Geometric Mechanics

2012/2013

1st Test - 4 February 2013 - 9:00

Recall that the Euler angles $(\theta, \varphi, \psi) : SO(3) \to (0, \pi) \times (0, 2\pi) \times (0, 2\pi)$ are the local coordinates defined by the parameterization

$$S(\theta, \varphi, \psi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $S : \mathbb{R} \to SO(3)$ describe the orientation of a rigid disk of radius $R > 0$, mass $M > 0$ and inertia tensor $I = \mathrm{diag}(I_1, I_1, I_3)$. It can be show that the body angular velocity Ω is given by

$$\Omega = (\dot\theta \cos\psi + \dot\varphi \sin\theta \sin\psi)\, e_1 + (-\dot\theta \sin\psi + \dot\varphi \sin\theta \cos\psi)\, e_2 + (\dot\psi + \dot\varphi \cos\theta)\, e_3.$$

1. Consider first only the rotational motion of the disk about its center.

(3/20) (a) Show that the kinetic energy $K : TSO(3) \to \mathbb{R}$ is

$$K\left(\theta, \varphi, \psi, v^\theta, v^\varphi, v^\psi\right) = \frac{I_1}{2}\left(\left(v^\theta\right)^2 + \left(v^\varphi\right)^2 \sin^2\theta\right) + \frac{I_3}{2}\left(v^\psi + v^\varphi \cos\theta\right)^2.$$

(3/20) (b) Prove that the Lagrangian function $L = K$ is hyper-regular.

(4/20) (c) Determine the Hamiltonian function $H : T^*SO(3) \to \mathbb{R}$ and show that it is completely integrable.

2. Now assume that the disk is rolling on the plane $z = 0$, and let (x, y, z) be the Cartesian coordinates of the center of mass, so that the Lagrangian has an additional term

$$\frac{1}{2}M\left((v^x)^2 + (v^y)^2 + (v^z)^2\right) - Mgz$$

(g is the constant gravitational acceleration). Besides the holonomic constraint that the disk should touch the plane, $z = R\sin\theta$, we have the constraint of rolling without slipping, given by the kernels of the 1-forms

$$\alpha^1 = dx - R\sin\theta\sin\varphi\, d\theta + R\cos\theta\cos\varphi\, d\varphi + R\cos\varphi\, d\psi;$$
$$\alpha^2 = dy + R\sin\theta\cos\varphi\, d\theta + R\cos\theta\sin\varphi\, d\varphi + R\sin\varphi\, d\psi.$$

(3/20) (a) Prove this constraint is non-holonomic.

(4/20) (b) Write the equations of motion assuming a perfect reaction force.

(3/20) (c) Find all motions satisfying $\dot x = \dot y = \dot z = 0$.

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2[nd] Test - 4 February 2013 - 9:00

1. Recall that the cosmological FLRW models are given by Lorentzian metrics of the form

$$g = -dt \otimes dt + a^2(t)\left(\frac{1}{1-kr^2}dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta d\varphi \otimes d\varphi\right),$$

where $k \in \{-1, 0, 1\}$. Consider two galaxies in a FLRW model, whose spatial locations can be assumed to be $r = 0$ and $(r, \theta, \varphi) = (r_1, \theta_1, \varphi_1)$. Show that:

(4/20) (a) The spatial distance $d(t)$ between the two galaxies along the spatial Riemannian manifold of constant t satisfies the *Hubble law*

$$\dot{d} = Hd,$$

where $H = \dot{a}/a$ is the Hubble constant.

(4/20) (b) The family (reparameterized) null geodesics connecting the first galaxy to the second galaxy can be written as

$$(t, r, \theta, \varphi) = (t(r, t_0), r, \theta_1, \varphi_1) \qquad (0 \leq r \leq r_1),$$

where $(t(r, t_0))$ is the solution of

$$\frac{dt}{dr} = \frac{a(t)}{\sqrt{1-kr^2}}, \qquad t(0, t_0) = t_0.$$

(4/20) (c) Light emitted by the first galaxy with period T is measured by the second galaxy to have period $T' = (a(t_1)/a(t_0))T$, that is,

$$\frac{\partial t}{\partial t_0}(r_1, t_0) = \frac{a(t_1)}{a(t_0)},$$

where $t_1 = t(r_1, t_0)$.

2. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold, and suppose that $H \in C^\infty(M)$ has a local minimum at $p \in M$. Show that:

(4/20) (a) If the minimum is strict (that is, $H(p) < H(q)$ for all $q \neq p$ in some open neighborhood of p) then p is a stable fixed point for the flow of X_H (that is, for each open neighborhood V of p there exists another open neighborhood $U \subset V$ of p such that any integral curve of X_H with initial condition in U remains in V).

(4/20) (b) If the minimum is not strict (that is, $H(p) \leq H(q)$ for all q in some open neighborhood of p) then p may be unstable.