Geometric Mechanics

2012/2013

1st Test - 13 November 2012 - 11:00

1. Consider the Lagrangian function $L : T\mathbb{R}^2 \to \mathbb{R}$ given in polar coordinates (r, θ) by

$$L(r, \theta, v^r, v^\theta) = \frac{1}{2}\left((v^r)^2 + r^2(v^\theta)^2\right) + r^2 v^\theta + \frac{1}{2}r^2,$$

describing the motion of a free particle on a rotating frame (in appropriate units).

(3/20) (a) Write the Euler-Lagrange equations, and prove that there exist circular orbits for any radius $r_0 > 0$.

(3/20) (b) Show that L is hyper-regular and compute the Hamiltonian function $H : T\mathbb{R}^2 \to \mathbb{R}$.

(3/20) (c) Determine the Hamiltonian function $H : T^*\mathbb{R}^2 \to \mathbb{R}$ and show that it is completely integrable.

(3/20) (d) Compute the motion of the particle subject to the semi-holonomic constraint given by $dr - r d\theta = 0$ (assuming a perfect reaction force).

2. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and let $\langle\langle \cdot, \cdot \rangle\rangle$ be a bi-invariant metric on G (that is, $\langle\langle \cdot, \cdot \rangle\rangle$ is both left- and right-invariant).

(4/20) (a) Use Noether's theorem to prove that the geodesics $S : \mathbb{R} \to G$ of $\langle\langle \cdot, \cdot \rangle\rangle$ with $S(0) = S_0$ are given by $S(t) = S_0 \exp(At)$ for some $A \in \mathfrak{g}$.

(4/20) (b) When is the metric defined on $SO(3)$ by a rigid body with a fixed point bi-invariant? *Hint:* Use Euler's equation.