# Geometric Mechanics

## Homework 8

*Due on November 16*

1. Recall that the (completely integrable) Hamiltonian $H : T^*\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ for the Kepler problem is given in local coordinates by

$$H(r, \theta, p_r, p_\theta) = \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} - \frac{1}{r}.$$

   (a) Show that the projection of the invariant set

$$L_{(E,l)} := H^{-1}(E) \cap {p_\theta}^{-1}(l)$$

   on $\mathbb{R}^2$ is given in polar coordinates $(r, \theta)$ by

$$\frac{l^2}{2mr^2} - \frac{1}{r} \leq E.$$

   (b) Conclude that there exist circular orbits of any radius, and that these orbits are stable.

2. Consider the sequence formed by the first digit of the decimal expansion of each of the integers $2^n$ for $n \in \mathbb{N}_0$:

$$1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, \ldots$$

   The purpose of this exercise is to answer the following question: is there a $7$ in this sequence?

   (a) Show that if $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then

$$\lim_{n \to +\infty} \frac{1}{n+1} \sum_{k=0}^{n} e^{2\pi i \nu k} = 0.$$

   (b) Prove the following discrete version of the Birkhoff Ergodic Theorem: if a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period $1$ and $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then for all $x \in \mathbb{R}$

$$\lim_{n \to +\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(x + \nu k) = \int_0^1 f(x) dx.$$

   (c) Show that $\log 2$ is an irrational multiple of $\log 10$.

   (d) Is there a $7$ in the sequence above?