Geometric Mechanics

2018/2019

1st Exam - 18 January 2019 - 13:00

1. Let $M \subset \mathbb{R}^3$ be the surface of revolution given in cylindrical coordinates (ρ, φ, z) by

$$\rho = f(z),$$

where $f : \mathbb{R} \to \mathbb{R}^+$ is a smooth function, and consider the metric induced on M by the Euclidean metric of $\mathbb{R}^3$,

$$g_{\text{Euclidean}} = d\rho \otimes d\rho + \rho^2 d\varphi \otimes d\varphi + dz \otimes dz.$$

(2/20) (a) Show that the equations for the geodesics of M are the Euler-Lagrange equations determined by the Lagrangian $L : TM \to \mathbb{R}$ given in local coordinates by

$$L(\varphi, z, v^\varphi, v^z) = \frac{1}{2}\left((f(z))^2 (v^\varphi)^2 + \left((f'(z))^2 + 1\right)(v^z)^2\right).$$

(2/20) (b) Check that the curves given in local coordinates by $\varphi = \varphi_0$ (for any constant $\varphi_0 \in \mathbb{R}$) or by $f'(z) = 0$ are images of geodesics.

(2/20) (c) Compute the Legendre transformation, show that L is hyper-regular and write an expression in local coordinates for the Hamiltonian $H : T^*M \to \mathbb{R}$.

(2/20) (d) Show that H is completely integrable.

(2/20) (e) Prove that if $f'(z_0) = 0$ and $f''(z_0) < 0$ then any geodesic whose image is $z = z_0$ is stable, that is, if its initial condition is perturbed by a sufficiently small amount then the resulting geodesic remains close to $z = z_0$.

(2/20) (f) Show that if $f'(z_0) = 0$ and $f''(z_0) > 0$ then there exist geodesics $c : \mathbb{R} \to M$ accumulating on $z = z_0$, that is, satisfying $z(c(t)) \neq z_0$ and $\lim_{t \to +\infty} z(c(t)) = z_0$.

(2/20) (g) Consider now an ice skate moving on M in the particular case $f(z) = 1$, correponding to the configuration space $M \times S^1$, kinetic energy

$$K(\varphi, z, \psi, v^\varphi, v^z, v^\psi) = \frac{m}{2}\left((v^\varphi)^2 + (v^z)^2\right) + \frac{I}{2}(v^\psi)^2,$$

potential energy $U = mgz$ (where $m, I, g > 0$) and non-holonomic constraint

$$\Sigma = \operatorname{span}\left\{\cos\psi \frac{\partial}{\partial\varphi} + \sin\psi \frac{\partial}{\partial z}, \frac{\partial}{\partial\psi}\right\}.$$

Show that this is a true non-holonomic constraint, write the equations of motion and obtain a particular solution of these equations.

(2/20) **2.** Let M be an n-dimensional differentiable manifold, $\omega \in \Omega^2(T^*M)$ the canonical symplectic form, $H \in C^\infty(T^*M)$ a smooth function and $E \in \mathbb{R}$ a regular value of H. Show that if $G \in C^\infty(T^*M)$ satisfies $\{G, H\} = 1$ then $\Omega = \iota(X_G)\omega^n$ is a volume form on the $(2n-1)$-dimensional submanifold $H^{-1}(E) \subset T^*M$, preserved by the flow of X_H, and independent of the choice of G.

(**Remark:** This volume form can be used to prove the Poicaré Recurrence Theorem on a single energy level; the existence of a local function G as above is guaranteed by Darboux's Theorem).

3. Recall that the Schwarzschild metric restricted to a radial line is given by

$$g_{\text{Schwarzschild}} = -\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr,$$

where $m > 0$ is the mass of the spherically symmetric object generating the field.

(2/20) (a) Check that the equations for radial timelike geodesics can be written as

$$\begin{cases} \left(1 - \dfrac{2m}{r}\right) \dfrac{dt}{d\tau} = E \\ \\ \left(\dfrac{dr}{d\tau}\right)^2 = E^2 - 1 + \dfrac{2m}{r} \end{cases},$$

where τ is the proper time and $E > 0$ is a constant.

(2/20) (b) Show that a free-falling particle that is thrown upwards from $r = r_0$, reaches a maximum height $r = r_1$, and falls back to $r = r_0$, measures a proper time

$$\Delta\tau = 2 \int_{r_0}^{r_1} \frac{dr}{\sqrt{\frac{2m}{r} - \frac{2m}{r_1}}},$$

corresponding to a coordinate time

$$\Delta t = 2 \int_{r_0}^{r_1} \frac{\sqrt{1 - \frac{2m}{r_1}}\, dr}{\left(1 - \frac{2m}{r}\right)\sqrt{\frac{2m}{r} - \frac{2m}{r_1}}}.$$

Use this result to prove that

$$\Delta\tau > \Delta t \sqrt{1 - \frac{2m}{r_0}},$$

that is, the particle measures a longer proper time than the stationary observer sitting at $r = r_0$. (**Hint:** Compare the derivatives of $\Delta\tau$ and $\Delta t\sqrt{1 - \frac{2m}{r_0}}$ with respect to r_0 using $\Delta t \geq 2\int_{r_0}^{r_1} \frac{r_0\, dr}{\sqrt{1 - \frac{2m}{r_1}}\sqrt{2m}\sqrt{r_1 - r}} \geq \frac{2{r_0}^2}{m\sqrt{1 - \frac{2m}{r_1}}}\sqrt{\frac{2m}{r_0} - \frac{2m}{r_1}}$).