Geometric Mechanics

Homework 1

Due on September 28

1. Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold with Levi-Civita connection ∇ and let $\langle\langle\cdot,\cdot\rangle\rangle = e^{2\rho}\langle\cdot,\cdot\rangle$ be a metric conformally related to $\langle\cdot,\cdot\rangle$ (where $\rho \in C^\infty(M)$). Show that the Levi-Civita connection $\widetilde{\nabla}$ of $\langle\langle\cdot,\cdot\rangle\rangle$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + d\rho(X)Y + d\rho(Y)X - \langle X, Y\rangle \operatorname{grad}\rho$$

for all $X, Y \in \mathfrak{X}(M)$, where the gradient is taken with respect to $\langle\cdot,\cdot\rangle$. (**Hint:** Use the Koszul formula).

2. Prove that a curve $c : I \subset \mathbb{R} \to M$ is a reparameterized geodesic of a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ if and only if it satisfies

$$\frac{D\dot{c}}{dt} = f(t)\,\dot{c}$$

for some differentiable function $f : I \to \mathbb{R}$.

3. Recall that the hyperbolic plane is the upper half plane

$$H = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\}$$

with the Riemannian metric

$$\langle\cdot,\cdot\rangle = \frac{1}{y^2}\,(dx \otimes dx + dy \otimes dy)$$

Use the local coordinate expression of Newton's equation to compute the Christoffel symbols for the Levi-Civita connection of $(H, \langle\cdot,\cdot\rangle)$ in the coordinates (x, y).