Geometric Mechanics

2012/2013

1^{st} Test - 21 January 2012 - 9:00

Recall that the Euler angles $(\theta, \varphi, \psi) : SO(3) \to (0, \pi) \times (0, 2\pi) \times (0, 2\pi)$ are the local coordinates defined by the parameterization

$$S(\theta, \varphi, \psi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $S : \mathbb{R} \to SO(3)$ describe the orientation of a rigid sphere of radius $R > 0$, mass $M > 0$ and inertia tensor $I = \operatorname{diag}(I_1, I_1, I_1)$. It can be show that the space angular velocity $\omega = S\Omega$ is given by

$$\omega = (\dot\theta\cos\varphi + \dot\psi\sin\theta\sin\varphi)\, e_1 + (\dot\theta\sin\varphi - \dot\psi\sin\theta\cos\varphi)\, e_2 + (\dot\varphi + \dot\psi\cos\theta)\, e_3.$$

1. Consider first only the rotational motion of the sphere about its center.

(3/20) (a) Show that the kinetic energy $K : TSO(3) \to \mathbb{R}$ is

$$K\big(\theta, \varphi, \psi, v^\theta, v^\varphi, v^\psi\big) = \frac{I_1}{2}\left(\big(v^\theta\big)^2 + (v^\varphi)^2 + \big(v^\psi\big)^2 + 2v^\varphi v^\psi\cos\theta\right).$$

(3/20) (b) Prove that the Lagrangian function $L = K$ is hyper-regular.

(4/20) (c) Determine the Hamiltonian function $H : T^*SO(3) \to \mathbb{R}$ and show that it is completely integrable.

2. Now assume that the sphere is rolling on a horizontal plane, and let (x, y) be the Cartesian coordinates of the contact point, so that the kinetic energy has an additional term

$$\frac{1}{2}M\left((v^x)^2 + (v^y)^2\right).$$

(3/20) (a) Show that the constraint of rolling without slipping, that is, that the velocity of the contact point is zero, is given by the kernels of the 1-forms

$$\alpha^1 = dx - R\sin\varphi\, d\theta + R\sin\theta\cos\varphi\, d\psi, \qquad \alpha^2 = dy + R\cos\varphi\, d\theta + R\sin\theta\sin\varphi\, d\psi.$$

(3/20) (b) Prove this constraint is non-holonomic.

(4/20) (c) Write the equations of motion assuming a perfect reaction force, and find all solutions satisfying $x = vt$ and $y = \dot\theta = \dot\varphi = 0$.

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2^{nd} Test - 21 January 2012 - 9:00

1. Recall that the Schwarzschild metric for the equatorial plane $\theta = \frac{\pi}{2}$ is given by

$$g = -\left(1 - \frac{2m}{r}\right) dt \otimes dt + \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr + r^2 d\varphi \otimes d\varphi.$$

(4/20) (a) Show that the conditions for a curve of constant r to be a timelike geodesic parameterized by its proper time are

$$\begin{cases} \ddot{t} = 0 \\ \ddot{\varphi} = 0 \\ r\dot{\varphi}^2 = \frac{m}{r^2}\dot{t}^2 \\ \left(1 - \frac{3m}{r}\right)\dot{t}^2 = 1 \end{cases}$$

(b) Conclude that massive particles can orbit the central mass in circular orbits for all $r > 3m$, and compute the period of these orbits as measured by:

(3/20) i. A stationary observer;

(3/20) ii. The orbiting particle.

(4/20) (c) Show that there exists a circular null geodesic for $r = 3m$.

2. Recall that the upper half plane $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ has a Lie group structure, given by the operation

$$(a, b) \cdot (x, y) = (bx + a, by).$$

(3/20) (a) Show that the formula

$$(a, b) \cdot (x, y, p_x, p_y) = \left(bx + a, by, \frac{p_x}{b}, \frac{p_y}{b}\right)$$

defines a Poisson action on T^*H (with the canonical symplectic structure).

Hint: Show that this map preserves the canonical symplectic form.

(3/20) (b) Check that the functions

$$F(x, y, p_x, p_y) = yp_x \quad \text{and} \quad G(x, y, p_x, p_y) = yp_y$$

are H-invariant, and use this to obtain the quotient Poisson structure on T^*H/H. Is this quotient Poisson manifold a symplectic manifold?