Geometric Mechanics

2018/2019

2^{nd} Exam - 8 February 2019 - 13:00

1. Consider a particle of mass $m > 0$ moving on the surface of revolution $M \subset \mathbb{R}^3$ given in cylindrical coordinates (ρ, φ, z) by

$$z = f(\rho),$$

where $f : \mathbb{R}^+ \to \mathbb{R}$ is a smooth function, under the action of the constant gravitational force associated to the potential energy $U = mgz$ (with $g > 0$ constant).

(2/20) (a) Show that the particle's equations of motion are the Euler-Lagrange equations determined by the Lagrangian $L : TM \to \mathbb{R}$ given in local coordinates by

$$L(\rho, \varphi, v^\rho, v^\varphi) = \frac{m}{2}\left(\left(1 + (f'(\rho))^2\right)(v^\rho)^2 + \rho^2 (v^\varphi)^2\right) - mgf(\rho).$$

(2/20) (b) Check that the curves given in local coordinates by $\rho = \rho_0$ (for $\rho_0 \in \mathbb{R}^+$ constant) are images of motions if and only if $f'(\rho_0) > 0$. Can you interpret this result?

(2/20) (c) Compute the Legendre transformation, show that L is hyper-regular and write an expression in local coordinates for the Hamiltonian $H : T^*M \to \mathbb{R}$.

(2/20) (d) Show that H is completely integrable.

(2/20) (e) Prove that if $f'(\rho_0) > 0$ and $f''(\rho_0) + \frac{3}{\rho_0} f'(\rho_0) > 0$ then any motion whose image is $\rho = \rho_0$ is stable, that is, if its initial condition is perturbed by a sufficiently small amount then the resulting motion remains close to $\rho = \rho_0$.

(2/20) (f) Show that the (perfect) reaction force is given by

$$\mu^{-1}(\mathcal{R})(\rho, \varphi, v^\rho, v^\varphi) = \frac{f''(\rho)(v^\rho)^2 + \rho f'(\rho)(v^\varphi)^2 + g}{\sqrt{1 + f'(\rho))^2}}\, \mathbf{n}(\rho, \varphi),$$

where

$$\mathbf{n}(\rho, \varphi) = \frac{1}{\sqrt{1 + f'(\rho))^2}}(-f'(\rho)\cos\varphi, -f'(\rho)\sin\varphi, 1)$$

is the Euclidean upward-pointing unit normal to M at the point parameterized by (ρ, φ), written in Cartesian coordinates. (**Hint:** Compute the acceleration in $\mathbb{R}^3$ of the curve given in Cartesian coordinates by $c(t) = (\rho(t)\cos\varphi(t), \rho(t)\sin\varphi(t), f(\rho(t)))$ and project it along $\mathbf{n}(\rho, \varphi)$).

(2/20) **2.** The **pseudo-rigid body** is the mechanical system $(SL(3), \langle\cdot,\cdot\rangle, -dU)$, where the configuration space is

$$SL(3) = \{S \in \mathcal{M}_{3\times 3}(\mathbb{R}) : \det S = 1\},$$

the kinetic energy is

$$K(V) = \frac{1}{2}\langle V, V\rangle = \frac{1}{2}\operatorname{tr}(VV^t),$$

and the potential energy U depends only on the (unordered) eigenvalues of SS^t. Show that the Lagrangian function $L : TSL(3) \to \mathbb{R}$ for this mechanical system is invariant under the $SO(3)$ actions $A, B : SO(3) \times SL(3) \to SL(3)$ given by

$$A(R,S) = RS \qquad \text{and} \qquad B(R,S) = SR^t,$$

and use Noether's theorem to show that if $S : \mathbb{R} \to SL(3)$ is a motion of the pseudo-rigid body then the antisymmetric parts of both $\dot{S}S^t$ and $S^t\dot{S}$ are constant.

3. The Schwarzschild-de Sitter metric is the spherically symmetric solution of the vacuum Einstein equations with a cosmological constant $\Lambda > 0$ (known to produce a repulsive gravitational effect) given by the metric

$$g_{\text{SdS}} = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right) dt \otimes dt + \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr \otimes dr + r^2\left(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi\right),$$

where $m > 0$ is the mass of the spherically symmetric object generating the field and we assume that $\Lambda < \frac{1}{9m^2}$.

(2/20) (a) Check that the stationary observers sitting at $r = \left(\frac{3m}{\Lambda}\right)^{\frac{1}{3}}$ correspond to timelike geodesics. Can you interpret this result?

(2/20) (b) Show that the conditions a curve in the equatorial plane $\theta = \frac{\pi}{2}$ to be a circular timelike geodesic of radius r parameterized by proper time are

$$\begin{cases} \ddot{t} = 0 \\ \ddot{\varphi} = 0 \\ \dot{\varphi}^2 = \left(\frac{m}{r^3} - \frac{\Lambda}{3}\right)\dot{t}^2 \\ \left(1 - \frac{3m}{r}\right)\dot{t}^2 = 1 \end{cases}$$

Conclude that massive particles can orbit the central mass in equatorial circular orbits only for $3m < r < \left(\frac{3m}{\Lambda}\right)^{\frac{1}{3}}$. Can you interpret this result?

(2/20) (c) Compute the period of an equatorial circular orbit as measured by:

(i) A stationary observer placed at the same altitude as the orbit;

(ii) The observer in orbit.