do you explain the asymmetry in the twin's ages? Notice that, from Bob's point of view, he is the one who is stationary, while the the Earth moves away and back again!
(c) Imagine that each twin has a very powerful telescope. What does each of them **see**? In particular, how much time elapses for each of them as they see their twin experiencing one year?