Geometric Mechanics

Homework 10

Due on November 30

1. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold, $\alpha \in \Omega^1(M)$ a 1-form and $U \in C^\infty(M)$ a differentiable function.

   (a) Show that the Euler-Lagrange equations for the Lagrangian $L : TM \to \mathbb{R}$ given by

   $$L(v) = \frac{1}{2}\langle v, v \rangle + \iota(v)\alpha_p - U(p)$$

   for $v \in T_pM$ yield the motions of the mechanical system $(M, \langle \cdot, \cdot \rangle, \mathcal{F})$, where

   $$\mathcal{F}(v) = -(dU)_p - \iota(v)(d\alpha)_p$$

   for $v \in T_pM$.

   (b) Show that the mechanical energy $E = K + U$ is conserved along the motions of $(M, \langle \cdot, \cdot \rangle, \mathcal{F})$.

   (**Remark:** This is called a **conservative mechanical system with a magnetic term**).

   (c) Show that L is hyper-regular and compute the Legendre transformation.

   (d) Find the Hamiltonian $H : T^*M \to \mathbb{R}$ and write Hamilton's equations.