Geometric Mechanics

Homework 10

Due on November 30

1. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold, B the Poisson bivector and $(x^1, \ldots, x^n)$ local coordinates on M. Show that:

   (a) B can be written in these local coordinates as
   $$B = \sum_{i,j=1}^{n} B^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},$$
   where $B^{ij} = \{x^i, x^j\}$ for $i, j = 1, \ldots, n$;

   (b) The Hamiltonian vector field generated by $F \in C^\infty(M)$ can be written as
   $$X_F = \sum_{i,j=1}^{p} B^{ij} \frac{\partial F}{\partial x^i} \frac{\partial}{\partial x^j};$$

   (c) The components of B must satisfy
   $$\sum_{l=1}^{n} \left(B^{il} \frac{\partial B^{jk}}{\partial x^l} + B^{jl} \frac{\partial B^{ki}}{\partial x^l} + B^{kl} \frac{\partial B^{ij}}{\partial x^l} \right) = 0$$
   for all $i, j, k = 1, \ldots, n$;

   (d) If $\{\cdot,\cdot\}$ arises from a symplectic form ω then $(B^{ij}) = -(\omega_{ij})^{-1}$;

   (e) If B is nondegenerate then it arises from a symplectic form.

2. Recall that the the Lagrangian $L : TSO(3) \to \mathbb{R}$ for the Euler top is given by
   $$L = \frac{1}{2} \langle I\Omega, \Omega \rangle,$$
   where Ω are the left-invariant coordinates on the fibers resulting from the usual identifications $T_S SO(3) = dL_S(\mathfrak{so}(3)) \cong \mathfrak{so}(3) \cong \mathbb{R}^3$.

   (a) Show that if we use the Euclidean inner product $\langle \cdot,\cdot \rangle$ to identify $(\mathbb{R}^3)^*$ with $\mathbb{R}^3$ then the Legendre transformation is written $P = I\Omega$, where P are the corresponding left-invariant coordinates on $T^*SO(3)$. Determine the Hamiltonian $H : T^*SO(3) \to \mathbb{R}$.

   (b) The Poisson bracket on the reduced Poisson manifold $T^*SO(3)/SO(3) \cong \mathbb{R}^3$ (where the action is by the lift of the left translation) can be shown to be given by
   $$\{F, G\}(P) = \langle P, \operatorname{grad} F \times \operatorname{grad} G \rangle.$$
   (**Lie-Poisson reduction**). Determine X_H on the reduced manifold, and show that the equations $\dot{P} = X_H$ are precisely the Euler equations. What are the symplectic leaves?