Geometric Mechanics

Homework 11

Due on December 7

1. Let $(M, \langle \cdot, \cdot \rangle)$ be a compact Riemannian manifold. Show that for each normal ball $B \subset M$ and each $T > 0$ there exist geodesics $c : \mathbb{R} \to M$ with $\|\dot{c}(t)\| = 1$ such that $c(0) \in B$ and $c(t) \in B$ for some $t \geq T$.

   (**Remark:** Actually, almost all geodesics with initial point in B satisfy this property; can you think of an example of a compact Riemannian manifold containing a geodesic that **does not** return to B?)

2. Recall that the Lagrange top is the mechanical system determined by the Lagrangian function $L : TSO(3) \to \mathbb{R}$ given in local coordinates by

$$L = \frac{I_1}{2}\left(\left(v^\theta\right)^2 + \left(v^\varphi\right)^2 \sin^2\theta\right) + \frac{I_3}{2}\left(v^\psi + v^\varphi \cos\theta\right)^2 - Mgl\cos\theta,$$

   where (θ, φ, ψ) are the Euler angles, M is the top's mass and l is the distance from the fixed point to the center of mass.

   (a) Compute the Legendre transformation, show that L is hyper-regular and write an expression in local coordinates for the Hamiltonian $H : T^*SO(3) \to \mathbb{R}$.

   (b) Prove that H is completely integrable.

   (c) Find all solutions with constant θ, $\dot{\varphi}$ and $\dot{\psi}$, and argue that they are stable for $|\dot{\varphi}| \ll |\dot{\psi}|$ if $|\dot{\psi}|$ is large enough.