Geometric Mechanics

2016/2017

2^{nd} Exam - 3 February 2017 - 10:00

1. The **anti-de Sitter solution** of Eintein's field equations is given in spherical coordinates (t, r, θ, φ) by the Lorentzian metric

$$g = -\cosh^2 r\, dt^2 + dr^2 + \sinh^2 r \left(d\theta^2 + \sin^2 \theta d\varphi^2\right).$$

(2/20) (a) Obtain the geodesic Lagrangian and find three constants of motion.

(2/20) (b) Show that geodesics whose initial velocity is tangent to the hypersurface $\theta = \frac{\pi}{2}$ remain on this hypersurface.

(2/20) (c) Show that the circular geodesics on the hypersurface $\theta = \frac{\pi}{2}$ are given by

$$\begin{cases} \ddot{t} = 0; \\ \ddot{\varphi} = 0; \\ \dfrac{d\varphi}{dt} = \pm 1. \end{cases}$$

(d) Compute the period of a circular orbit as measured by:

(1/20) (i) A stationary clock;

(1/20) (ii) The orbiting clock.

(e) Show that a clock dropped radially will return to the initial position, and compute the period of this motion as measured by:

(1/20) (i) A stationary clock;

(1/20) (ii) The free-falling clock.

(**Hint:** Write the metric of the submanifold of constant (θ, φ) using the new coordinate $z = \sinh r$.)

(2/20) **2.** Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. Recall that a vector field $X \in \mathfrak{X}(M)$ is called a **Killing vector field** if

$$X_p = \frac{d}{dt}_{|t=0} \psi_t(p),$$

where $\psi_t : M \to M$ is a 1-parameter family of isometries. Show that if $c : I \subset \mathbb{R} \to M$ is the motion of a free particle on M subject to the perfect reaction force determined by the non-holonomic constraint Σ, and X is compatible with Σ (that is, $X_p \in \Sigma_p$ for all $p \in M$), then the quantity

$$J(t) = \left\langle \dot{c}(t), X_{c(t)} \right\rangle$$

is constant along the motion.

3. Let (M,ω) be a symplectic manifold, $H \in C^\infty(M)$ a completely integrable Hamiltonian, $F_1,\ldots,F_n \in C^\infty(M)$ first integrals in involution, independent on some dense open set, and $(x^1,\ldots,x^n,p_1,\ldots,p_n)$ Darboux coordinates in a neighborhood of some invariant torus such that $(x^1,\ldots,x^n,F_1,\ldots,F_n)$ are also local coordinates (hence $(x^1,\ldots,x^n)$ are coordinates on T^n, which we take to be $\mathbb{R}^n/\mathbb{Z}^n$). Show that:

(2/20) (a) The matrix with components

$$\frac{\partial p_i}{\partial x^j}(x,F)$$

$(i,j = 1,\ldots,n)$ is symmetric, and therefore the system of first order partial differential equations

$$\frac{\partial S}{\partial x^i} = p_i(x,F)$$

$(i = 1,\ldots,n)$ is integrable in $\mathbb{R}^n$ for each fixed$(F_1,\ldots,F_n)$.

(**Hint:** Start by proving that ω vanishes when restricted to the tangent space of an invariant torus.)

(2/20) (b) The functions

$$y^i = \frac{\partial S}{\partial F_i}$$

$(i = 1,\ldots,n)$ complete the functions $F_1,\ldots,F_n$ into a system of Darboux coordinates, that is,

$$\omega = \sum_{i=1}^{n} dF_i \wedge dy^i.$$

(2/20) (c) If γ_i is an integral curve of $\frac{\partial}{\partial x^i}$ on T^n then

$$\oint_{\gamma_i} dy^j = \frac{\partial}{\partial F_j}\oint_{\gamma_i}\theta,$$

where

$$\theta = \sum_{i=1}^{n} p_i dx^i.$$

(2/20) (d) The functions

$$J_i = \oint_{\gamma_i}\theta$$

are also first integrals and involution, and with this choice of first integrals the coordinates $(y^1,\ldots,y^n)$ are standard coordinates on T^n.

(**Remark:** The coordinates $(y^1,\ldots,y^n,J_1,\ldots,J_n)$ are called **action-angle coordinates**. Note that by using the procedure outlined above one can solve Hamilton's equations "by quadratures", that is, by simply calculating definite and indefinite integrals, namely by computing first $(J_1,\ldots,J_n)$ and then $S(x,J)$. This is the origin of the designation "completely integrable".)