Geometric Mechanics

Homework 3

Due on October 12

1. Prove that there exists a linear isomorphism $\Omega : \mathfrak{so}(3) \to \mathbb{R}^3$ such that

$$A\xi = \Omega(A) \times \xi$$

for all $\xi \in \mathbb{R}^3$ and $A \in \mathfrak{so}(3)$. Show that moreover

$$\Omega([A, B]) = \Omega(A) \times \Omega(B)$$

for all $A, B \in \mathfrak{so}(3)$ (that is, Ω is a Lie algebra isomorphism between $\mathfrak{so}(3)$ and $(\mathbb{R}^3, \times)$).

2. A general rigid body is any mechanical system of the form $(\mathbb{R}^3 \times SO(3), \langle\langle\langle \cdot, \cdot \rangle\rangle\rangle, \mathcal{F})$, with

$$\langle\langle\langle (v, V), (w, W) \rangle\rangle\rangle := \int_{\mathbb{R}^3} \langle v + V\xi, w + W\xi \rangle \, dm$$

for all $(v, V), (w, W) \in T_{(x,S)}\mathbb{R}^3 \times SO(3)$ and $(x, S) \in \mathbb{R}^3 \times SO(3)$, where $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product on $\mathbb{R}^3$ and m is a positive finite measure on $\mathbb{R}^3$ not supported on any straight line and satisfying $\int_{\mathbb{R}^3} \|\xi\|^2 dm < +\infty$. We further assume

$$\int_{\mathbb{R}^3} \xi \, dm = 0,$$

that is, the center of mass of the reference configuration is placed at the origin.

(a) Show that the kinetic energy of the rigid body is

$$K(v, V) = \frac{1}{2} M \langle v, v \rangle + \frac{1}{2} \langle\langle V, V \rangle\rangle,$$

where $M = m(\mathbb{R}^3)$ is the total mass of the rigid body and $\langle\langle \cdot, \cdot \rangle\rangle$ is the metric for the rigid body with a fixed point determined by m.

(b) The motion of a rigid body falling in a constant gravitational field $-ge_z$ is determined by the conservative force given by the potential energy

$$U(x, S) = \int_{\mathbb{R}^3} V(x + S\xi) \, dm,$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is the function $V(x) = \langle x, ge_z \rangle$ (called the gravitational potential). Show that

$$U(x, S) = Mg\langle x, e_z \rangle,$$

and use this to argue that the projection of any motion on $SO(3)$ is a geodesic of $(SO(3), \langle\langle \cdot, \cdot \rangle\rangle)$. Would this still be true for a non-constant gravitational field (i.e. for a non-linear gravitational potential)?

(c) Compute the motion of the rigid body's center of mass (i.e. the projection of the motion on $\mathbb{R}^3$).