Geometric Mechanics

Homework 2

Due on October 9

1. Let M be a Riemannian manifold with Levi-Civita connection $\widetilde{\nabla}$, and let N be a submanifold endowed with the induced metric and Levi-Civita connection ∇. Let $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)$ be local extensions of $X, Y \in \mathfrak{X}(N)$.

   (a) Show that
   $$\nabla_X Y = \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}\right)^{\top},$$
   where $^{\top} : TM|_N \to TN$ is the orthogonal projection. (**Hint:** Use the Koszul formula).

   (b) Use this result to determine the geodesics of the sphere $S^n \subset \mathbb{R}^{n+1}$.

   (c) Recall that the second fundamental form of N is the map $B : T_pN \times T_pN \to (T_pN)^{\perp}$ defined at each point $p \in N$ by
   $$B(X, Y) := \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} - \nabla_X Y = \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y}\right)^{\perp}.$$
   Show that B is well defined, symmetric and bilinear.

2. Use spherical coordinates to write the equations of motion for the **spherical pendulum** of length l, i.e. a particle of mass $m > 0$ moving in $\mathbb{R}^3$ subject to a constant gravitational acceleration g and the holonomic constraint
   $$N = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = l^2\}.$$
   What are the equilibrium points? Which parallels of N are possible trajectories of the particle?